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BCK-filters Based on Fuzzy Points with Thresholds

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Abstract. The notions of $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy BCK-filters and fuzzy BCK-filters with thresholds are introduced, and several related properties are investigated. Characterizations of such notions are displayed, and implication-based fuzzy BCK-filters are discussed.

1. Introduction

Murali [11] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy subset, which is mentioned in [12], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [1, 2] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation ($\in$) and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee\, \mathrm{q})$-fuzzy subgroup. In particular, $(\in, \in \vee\, \mathrm{q})$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Jun and Song [9] discussed general forms of fuzzy interior ideals

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in semigroups. Also, Jun [3, 4] introduced the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigated related results. Using more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy subset, Jun [5] dealt with generalizations of results which are obtained in the papers [3, 4]. As a generalization of $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebras, he introduced the notions of $(\in, \mathrm{q}_k)$-fuzzy subalgebras and $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy subalgebras in a BCK/BCI-algebra X, and investigated several properties. He gave characterizations of $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy subalgebra in a BCK/BCI-algebra X which are generalization of characterizations of $(\in, \in\vee\,\mathrm{q})$-fuzzy subalgebra. Meng [10] introduced the notion of BCK-filters in BCK-algebras, and Jun et al. [8] considered the fuzzy theory of BCK-filters. Jun [6, 7] introduced the notion of $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy BCK-filters, and investigated related properties. He provided many characterizations of $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy BCK-filters, and discussed relations between a fuzzy BCK-filter and an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy BCK-filter. In this paper, we introduce the notions of $(\overline{\in}, \overline{\in}\vee\overline{\mathrm{q}_k})$-fuzzy BCK-filters and fuzzy BCK-filters with thresholds, and investigate related properties. We consider their characterizations, and discuss implication-based fuzzy BCK-filters.

2. Preliminaries

A BCK-algebra is an important calss of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

A nonempty set X with a constant 0 and a binary operation denoted by juxtaposition is called a *BCK-algebra* if for all $x, y, z \in X$ the following conditions hold:

(I) $((xy)(xz))(zy) = 0$,

(II) $(x(xy))y = 0$,

(III) $xx = 0$,

(IV) $0x = 0$,

(V) $xy = 0$ and $yx = 0$ imply $x = y$.

A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $xy = 0$. This ordering is called *BCK-ordering.* The following statements are true in any BCK-algebra X.

(a1) $x0 = x$.

(a2) $(xy)z = (xz)y$.

(a3) $xy \leq x$.

(a4) $(xy)z \leq (xz)(yz)$.

(a5) $x \leq y$ implies $xz \leq yz$ and $zy \leq zx$.

If there is a special element e of a BCK-algebra X satisfying $x \leq e$ for all $x \in X$, then e is called *unit* of X. A BCK-algebra with unit is said to be *bounded*. In a bounded BCK-algebra X, we denote ex by x^* for every $x \in X$.

In a bounded BCK-algebra, we have

(a6) $e^* = 0$ and $0^* = e$.

(a7) $y \leq x$ implies $x^* \leq y^*$.

(a8) $x^* y^* \leq yx$.

A nonempty subset F of a bounded BCK-algebra X is called a *BCK-filter* (see [10]) of X if it satisfies:

(F1) $e \in F$,

(F2) $(x^* y^*)^* \in F$ and $y \in F$ imply $x \in F$ for all $x, y \in X$.

Now we review some fuzzy logic concepts. A *fuzzy subset* of X is a function $\mathcal{A} : X \to [0,1]$. We shall use the notation $C(\mathcal{A}; t)$, called a *closed t-cut* of $\mathcal{A}$, for $\{x \in X \mid \mathcal{A}(x) \geq t\}$ where $t \in [0,1]$.

A fuzzy subset $\mathcal{A}$ of a set X of the form

$$\mathcal{A}(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{1.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by $[x;t]$. For a fuzzy subset $\mathcal{A}$ of a set X, we say that a fuzzy point $[x;t]$ is

(o1) *contained* in $\mathcal{A}$, denoted by $[x;t] \in \mathcal{A}$, ([12]) if $\mathcal{A}(x) \geq t$.

(o2) *quasi-coincident* with $\mathcal{A}$, denoted by $[x;t]\,\mathrm{q}\,\mathcal{A}$, ([12]) if $\mathcal{A}(x) + t > 1$.

For a fuzzy point $[x;t]$ and a fuzzy subset $\mathcal{A}$ of a set X, we say that

(o3) $[x;t] \in\!\vee\,\mathrm{q}\,\mathcal{A}$ if $[x;t] \in \mathcal{A}$ or $[x;t]\,\mathrm{q}\,\mathcal{A}$.

(o4) $[x;t] \in\!\wedge\,\mathrm{q}\,\mathcal{A}$ if $[x;t] \in \mathcal{A}$ and $[x;t]\,\mathrm{q}\,\mathcal{A}$.

(o5) $[x;t]\,\overline{\alpha}\,\mathcal{A}$ if $[x;t]\,\alpha\,\mathcal{A}$ does not hold for $\alpha \in \{\in, \mathrm{q}, \in\!\vee\,\mathrm{q}, \in\!\wedge\,\mathrm{q}\}$.

In what follows let X denote a bounded BCK-algebra.

A fuzzy subset $\mathcal{A}$ of X is called a *fuzzy BCK-filter* of X (see [8]) if it satisfies:

(b1) $(\forall x \in X)\ (\mathcal{A}(e) \geq \mathcal{A}(x))$,

(b2) $(\forall x, y \in X)\ (\mathcal{A}(x) \geq \min\{\mathcal{A}((x^* y^*)^*\}, \mathcal{A}(y)))$.

3. $(\in, \in\!\vee\,\mathrm{q}_k)$-fuzzy BCK-filters

Let k denote an arbitrary element of $[0,1)$ unless otherwise specified. For a fuzzy point $[x;t]$ and a fuzzy subset $\mathcal{A}$ of X, we say that (o6) $[x;t]\,\mathrm{q}_k\,\mathcal{A}$ if $\mathcal{A}(x)+t+k>1$.

(o7) $[x;t]\in\vee\,\mathrm{q}_k\,\mathcal{A}$ if $[x;t]\in\mathcal{A}$ or $[x;t]\,\mathrm{q}_k\,\mathcal{A}$.

(o8) $[x;t]\in\wedge\,\mathrm{q}_k\,\mathcal{A}$ if $[x;t]\in\mathcal{A}$ and $[x;t]\,\mathrm{q}_k\,\mathcal{A}$.

(o9) $[x;t]\,\overline{\alpha}\,\mathcal{A}$ if $[x;t]\,\alpha\,\mathcal{A}$ does not hold for $\alpha\in\{\mathrm{q}_k,\in\vee\,\mathrm{q}_k,\in\wedge\,\mathrm{q}_k\}$.

Definition 3.1([6]). A fuzzy subset $\mathcal{A}$ of X is called an $(\in,\in\vee\,\mathrm{q}_k)$-*fuzzy BCK-filter* of X if it satisfies:

(d1) $[x;t]\in\mathcal{A}\;\Rightarrow\;[e;t]\in\vee\,\mathrm{q}_k\,\mathcal{A}$.

(d2) $[(x^*y^*)^*;t_1]\in\mathcal{A},\ [y;t_2]\in\mathcal{A}\;\Rightarrow\;[x;\min\{t_1,t_2\}]\in\vee\,\mathrm{q}_k\,\mathcal{A}$

for all $x,y\in X$ and $t,t_1,t_2\in(0,1]$.

An $(\in,\in\vee\,\mathrm{q}_k)$-fuzzy BCK-filter of X with $k=0$ is called an $(\in,\in\vee\,\mathrm{q})$-*fuzzy BCK-filter* of X.

Lemma 3.2([6]). *A fuzzy subset $\mathcal{A}$ of X is an $(\in,\in\vee\,\mathrm{q}_k)$-fuzzy BCK-filter of X if and only if it satisfies:*

(1) $(\forall x\in X)\,(\mathcal{A}(e)\geq\min\{\mathcal{A}(x),\frac{1-k}{2}\})$,

(2) $(\forall x,y\in X)\,(\mathcal{A}(x)\geq\min\{\mathcal{A}((x^*y^*)^*),\mathcal{A}(y),\frac{1-k}{2}\})$.

For any fuzzy subset $\mathcal{A}$ of X and any $t\in(0,1]$, we consider four subsets:

$$Q(\mathcal{A};t):=\{x\in R\mid[x;t]\mathrm{q}\mathcal{A}\},\quad[\mathcal{A}]_t:=\{x\in R\mid[x;t]\in\vee\,\mathrm{q}\,\mathcal{A}\},$$

$$Q^k(\mathcal{A};t):=\{x\in R\mid[x;t]\mathrm{q}_k\mathcal{A}\},\quad[\mathcal{A}]_t^k:=\{x\in R\mid[x;t]\in\vee\,\mathrm{q}_k\mathcal{A}\}.$$

It is clear that $[\mathcal{A}]_t=C(\mathcal{A};t)\cup Q(\mathcal{A};t)$ and $[\mathcal{A}]_t^k=C(\mathcal{A};t)\cup Q^k(\mathcal{A};t)$.

Theorem 3.3. *If $\mathcal{A}$ is an $(\in,\in\vee\,\mathrm{q}_k)$-fuzzy BCK-filter of X, then*

$$\text{(3.1)}\qquad\Big(\forall t\in(\tfrac{1-k}{2},1]\Big)\ \Big(Q^k(\mathcal{A};t)\neq\emptyset\;\Rightarrow\;Q^k(\mathcal{A};t)\text{ is a BCK-filter of }X\Big).$$

Proof. Assume that $\mathcal{A}$ is an $(\in,\in\vee\,\mathrm{q}_k)$-fuzzy BCK-filter of X and let $t\in(\frac{1-k}{2},1]$ be such that $Q^k(\mathcal{A};t)\neq\emptyset$. Then there exists $x\in Q^k(\mathcal{A};t)$, and so $\mathcal{A}(x)+t+k>1$. It follows from Lemma 3.2(1) that

$$\begin{array}{rcl}\mathcal{A}(e)&\geq&\min\{\mathcal{A}(x),\frac{1-k}{2}\}\\&=&\left\{\begin{array}{ll}\frac{1-k}{2}&\text{if }\mathcal{A}(x)\geq\frac{1-k}{2},\\\mathcal{A}(x)&\text{if }\mathcal{A}(x)<\frac{1-k}{2},\end{array}\right.\\&>&1-t-k,\end{array}$$

that is, $e \in Q^k(\mathcal{A};t)$. Let $x, y \in X$ be such that $(x^*y^*)^* \in Q^k(\mathcal{A};t)$ and $y \in Q^k(\mathcal{A};t)$. Then $\mathcal{A}((x^*y^*)^*) + t + k > 1$ and $\mathcal{A}(y) + t + k > 1$. Using Lemma 3.2(2), we have

$$\begin{aligned} \mathcal{A}(x) &\geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \tfrac{1-k}{2}\} \\ &= \begin{cases} \frac{1-k}{2} & \text{if } \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \geq \frac{1-k}{2}, \\ \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} & \text{if } \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} < \frac{1-k}{2}, \end{cases} \\ &> 1 - t - k, \end{aligned}$$

that is, $x \in Q^k(\mathcal{A};t)$. Therefore $Q^k(\mathcal{A};t)$ is a BCK-filter of X. □

Corollary 3.4. *If* $\mathcal{A}$ *is an* $(\in, \in\vee\,\mathrm{q}\,)$*-fuzzy BCK-filter of* X*, then*

(3.2) $$\Big(\forall t \in (0.5, 1]\Big) \Big(Q(\mathcal{A};t) \neq \emptyset \;\Rightarrow\; Q(\mathcal{A};t) \text{ is a BCK-filter of } X\Big).$$

Theorem 3.5. *For any fuzzy subset* $\mathcal{A}$ *of* X*, the following are equivalent:*

(1) $\mathcal{A}$ *is an* $(\in, \in\vee\,\mathrm{q}_k)$*-fuzzy BCK-filter of* X.

(2) $(\forall t \in (0,1])$ $\Big([\mathcal{A}]_t^k \neq \emptyset \implies [\mathcal{A}]_t^k \text{ is a BCK-filter of } X\Big)$.

We call $[\mathcal{A}]_t^k$ an $\in\vee\,\mathrm{q}_k$*-level BCK-filter* of $\mathcal{A}$.

Proof. Assume that $\mathcal{A}$ is an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy BCK-filter of X and let $t \in (0,1]$ be such that $[\mathcal{A}]_t^k \neq \emptyset$. Then there exists $x \in [\mathcal{A}]_t^k$ and so $x \in C(\mathcal{A};t)$ or $x \in Q^k(\mathcal{A};t)$, i.e., $\mathcal{A}(x) \geq t$ or $\mathcal{A}(x) + t + k > 1$. Using Lemma 3.2(1), we obtain

$$\mathcal{A}(e) \geq \min\{\mathcal{A}(x), \tfrac{1-k}{2}\} = \begin{cases} \frac{1-k}{2} & \text{if } \mathcal{A}(x) > \frac{1-k}{2}, \\ \mathcal{A}(x) & \text{if } \mathcal{A}(x) \leq \frac{1-k}{2}. \end{cases}$$

Assume that $\mathcal{A}(x) \leq \frac{1-k}{2}$. If $\mathcal{A}(x) \geq t$, then $\mathcal{A}(e) \geq \mathcal{A}(x) \geq t$ and so $e \in C(\mathcal{A};t)$. If $\mathcal{A}(x) + t + k > 1$, then $\mathcal{A}(e) \geq \mathcal{A}(x) > 1 - t - k$ and thus $e \in Q^k(\mathcal{A};t)$. Hence $e \in C(\mathcal{A};t) \cup Q^k(\mathcal{A};t) = [\mathcal{A}]_t^k$. Suppose that $\mathcal{A}(x) > \frac{1-k}{2}$. Then either $\mathcal{A}(e) \geq \frac{1-k}{2} \geq t$ or $\mathcal{A}(e) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$. Thus $e \in C(\mathcal{A};t) \cup Q^k(\mathcal{A};t) = [\mathcal{A}]_t^k$. Let $x, y \in X$ be such that $(x^*y^*)^* \in [\mathcal{A}]_t^k$ and $y \in [\mathcal{A}]_t^k$. Then $\mathcal{A}((x^*y^*)^*) \geq t$ or $\mathcal{A}((x^*y^*)^*) + t + k > 1$, and $\mathcal{A}(y) \geq t$ or $\mathcal{A}(y) + t + k > 1$. We can consider four cases:

(3.3) $$\mathcal{A}((x^*y^*)^*) \geq t \text{ and } \mathcal{A}(y) \geq t,$$

(3.4) $$\mathcal{A}((x^*y^*)^*) \geq t \text{ and } \mathcal{A}(y) + t + k > 1,$$

(3.5) $$\mathcal{A}((x^*y^*)^*) + t + k > 1 \text{ and } \mathcal{A}(y) \geq t,$$

(3.6) $$\mathcal{A}((x^*y^*)^*) + t + k > 1 \text{ and } \mathcal{A}(y) + t + k > 1.$$

For the first case, Lemma 3.2(2) implies that

$$\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \tfrac{1-k}{2}\} \geq \min\{t, \tfrac{1-k}{2}\} = \begin{cases} \frac{1-k}{2} & \text{if } t > \frac{1-k}{2}, \\ t & \text{if } t \leq \frac{1-k}{2}, \end{cases}$$

and so $\mathcal{A}(x) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, i.e. $[x;t]\mathrm{q}_k\mathcal{A}$, or $x \in C(\mathcal{A};t)$. Therefore $x \in C(\mathcal{A};t) \cup Q^k(\mathcal{A};t) = [\mathcal{A}]_t^k$. For the case (3.4), assume that $t > \frac{1-k}{2}$. Then $1 - t - k \leq 1 - t < \frac{1-k}{2}$. Thus, if $\min\{\mathcal{A}(y), \frac{1-k}{2}\} \leq \mathcal{A}((x^*y^*)^*)$, then

$$\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \tfrac{1-k}{2}\} = \min\{\mathcal{A}(y), \tfrac{1-k}{2}\} > 1 - t - k,$$

and if $\min\{\mathcal{A}(y), \frac{1-k}{2}\} > \mathcal{A}((x^*y^*)^*)$, then

$$\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \tfrac{1-k}{2}\} = \mathcal{A}((x^*y^*)^*) \geq t$$

by Lemma 3.2(2). Hence $x \in C(\mathcal{A};t) \cup Q^k(\mathcal{A};t) = [\mathcal{A}]_t^k$. Suppose that $t \leq \frac{1-k}{2}$. Then $1 - t \geq \frac{1-k}{2}$. Using Lemma 3.2(2), we obtain

$$\begin{aligned} \mathcal{A}(x) &\geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \tfrac{1-k}{2}\} \\ &= \begin{cases} \min\{\mathcal{A}((x^*y^*)^*), \frac{1-k}{2}\} \geq t & \text{if } \min\{\mathcal{A}((x^*y^*)^*), \frac{1-k}{2}\} \leq \mathcal{A}(y), \\ \mathcal{A}(y) > 1 - t - k & \text{if } \min\{\mathcal{A}((x^*y^*)^*), \frac{1-k}{2}\} > \mathcal{A}(y), \end{cases} \end{aligned}$$

which implies that $x \in C(\mathcal{A};t) \cup Q^k(\mathcal{A};t) = [\mathcal{A}]_t^k$. We have similar result for the case (3.5). For the final case, assume that $t > \frac{1-k}{2}$. Then $1 - t - k \leq 1 - t < \frac{1-k}{2}$. If $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \geq \frac{1-k}{2}$, then $\mathcal{A}(x) \geq \frac{1-k}{2} > 1 - t - k$. If $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} < \frac{1-k}{2}$, then

$$\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} > 1 - t - k.$$

Hence $x \in Q^k(\mathcal{A};t) \subseteq [\mathcal{A}]_t^k$. Suppose that $t \leq \frac{1-k}{2}$. If $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \geq \frac{1-k}{2}$, then $\mathcal{A}(x) \geq \frac{1-k}{2} \geq t$. If $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} < \frac{1-k}{2}$, then

$$\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} > 1 - t - k.$$

Thus $x \in C(\mathcal{A};t) \cup Q^k(\mathcal{A};t) = [\mathcal{A}]_t^k$. Therefore $[\mathcal{A}]_t^k$ is a BCK-filter of X.

Conversely, suppose that (2) is valid. If there exists $a \in X$ such that $\mathcal{A}(e) < \min\{\mathcal{A}(a), \frac{1-k}{2}\}$, then $\mathcal{A}(e) < t_e \leq \min\{\mathcal{A}(a), \frac{1-k}{2}\}$ for some $t_e \in (0, \frac{1-k}{2}]$. Thus $a \in C(\mathcal{A};t_e) \subseteq [\mathcal{A}]_{t_e}^k$ and $e \notin C(\mathcal{A};t_e)$. Also, we have $\mathcal{A}(e) + t_e < 2t_e \leq 1 - k$, and so $[e;t_e]\,\overline{\mathrm{q}_k}\,\mathcal{A}$, i.e., $e \notin Q^k(\mathcal{A};t_e)$. Therefore $e \notin [\mathcal{A}]_{t_e}^k$, a contradiction. Therefore $\mathcal{A}(e) \geq \min\{\mathcal{A}(x), \frac{1-k}{2}\}$ for all $x \in X$. Suppose there exist $a, b \in X$ such that $\mathcal{A}(a) < \min\{\mathcal{A}((a^*b^*)^*), \mathcal{A}(b), \frac{1-k}{2}\}$. Then $\mathcal{A}(a) < t_a \leq \min\{\mathcal{A}((a^*b^*)^*), \mathcal{A}(b), \frac{1-k}{2}\}$ for some $t_a \in (0, \frac{1-k}{2}]$. It follows that $(a^*b^*)^* \in C(\mathcal{A};t_a) \subseteq [\mathcal{A}]_{t_a}^k$ and $b \in C(\mathcal{A};t_a) \subseteq$

$[\mathcal{A}]^k_{t_a}$ so from (F2) that $a \in [\mathcal{A}]^k_{t_a}$. Thus $\mathcal{A}(a) \geq t_a$ or $\mathcal{A}(a) + t_a + k > 1$, a contradiction. Hence $\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \frac{1-k}{2}\}$ for all $x, y \in X$. Using Lemma 3.2, we conclude that $\mathcal{A}$ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy BCK-filter of X. □

Corollary 3.6. *For any fuzzy subset $\mathcal{A}$ of X, the following are equivalent:*

(1) *$\mathcal{A}$ is an $(\in, \in\vee\, \mathrm{q}\,)$-fuzzy BCK-filter of X.*

(2) $(\forall t \in (0,1])\ \Big([\mathcal{A}]_t \neq \emptyset \implies [\mathcal{A}]_t$ is a BCK-filter of $X\Big)$.

A fuzzy subset $\mathcal{A}$ of X is said to be *proper* if $\mathrm{Im}(\mathcal{A})$ has at least two elements. Two fuzzy subsets are said to be *equivalent* if they have same family of closed t-cuts. Otherwise, they are said to be *non-equivalent*.

Theorem 3.7. *Let $\mathcal{A}$ be an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy BCK-filter of X such that $\#\{\mathcal{A}(x) \mid \mathcal{A}(x) < \frac{1-k}{2}\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy BCK-filters of X such that $\mathcal{A}$ can be expressed as the union of them.*

Proof. Let $\{\mathcal{A}(x) \mid \mathcal{A}(x) < \frac{1-k}{2}\} = \{t_1, t_2, \cdots, t_n\}$, where $t_1 > t_2 > \cdots > t_n$ and $n \geq 2$. Then the chain of $\in\vee\, \mathrm{q}_k$-level BCK-filters of $\mathcal{A}$ is

$$[\mathcal{A}]^k_{\frac{1-k}{2}} \subseteq [\mathcal{A}]^k_{t_1} \subseteq [\mathcal{A}]^k_{t_2} \subseteq \cdots \subseteq [\mathcal{A}]^k_{t_n} = X.$$

Let $\mathcal{B}$ and $\mathcal{C}$ be fuzzy subsets of X defined by

$$\mathcal{B}(x) = \begin{cases} t_1 & \text{if } x \in [\mathcal{A}]^k_{t_1}, \\ t_2 & \text{if } x \in [\mathcal{A}]^k_{t_2} \setminus [\mathcal{A}]^k_{t_1}, \\ \cdots & \\ t_n & \text{if } x \in [\mathcal{A}]^k_{t_n} \setminus [\mathcal{A}]^k_{t_{n-1}}, \end{cases}$$

and

$$\mathcal{C}(x) = \begin{cases} \mathcal{A}(x) & \text{if } x \in [\mathcal{A}]^k_{\frac{1-k}{2}}, \\ k & \text{if } x \in [\mathcal{A}]^k_{t_2} \setminus [\mathcal{A}]^k_{\frac{1-k}{2}}, \\ t_3 & \text{if } x \in [\mathcal{A}]^k_{t_3} \setminus [\mathcal{A}]^k_{t_2}, \\ \cdots & \\ t_n & \text{if } x \in [\mathcal{A}]^k_{t_n} \setminus [\mathcal{A}]^k_{t_{n-1}}, \end{cases}$$

respectively, where $t_3 < k < t_2$. Then $\mathcal{B}$ and $\mathcal{C}$ are $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy BCK-filters of X, and $\mathcal{B}, \mathcal{C} \leq \mathcal{A}$. The chains of $\in\vee\, \mathrm{q}_k$-level BCK-filters of $\mathcal{B}$ and $\mathcal{C}$ are, respectively, given by

$$[\mathcal{A}]^k_{t_1} \subseteq [\mathcal{A}]^k_{t_2} \subseteq \cdots \subseteq [\mathcal{A}]^k_{t_n}$$

and

$$[\mathcal{A}]^k_{\frac{1-k}{2}} \subseteq [\mathcal{A}]^k_{t_2} \subseteq \cdots \subseteq [\mathcal{A}]^k_{t_n}.$$

Therefore $\mathcal{B}$ and $\mathcal{C}$ are non-equivalent and clearly $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$. This completes the proof. □

Corollary 3.8. *Let $\mathcal{A}$ be an $(\in, \in\vee\,\mathrm{q})$-fuzzy BCK-filter of X such that $\#\{\mathcal{A}(x) \mid \mathcal{A}(x) < 0.5\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in\vee\,\mathrm{q})$-fuzzy BCK-filters of X such that $\mathcal{A}$ can be expressed as the union of them.*

It is well known that a fuzzy subset $\mathcal{A}$ of X is a fuzzy BCK-filter of X if and only if the non-empty closed t-cut $C(\mathcal{A}; t)$, $t \in (0, 1]$, of $\mathcal{A}$ is a BCK-filter of X. Note that for a fuzzy subset $\mathcal{A}$ of X, the non-empty closed t-cut $C(\mathcal{A}; t)$, $t \in (0, \frac{1-k}{2}]$, of $\mathcal{A}$ is a BCK-filter of X if and only if $\mathcal{A}$ is an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy BCK-filter of X (see [6]).

Since it is natural to consider the number $t \in (\frac{1-k}{2}, 1]$ for which $C(\mathcal{A}; t)$ is a BCK-filter of X, we consider a new kind of a fuzzy BCK-filter as follows.

Definition 3.9. A fuzzy subset $\mathcal{A}$ of X is called an $(\overline{\in}, \overline{\in}\vee\overline{\mathrm{q}_k})$*-fuzzy BCK-filter* of X if it satisfies:

(d3) $[e; t]\,\overline{\in}\,\mathcal{A} \;\Rightarrow\; [x; t]\,\overline{\in}\vee\overline{\mathrm{q}_k}\,\mathcal{A}$,

(d4) $[x; \min\{t_1, t_2\}]\,\overline{\in}\,\mathcal{A} \;\Rightarrow\; [(x^*y^*)^*; t_1]\,\overline{\in}\vee\overline{\mathrm{q}_k}\,\mathcal{A}$ or $[y; t_2]\,\overline{\in}\vee\overline{\mathrm{q}_k}\,\mathcal{A}$

for all $x, y \in X$ and $t, t_1, t_2 \in (0, 1)$.

An $(\overline{\in}, \overline{\in}\vee\overline{\mathrm{q}_k})$-fuzzy BCK-filter of X with $k = 0$ is called an $(\overline{\in}, \overline{\in}\vee\overline{\mathrm{q}})$*-fuzzy BCK-filter* of X.

Example 3.10. Consider a bounded BCK-algebra $X = \{0, a, b, e\}$ with a Cayley table which is given by Table 1. Define a fuzzy set $\mathcal{A}$ in X as follows:

$$\mathcal{A} : X \to [0, 1], \quad x \mapsto \begin{cases} 0.37 & \text{if } x = 0, \\ 0.8 & \text{if } x \in \{a, e\}, \\ 0.2 & \text{if } x = b. \end{cases}$$

By routine calculations, we know that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in}\vee\overline{\mathrm{q}_{0.26}})$-fuzzy BCK-filter of X. But it is not an $(\overline{\in}, \overline{\in}\vee\overline{\mathrm{q}_{0.28}})$-fuzzy BCK-filter of X since $[b; \min\{0.37, 0.365\}]\,\overline{\in}\,\mathcal{A}$,

	0	a	b	e
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
e	e	b	a	0

Table 1: Cayley table

$[(b^*0^*)^*; 0.37] \in \wedge\, q_{0.28}\, \mathcal{A}$ and $[0; 0.365] \in \wedge\, q_{0.28}\, \mathcal{A}$.

Theorem 3.11. *A fuzzy subset $\mathcal{A}$ of X is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy BCK-filter of X if it satisfies:*

(1) $\max\{\mathcal{A}(e), \frac{1-k}{2}\} \geq \mathcal{A}(x)$,

(2) $\max\{\mathcal{A}(x), \frac{1-k}{2}\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$

for all $x, y \in X$.

Proof. Assume that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy BCK-filter of X. If there exists $a \in X$ such that $\max\{\mathcal{A}(e), \frac{1-k}{2}\} < t = \mathcal{A}(a)$, then $\frac{1-k}{2} < t \leq 1$, $[e; t]\, \overline{\in}\, \mathcal{A}$ and $[a; t] \in \mathcal{A}$. It follows from (d3) that $[a; t]\, \overline{q_k}\, \mathcal{A}$. Hence $\mathcal{A}(a) \geq t$ and $\mathcal{A}(a) + t + k \leq 1$, which imply that $t \leq \frac{1-k}{2}$. This is a contradiction, and so $\max\{\mathcal{A}(e), \frac{1-k}{2}\} \geq \mathcal{A}(x)$ for all $x \in X$. Now, suppose that Theorem 3.11(2) is not valid. Then there exist $a, b \in X$ such that

$$\max\{\mathcal{A}(a), \tfrac{1-k}{2}\} < t_a = \min\{\mathcal{A}((a^*b^*)^*), \mathcal{A}(b)\}.$$

Thus $\frac{1-k}{2} < t_a \leq 1$, $[a; t_a]\, \overline{\in}\, \mathcal{A}$, $[(a^*b^*)^*; t_a] \in \mathcal{A}$ and $[b; t_a] \in \mathcal{A}$. Using (d4), we have $[(a^*b^*)^*; t_a]\, \overline{q_k}\, \mathcal{A}$ or $[b; t_a]\, \overline{q_k}\, \mathcal{A}$. It follows that $\mathcal{A}((a^*b^*)^*) \geq t_a$ and $\mathcal{A}((a^*b^*)^*) + t_a + k \leq 1$, or $\mathcal{A}(b) \geq t_a$ and $\mathcal{A}(b) + t_a + k \leq 1$. Hence $t_a \leq \frac{1-k}{2}$, a contradiction. Therefore 3.11(2) is valid.

Conversely, let $\mathcal{A}$ be a fuzzy subset of X that satisfies conditions (1) and (2) of Theorem 3.11. Let $x, y \in X$ and $t \in (0, 1)$ be such that $[e; t]\, \overline{\in}\, \mathcal{A}$. Then $\mathcal{A}(e) < t$. If $\mathcal{A}(e) \geq \mathcal{A}(x)$, then $\mathcal{A}(x) < t$ and so $[x; t]\, \overline{\in} \vee \overline{q_k}\, \mathcal{A}$. If $\mathcal{A}(e) < \mathcal{A}(x)$, then $\mathcal{A}(x) \leq \frac{1-k}{2}$ by Theorem 3.11(1). Assume that $[x; t] \in \mathcal{A}$. Then $t \leq \mathcal{A}(x) \leq \frac{1-k}{2}$, and thus

$$\mathcal{A}(x) + t + k \leq 2\mathcal{A}(x) + k \leq 1,$$

i.e., $[x; t]\, \overline{q_k}\, \mathcal{A}$. Hence $[x; t]\, \overline{\in} \vee \overline{q_k}\, \mathcal{A}$. Let $x, y \in X$ and $t_1, t_2 \in (0, 1)$ be such that $[x; \min\{t_1, t_2\}]\, \overline{\in}\, \mathcal{A}$. Then $\mathcal{A}(x) < \min\{t_1, t_2\}$. If

$$\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\},$$

then $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} < \min\{t_1, t_2\}$, and so $\mathcal{A}((x^*y^*)^*) < t_1$ or $\mathcal{A}(y) < t_2$. Thus $[(x^*y^*)^*; t_1]\, \overline{\in}\, \mathcal{A}$ or $[y; t_2]\, \overline{\in}\, \mathcal{A}$, which imply that $[(x^*y^*)^*; t_1]\, \overline{\in} \vee \overline{q_k}\, \mathcal{A}$ or $[y; t_2]\, \overline{\in} \vee \overline{q_k}\, \mathcal{A}$. If $\mathcal{A}(x) < \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$, then

$$\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \leq \tfrac{1-k}{2}$$

by Theorem 3.11(2). Setting $[(x^*y^*)^*; t_1] \in \mathcal{A}$ and $[y; t_2] \in \mathcal{A}$, we have $t_1 \leq \mathcal{A}((x^*y^*)^*) \leq \frac{1-k}{2}$ or $t_2 \leq \mathcal{A}(y) \leq \frac{1-k}{2}$. Hence

$$\mathcal{A}((x^*y^*)^*) + t_1 + k \leq 2\mathcal{A}((x^*y^*)^*) + k \leq 1,$$

i.e., $[(x^*y^*)^*; t_1]\, \overline{\mathrm{q}_k}\, \mathcal{A}$, or $\mathcal{A}(y) + t_2 + k \leq 2\mathcal{A}(y) + k \leq 1$, i.e., $[y; t_2]\, \overline{\mathrm{q}_k}\, \mathcal{A}$. Therefore $[(x^*y^*)^*; t_1]\, \overline{\in \vee\, \mathrm{q}_k}\, \mathcal{A}$ or $[y; t_2]\, \overline{\in \vee\, \mathrm{q}_k}\, \mathcal{A}$. Consequently, $\mathcal{A}$ is an $(\overline{\in}, \overline{\in \vee\, \mathrm{q}_k})$-fuzzy BCK-filter of X. □

Corollary 3.12. *A fuzzy subset $\mathcal{A}$ of X is an $(\overline{\in}, \overline{\in \vee\, \mathrm{q}})$-fuzzy BCK-filter of X if it satisfies:*

(1) $\max\{\mathcal{A}(e), 0.5\} \geq \mathcal{A}(x)$,

(2) $\max\{\mathcal{A}(x), 0.5\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$

for all $x, y \in X$.

Corollary 3.13. *Every $(\overline{\in}, \overline{\in \vee\, \mathrm{q}_k})$-fuzzy BCK-filter of X satisfies the following assertion:*

$$(\forall x, y \in X)\ (x^*y^* = 0\ \Rightarrow\ \max\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}(e), \mathcal{A}(y)\}).$$

Proof. It is straightforward by Theorem 3.11(2) and (a6). □

Corollary 3.14. *Every $(\overline{\in}, \overline{\in \vee\, \mathrm{q}})$-fuzzy BCK-filter of X satisfies the following assertion:*

$$(\forall x, y \in X)\ (x^*y^* = 0\ \Rightarrow\ \max\{\mathcal{A}(x), 0.5\} \geq \min\{\mathcal{A}(e), \mathcal{A}(y)\}).$$

Theorem 3.15. *If $0 < r < k \leq 1$, then every $(\overline{\in}, \overline{\in \vee\, \mathrm{q}_k})$-fuzzy BCK-filter is an $(\overline{\in}, \overline{\in \vee\, \mathrm{q}_r})$-fuzzy BCK-filter.*

Proof. Let $\mathcal{A}$ be an $(\overline{\in}, \overline{\in \vee \mathrm{q}_k})$-fuzzy BCK-filter of X. Using (1) and (2) of Theorem 3.11, we have

$$\max\{\mathcal{A}(e), \tfrac{1-r}{2}\} \geq \max\{\mathcal{A}(e), \tfrac{1-k}{2}\} \geq \mathcal{A}(x)$$

and

$$\max\{\mathcal{A}(x), \tfrac{1-r}{2}\} \geq \max\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$$

for all $x, y \in X$. Hence $\mathcal{A}$ is an $(\overline{\in}, \overline{\in \vee \mathrm{q}_r})$-fuzzy BCK-filter of X by Theorem 3.11. □

Example 3.10 shows that the converse of Theorem 3.15 may not be true.

Theorem 3.16. *For a fuzzy subset $\mathcal{A}$ of X, the following are equivalent:*

(1) *$\mathcal{A}$ is an $(\overline{\in}, \overline{\in \vee\, \mathrm{q}_k})$-fuzzy BCK-filter of X.*

(2) $(\forall t \in (\frac{1-k}{2}, 1])$ $(C(\mathcal{A}; t) \neq \emptyset \Rightarrow C(\mathcal{A}; t)$ *is a BCK-filter of* $X)$.

Proof. Assume that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy BCK-filter of X. Let $t \in (\frac{1-k}{2}, 1]$ be such that $C(\mathcal{A}; t) \neq \emptyset$. Using Theorem 3.11(1), we get $\mathcal{A}(x) \leq \max\{\mathcal{A}(e), \frac{1-k}{2}\}$ for all $x \in C(\mathcal{A}; t)$. It follows that

$$t \leq \mathcal{A}(x) \leq \max\{\mathcal{A}(e), \tfrac{1-k}{2}\} = \mathcal{A}(e)$$

so that $e \in C(\mathcal{A}; t)$. Let $x, y \in X$ be such that $(x^*y^*)^* \in C(\mathcal{A}; t)$ and $y \in C(\mathcal{A}; t)$. Then $\mathcal{A}((x^*y^*)^*) \geq t$ and $\mathcal{A}(y) \geq t$. It follows from Theorem 3.11(2) that

$$\max\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \geq t$$

so that $\mathcal{A}(x) \geq t$ since $t < \frac{1-k}{2}$. Hence $x \in C(\mathcal{A}; t)$, and therefore $C(\mathcal{A}; t)$ is a BCK-filter of X.

Conversely, let $\mathcal{A}$ be a fuzzy subset of X such that (2) is valid. If there exists $a \in X$ such that $\max\{\mathcal{A}(e), \frac{1-k}{2}\} < \mathcal{A}(a)$, then

$$\max\{\mathcal{A}(e), \tfrac{1-k}{2}\} < t_e \leq \mathcal{A}(a)$$

for some $t_e \in (\frac{1-k}{2}, 1]$. Hence $e \notin C(\mathcal{A}; t_e)$, a contradiction. Thus

$$\max\{\mathcal{A}(e), \tfrac{1-k}{2}\} \geq \mathcal{A}(x)$$

for all $x \in X$. Assume that

$$\max\{\mathcal{A}(a), \tfrac{1-k}{2}\} < \min\{\mathcal{A}((a^*b^*)^*), \mathcal{A}(b)\}$$

for some $a, b \in X$. Then there exists $t_a \in (\frac{1-k}{2}, 1]$ such that

$$\max\{\mathcal{A}(a), \tfrac{1-k}{2}\} < t_a \leq \min\{\mathcal{A}((a^*b^*)^*), \mathcal{A}(b)\}.$$

It follows that $(a^*b^*)^* \in C(\mathcal{A}; t_a)$ and $b \in C(\mathcal{A}; t_a)$ so from (F2) that $a \in C(\mathcal{A}; t_a)$, i.e., $\mathcal{A}(a) \geq t_a$. This is a contradiction, and so

$$\max\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$$

for all $x, y \in X$. Using Theorem 3.11, we conclude that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy BCK-filter of X. □

Corollary 3.17. *For a fuzzy subset* $\mathcal{A}$ *of* X, *the following are equivalent:*

(1) $\mathcal{A}$ *is an* $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}})$*-fuzzy BCK-filter of* X.

(2) $(\forall t \in (0.5, 1])$ $(C(\mathcal{A}; t) \neq \emptyset \Rightarrow C(\mathcal{A}; t)$ *is a BCK-filter of* $X)$.

For a fuzzy subset $\mathcal{A}$ of X, we consider the following set:

$$\Gamma := \{t \in (0,1] \mid C(\mathcal{A};t) \neq \emptyset \Rightarrow C(\mathcal{A};t) \text{ is a BCK-filter of } X\}.$$

Then

(1) If $\Gamma = (0,1]$, then $\mathcal{A}$ is a fuzzy BCK-filter of X.

(2) If $\Gamma = (0, \frac{1-k}{2}]$, then $\mathcal{A}$ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy BCK-filter of X.

(3) If $\Gamma = (\frac{1-k}{2}, 1]$, then $\mathcal{A}$ is an $(\overline{\in}, \overline{\in}\vee \overline{\mathrm{q}_k})$-fuzzy BCK-filter of X.

Now we have the following question:

Question. *If $\Gamma = (\varepsilon, \delta]$ where $\varepsilon < \delta$ in $(0,1]$, then what kind of a fuzzy BCK-filter is $\mathcal{A}$?, and what is the relation between them?*

To discuss this question, we introduce the following definition.

Definition 3.18. A fuzzy subset $\mathcal{A}$ of X is called a *fuzzy BCK-filter* of X with *thresholds* ε and δ where $\varepsilon, \delta \in (0,1]$ with $\varepsilon < \delta$ if it satisfies:

(d5) $(\forall x \in X)\ (\max\{\mathcal{A}(e), \varepsilon\} \geq \min\{\mathcal{A}(x), \delta\})$,

(d6) $(\forall x, y \in X)\ (\max\{\mathcal{A}(x), \varepsilon\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \delta\})$.

Example 3.19([6]). Consider a bounded BCK-algebra $X = \{0, x, y, z, e\}$ with a Cayley table (Table 2). Define a fuzzy set $\mathcal{A}$ in X as follows:

$$\mathcal{A} : X \to [0,1], \quad w \mapsto \begin{cases} 0.3 & \text{if } w = 0, \\ 0.2 & \text{if } w = x, \\ 0.1 & \text{if } w = y, \\ 0.4 & \text{if } w = z, \\ 0.7 & \text{if } w = e. \end{cases}$$

	0	x	y	z	e
0	0	0	0	0	0
x	x	0	0	0	0
y	y	x	0	x	0
z	z	z	z	0	0
e	e	z	z	x	0

Table 2: Cayley table

By routine calculations, we know that $\mathcal{A}$ is a fuzzy BCK-filter of X with thresholds $\varepsilon = 0.3$ and $\delta = 0.5$. But it is not a fuzzy BCK-filter of X with thresholds $\varepsilon = 0.15$ and $\delta = 0.5$ since

$$\max\{\mathcal{A}(y), 0.15\} = 0.15 < 0.2 = \min\{\mathcal{A}((y^*x^*)^*), \mathcal{A}(x), 0.5\}.$$

Theorem 3.20. *Let $\varepsilon_1, \varepsilon_2, \delta \in (0,1]$ such that $\varepsilon_1 < \varepsilon_2 < \delta$. Then every fuzzy BCK-filter with thresholds ε_1 and δ is a fuzzy BCK-filter with thresholds ε_2 and δ.*

Proof. Straightforward. □

Example 3.19 shows that the converse of Theorem 3.20 is not true.

Theorem 3.21. *Let $\mathcal{A}$ be a fuzzy subset of X and $\varepsilon, \delta \in (0,1]$ with $\varepsilon < \delta$. Then $\mathcal{A}$ is a fuzzy BCK-filter of X with thresholds ε and δ if and only if it satisfies:*

(3.7) $\quad (\forall t \in (\varepsilon, \delta])\ (C(\mathcal{A}; t) \neq \emptyset \ \Rightarrow \ C(\mathcal{A}; t)$ is a BCK-filter of X).

Proof. Assume that $\mathcal{A}$ is a fuzzy BCK-filter of X with thresholds ε and δ. Let $t \in (\varepsilon, \delta]$ be such that $C(\mathcal{A}; t) \neq \emptyset$. Then $\max\{\mathcal{A}(e), \varepsilon\} \geq \min\{\mathcal{A}(x), \delta\}$ for any $x \in C(\mathcal{A}; t)$, and so $\max\{\mathcal{A}(e), \varepsilon\} \geq \min\{t, \delta\} = t$. Since $\varepsilon < t$, it follows that $\mathcal{A}(e) \geq t$, i.e., $e \in C(\mathcal{A}; t)$. Let $x, y \in X$ be such that $(x^*y^*)^* \in C(\mathcal{A}; t)$ and $y \in C(\mathcal{A}; t)$. Then $\mathcal{A}((x^*y^*)^*) \geq t$ and $\mathcal{A}(y) \geq t$. It follows from (d6) that

$$\max\{\mathcal{A}(x), \varepsilon\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \delta\} \geq \min\{t, \delta\} = t$$

so that $\mathcal{A}(x) \geq t$ since $\varepsilon < t$. Hence $x \in C(\mathcal{A}; t)$, and therefore $C(\mathcal{A}; t)$ is a BCK-filter of X for all $t \in (\varepsilon, \delta]$.

Conversely, let $\mathcal{A}$ be a fuzzy subset of X satisfying (3.7). If there exists $a \in X$ such that $\max\{\mathcal{A}(e), \varepsilon\} < \min\{\mathcal{A}(a), \delta\}$, then

$$\max\{\mathcal{A}(e), \varepsilon\} < t_e \leq \min\{\mathcal{A}(a), \delta\}$$

for some $t_e \in (\varepsilon, \delta]$. Hence $e \notin C(\mathcal{A}; t_e)$ which is a contradiction. Therefore (d5) is valid. Assume that (d6) is not valid. Then there exist $a, b \in X$ such that $\max\{\mathcal{A}(a), \varepsilon\} < \min\{\mathcal{A}((a^*b^*)^*), \mathcal{A}(b), \delta\}$. It follows that

$$\max\{\mathcal{A}(a), \varepsilon\} < t_a \leq \min\{\mathcal{A}((a^*b^*)^*), \mathcal{A}(b), \delta\}$$

for some $t_a \in (\varepsilon, \delta]$ so that $(a^*b^*)^* \in C(\mathcal{A}; t_a)$ and $b \in C(\mathcal{A}; t_a)$, but $a \notin C(\mathcal{A}; t_a)$. This is impossible, and so $\max\{\mathcal{A}(x), \varepsilon\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \delta\}$ for all $x, y \in X$. Therefore $\mathcal{A}$ is a fuzzy BCK-filter of X with thresholds ε and δ. □

If we take $\varepsilon = 0$ and $\delta = 1$ in Theorem 3.21, we have the following corollary.

Corollary 3.22([8]). *Let $\mathcal{A}$ be a fuzzy subset of X. Then the following are equivalent:*

(1) $\mathcal{A}$ *is a fuzzy BCK-filter of* X.

(2) $(\forall t \in (0,1])\ (C(\mathcal{A};t) \neq \emptyset \ \Rightarrow\ C(\mathcal{A};t)$ *is a BCK-filter of* $X)$.

Theorem 3.23. *Let* $\mathcal{A}$ *be a fuzzy subset of* X *and* $\varepsilon, \delta \in (0,1]$ *with* $\varepsilon < \delta$. *Then*

(1) $\mathcal{A}$ *is a fuzzy BCK-filter of* X *if and only if* $\mathcal{A}$ *is a fuzzy BCK-filter of* X *with thresholds* $\varepsilon = 0$ *and* $\delta = 1$.

(2) $\mathcal{A}$ *is an* $(\in, \in\vee\, \mathrm{q}_k)$*-fuzzy BCK-filter of* X *if and only if* $\mathcal{A}$ *is a fuzzy BCK-filter of* X *with thresholds* $\varepsilon = 0$ *and* $\delta = \frac{1-k}{2}$.

(3) $\mathcal{A}$ *is an* $(\overline{\in}, \overline{\in}\vee\overline{\mathrm{q}_k})$*-fuzzy BCK-filter of* X *if and only if* $\mathcal{A}$ *is a fuzzy BCK-filter of* X *with thresholds* $\varepsilon = \frac{1-k}{2}$ *and* $\delta = 1$.

Proof. Straight forward. □

4. Implication-based fuzzy BCK-filters

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge$, $\vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x \in \mathcal{A}] = \mathcal{A}(x), \tag{4.1}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\}, \tag{4.2}$$

$$[\Phi \rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}, \tag{4.3}$$

$$[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{4.4}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{4.5}$$

The truth valuation rules given in (4.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{\mathrm{GR}}(a,b) = \begin{cases} 1 & \text{if } \ a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator (I_{G}):

$$I_{\mathrm{G}}(a,b) = \begin{cases} 1 & \text{if } \ a \leq b, \\ b & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator ($\overline{I}_{\rm G}$):

$$\overline{I}_{\rm G}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 1-a & \text{otherwise.} \end{cases}$$

Ying [13] introduced the concept of fuzzifying topology. We can expand his/her idea to BCK-algebras, and we define a fuzzifying BCK-filter as follows.

Definition 4.1. A fuzzy subset $\mathcal{A}$ of X is called a *fuzzifying BCK-filter* of X if it satisfies the following conditions:

(d7) $(\forall x \in X)\ (\models [x \in \mathcal{A}] \to [e \in \mathcal{A}])$,

(d8) $(\forall x, y \in X)\ (\models [(x^*y^*)^* \in \mathcal{A}] \wedge [y \in \mathcal{A}] \to [x \in \mathcal{A}])$.

Obviously, conditions (d7) and (d8) are equivalent to (b1) and (b2), respectively. Therefore a fuzzifying BCK-filter is an ordinary fuzzy BCK-filter. In [14], the concept of t-tautology is introduced, i.e.,

(4.6) $\models_t \Phi$ if and only if $[\Phi] \ge t$ for all valuations.

Definition 4.2. Let $\mathcal{A}$ be a fuzzy subset of X and $t \in (0,1]$. $\mathcal{A}$ is called a *t-implication-based fuzzy BCK-filter* of X if it satisfies:

(d9) $(\forall x \in X)\ (\models_t [x \in \mathcal{A}] \to [e \in \mathcal{A}])$,

(d10) $(\forall x, y \in X)\ (\models_t [(x^*y^*)^* \in \mathcal{A}] \wedge [y \in \mathcal{A}] \to [x \in \mathcal{A}])$.

Let I be an implication operator. Clearly, $\mathcal{A}$ is a t-implication-based fuzzy BCK-filter of X if and only if it satisfies

(1) $(\forall x \in X)\ (I(\mathcal{A}(x), \mathcal{A}(e)) \ge t)$,

(2) $(\forall x, y \in X)\ (I(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) \ge t)$.

Theorem 4.3. *For any fuzzy subset $\mathcal{A}$ of X, we have*

(1) *If $I = I_{\rm GR}$, then $\mathcal{A}$ is a 0.5-implication-based fuzzy BCK-filter of X if and only if $\mathcal{A}$ is a fuzzy BCK-filter of X with thresholds $\varepsilon = 0$ and $\delta = 1$.*

(2) *If $I = I_{\rm G}$, then $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy BCK-filter of X if and only if $\mathcal{A}$ is an $(\in, \in \vee\, {\rm q}_k)$-fuzzy BCK-filter of X.*

(3) *If $I = \overline{I}_{\rm G}$, then $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy BCK-filter of X if and only if $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{{\rm q}_k})$-fuzzy BCK-filter X.*

Proof. (1) Straightforward.

(2) Assume that $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy BCK-filter of X. Then $I_{\rm G}(\mathcal{A}(x), \mathcal{A}(e)) \ge \frac{1-k}{2}$ and $I_{\rm G}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) \ge \frac{1-k}{2}$. It follows that $\mathcal{A}(e) \ge \mathcal{A}(x)$ or $\mathcal{A}(x) \ge \mathcal{A}(e) \ge \frac{1-k}{2}$, and

$$\mathcal{A}(x) \ge \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \text{ or } \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \ge \mathcal{A}(x) \ge \tfrac{1-k}{2}.$$

Hence

$$\max\{\mathcal{A}(e), 0\} = \mathcal{A}(e) \geq \min\{\mathcal{A}(x), \tfrac{1-k}{2}\}$$

and

$$\max\{\mathcal{A}(x), 0\} = \mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \tfrac{1-k}{2}\}.$$

Therefore $\mathcal{A}$ is a fuzzy BCK-filter of X with thresholds $\varepsilon = 0$ and $\delta = \frac{1-k}{2}$, and hence $\mathcal{A}$ is an $(\in, \in\vee\, q_k)$-fuzzy BCK-filter of X by Theorem 3.23(2).

Conversely, suppose that $\mathcal{A}$ is an $(\in, \in\vee\, q_k)$-fuzzy BCK-filter of X. Then

$$\mathcal{A}(e) = \max\{\mathcal{A}(e), 0\} \geq \min\{\mathcal{A}(x), \tfrac{1-k}{2}\}$$

and

$$\mathcal{A}(x) = \max\{\mathcal{A}(x), 0\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \tfrac{1-k}{2}\}.$$

For the first case, if $\min\{\mathcal{A}(x), \frac{1-k}{2}\} = \mathcal{A}(x)$ then

$$I_{\mathrm{G}}(\mathcal{A}(x), \mathcal{A}(e)) = 1 \geq \tfrac{1-k}{2}.$$

If $\min\{\mathcal{A}(x), \frac{1-k}{2}\} = \frac{1-k}{2}$ then $\mathcal{A}(e) \geq \frac{1-k}{2}$ and so $I_{\mathrm{G}}(\mathcal{A}(x), \mathcal{A}(e)) \geq \frac{1-k}{2}$. For the second case, if $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \frac{1-k}{2}\} = \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$ then $\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$ and thus

$$I_{\mathrm{G}}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \tfrac{1-k}{2}.$$

Suppose that $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), \frac{1-k}{2}\} = \frac{1-k}{2}$. Then $\mathcal{A}(x) \geq \frac{1-k}{2}$, and hence

$$I_{\mathrm{G}}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) \geq \tfrac{1-k}{2}.$$

Therefore $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy BCK-filter of X.

(3) Suppose that $\mathcal{A}$ is an $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-fuzzy BCK-filter X. Then $\mathcal{A}$ is a fuzzy BCK-filter of X with thresholds $\varepsilon = \frac{1-k}{2}$ and $\delta = 1$ by Theorem 3.23(3). Thus

$$\max\{\mathcal{A}(e), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}(x), 1\}$$

and

$$\max\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), 1\}.$$

For the first case, if $\mathcal{A}(x) = 1$ then $\max\{\mathcal{A}(e), \frac{1-k}{2}\} = 1$ and thus

$$\overline{I}_{\mathrm{G}}(\mathcal{A}(x), \mathcal{A}(e)) = 1 \geq \tfrac{1-k}{2}.$$

If $\mathcal{A}(x) < 1$, then $\max\{\mathcal{A}(e), \frac{1-k}{2}\} \geq \mathcal{A}(x)$. Thus, if $\max\{\mathcal{A}(e), \frac{1-k}{2}\} = \mathcal{A}(e)$ then $\mathcal{A}(e) \geq \mathcal{A}(x)$ and so

$$\overline{I}_{\mathrm{G}}(\mathcal{A}(x), \mathcal{A}(e)) = 1 \geq \tfrac{1-k}{2}.$$

If $\max\{\mathcal{A}(e), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\mathcal{A}(x) \leq \frac{1-k}{2}$ which implies that

$$\overline{I}_{\mathrm{G}}(\mathcal{A}(x), \mathcal{A}(e)) = 1 \geq \tfrac{1-k}{2}$$

when $\mathcal{A}(e) \geq \mathcal{A}(x)$; and

$$\overline{I}_{\mathrm{G}}(\mathcal{A}(x), \mathcal{A}(e)) = 1 - \mathcal{A}(x) \geq \tfrac{1-k}{2}$$

when $\mathcal{A}(e) < \mathcal{A}(x)$. For the second case, if $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), 1\} = 1$, then

$$\overline{I}_{\mathrm{G}}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \tfrac{1-k}{2}.$$

Assume that $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), 1\} = \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$. Then

$$\max\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}.$$

If $\max\{\mathcal{A}(x), \frac{1-k}{2}\} = \mathcal{A}(x)$, then $\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$ and so

$$\overline{I}_{\mathrm{G}}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \tfrac{1-k}{2}.$$

If $\max\{\mathcal{A}(x), \frac{1-k}{2}\} = \frac{1-k}{2}$ then $\mathcal{A}(x) \leq \frac{1-k}{2}$ and $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \leq \frac{1-k}{2}$. Hence

$$\overline{I}_{\mathrm{G}}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 \geq \tfrac{1-k}{2}$$

when $\mathcal{A}(x) \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$; and

$$\overline{I}_{\mathrm{G}}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) = 1 - \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \geq \tfrac{1-k}{2}$$

when $\mathcal{A}(x) < \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}$. Consequently, $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy BCK-filter of X.

Conversely, suppose that $\mathcal{A}$ is a $\frac{1-k}{2}$-implication-based fuzzy BCK-filter of X. Then $\overline{I}_{\mathrm{G}}(\mathcal{A}(x), \mathcal{A}(e)) \geq \frac{1-k}{2}$ and $\overline{I}_{\mathrm{G}}(\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\}, \mathcal{A}(x)) \geq \frac{1-k}{2}$ for all $x, y \in X$. It follows that $\mathcal{A}(x) \leq \mathcal{A}(e)$ or $1 - \mathcal{A}(x) \geq \frac{1-k}{2}$, i.e., $\mathcal{A}(x) \leq \frac{1-k}{2}$; and $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \leq \mathcal{A}(x)$ or $\min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} \leq \frac{1-k}{2}$. Thus

$$\max\{\mathcal{A}(e), \tfrac{1-k}{2}\} \geq \mathcal{A}(x) = \min\{\mathcal{A}(x), 1\}$$

and

$$\max\{\mathcal{A}(x), \tfrac{1-k}{2}\} \geq \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y)\} = \min\{\mathcal{A}((x^*y^*)^*), \mathcal{A}(y), 1\}.$$

Hence $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}_k})$-fuzzy BCK-filter of X by Theorem 3.23(3). □

Corollary 4.4. *For any fuzzy subset $\mathcal{A}$ of X, we have*

(1) *If $I = I_{\mathrm{G}}$, then $\mathcal{A}$ is a 0.5-implication-based fuzzy BCK-filter of X if and only if $\mathcal{A}$ is an $(\in, \in \vee\, \mathrm{q})$-fuzzy BCK-filter of X.*

(2) *If $I = \overline{I}_{\mathrm{G}}$, then $\mathcal{A}$ is a 0.5-implication-based fuzzy BCK-filter of X if and only if $\mathcal{A}$ is an $(\overline{\in}, \overline{\in} \vee \overline{\mathrm{q}})$-fuzzy BCK-filter X.*

References

[1] S. K. Bhakat and P. Das, *On the definition of a fuzzy subgroup,* Fuzzy Sets and Systems **51**(1992), 235-241.

[2] S. K. Bhakat and P. Das, $(\in, \in\vee\mathrm{q})$*-fuzzy subgroup,* Fuzzy Sets and Systems **80**(1996), 359-368.

[3] Y. B. Jun, *On* (α,β)*-fuzzy subalgebras of BCK/BCI-algebras,* Bull. Korean Math. Soc. **42**(2005), 703-711.

[4] Y. B. Jun, *Fuzzy subalgebras of type* (α,β) *in BCK/BCI-algebras,* Kyungpook Math. J. **47**(2007), 403-410.

[5] Y. B. Jun, *Generalizations of* $(\in, \in\vee\mathrm{q})$*-fuzzy subalgebras in BCK/BCI-algebras,* Comput. Math. Appl. (to appear).

[6] Y. B. Jun, *New types of fuzzy BCK-filters,* Honam Math. J. **31**(2009), 157-166.

[7] Y. B. Jun, *New types of fuzzy BCK-filters - II,* Adv. Stud. Contemp. Math. (submitted).

[8] Y. B. Jun, S. M. Hong and J. Meng, *Fuzzy BCK-filters,* Math. Japon. **47**(1998), 45-49.

[9] Y. B. Jun and S. Z. Song, *Generalized fuzzy interior ideals in semigroups,* Inform. Sci. **176**(2006), 3079-3093.

[10] J. Meng, *BCK-filters,* Math. Japon. **44**(1996), 119-129.

[11] V. Murali, *Fuzzy points of equivalent fuzzy subsets,* Inform. Sci. **158**(2004), 277-288.

[12] P. M. Pu and Y. M. Liu, *Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence,* J. Math. Anal. Appl. **76**(1980), 571-599.

[13] M. S. Ying, *A new approach for fuzzy topology (I),* Fuzzy Sets and Systems **39** (1991), 303–321.

[14] M. S. Ying, *On standard models of fuzzy modal logics,* Fuzzy Sets and Systems **26**(1988), 357-363.