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Normal Pairs of Going-down Rings

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Abstract. Let (R, T) be a normal pair of commutative rings (i.e., $R \subseteq T$ is a unital extension of commutative rings, not necessarily integral domains, such that S is integrally closed in T for each ring S such that $R \subseteq S \subseteq T$) such that the total quotient ring of R is a von Neumann regular ring. Let $\mathcal{P}$ be one of the following ring-theoretic properties: going-down ring, extensionally going-down (EGD) ring, locally divided ring. Then R has $\mathcal{P}$ if and only if T has $\mathcal{P}$. An example shows that the "if" part of the assertion fails if $\mathcal{P}$ is taken to be the "divided domain" property.

1. Introduction

All rings and algebras considered in this note are commutative with $1 \neq 0$; all subrings, ring homomorphisms and algebra homomorphisms are unital. Our main concern here is to generalize the following result of the first-named author [10, Theorem 2.1 (a)]: if $R \subset T$ is a minimal extension of integral domains such that R is integrally closed in T, then R is a going-down domain if and only if T is a going-down domain. In fact, we generalize this result in three distinct ways, namely, by replacing the "minimal (ring) extension" hypothesis with the condition that (R, T) is a normal pair; by replacing the "integral domains" hypothesis with the condition that R and T are rings such that the total quotient ring of R is a

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von Neumann regular ring; and by replacing the "going-down domain" condition(s) with the condition that the ring(s) in question is a going-down ring (resp., an extensionally going-down (EGD) ring; resp., a locally divided ring).

Before proceeding further, some notation, definitions and background are in order. Let R be a ring. Then $\mathrm{Spec}(R)$ and $\mathrm{Min}(R)$ denote the set of prime ideals of R and the set of minimal prime ideals of R, respectively; $Z(R)$ denotes the set of zero-divisors of R; and $\mathrm{tq}(R) := R_{R\setminus Z(R)}$, the total quotient ring of R. For any subset X of R, it is convenient to define the set of *regular elements of* X as $\mathrm{reg}(X) := (R \setminus Z(R)) \cap X$. As in [15], R is said to be a *complemented ring* if $\mathrm{tq}(R)$ is a von Neumann regular ring; see [15] for an explanation of this terminology and for other terminology that some authors have used to describe this important type of ring. By an *overring* of R, we mean an R-subalgebra of $\mathrm{tq}(R)$, i.e., a ring T such that $R \subseteq T \subseteq \mathrm{tq}(R)$. Recall that a (proper) ring extension $R \subset T$ is called a *minimal* (*ring*) *extension* if there is no ring S such that $R \subset S \subset T$, i.e., if the inclusion map $R \hookrightarrow T$ is a "minimal (homo)morphism" in the sense of [16]. It is known that if $R \subset T$ is a minimal extension of (integral) domains and R is not a field, then T is (R-algebra isomorphic to) an overring of R [24, page 1738, lines 8–13]. If $R \subseteq T$ are rings, then (R, T) is said to be a *normal pair* if S is integrally closed in T for each ring S such that $R \subseteq S \subseteq T$. The most natural example of a normal pair arises when R is an arbitrary Prüfer domain and T is its quotient field (cf. [18, Theorems 26.1 (1) and 23.4]). Notice that if $R \subseteq T$ is a minimal extension, then (R, T) is a normal pair if and only if R is integrally closed in T. The concept of a normal pair (R, T) (arising from a ring extension $R \subseteq T$) was introduced in case T is a domain (resp., in case R is a complemented ring) by Davis [4] (resp., the authors [15]) who proved that if (R, T) is a normal pair in this context, then T is an overring of R [4, Proposition 4.1 (1)] (resp., [15, Proposition 3.4]).

Before stating the next result, two definitions are needed. First, as in [13], a ring R is said to be *almost quasilocal* if $\mathrm{Spec}(R)\setminus \mathrm{Min}(R)$ either is empty or has a unique maximal element. Each quasilocal ring is almost quasilocal, but the converse is false. Second, as in [19], if R is a ring and $P \in \mathrm{Spec}(R)$, we define the rings $R_{[P]} := \{x \in \mathrm{tq}(R) \mid xs \in R \text{ for some } s \in R \setminus P\}$ and $R_{(P)} := R_S$, where the multiplicatively closed set $S := \mathrm{reg}(R \setminus P)$. In case T is a domain (resp., in case R is a complemented ring), a normal pair (R, T) with R quasilocal (resp., almost quasilocal) was characterized as a a ring extension $R \subseteq T$ with R quasilocal (resp., almost quasilocal) for which there exists $M \in \mathrm{Spec}(R)$ such that $MT = M$, $T = R_M$ (resp., $T = R_{[M]}$) and R/M is a valuation domain [4, Theorem 1] (resp., [15, Theorem 3.8]).

We next state the two motivating results for this note. The first of these ([3, Proposition 2.1], [10, Corollary 2.5]) states that if $R \subset T$ is a minimal extension of domains such that R is integrally closed and T is a Prüfer domain, then R is a Prüfer domain. The second motivating result [10, Theorem 2.1 (a)] was mentioned above. It states that if $R \subset T$ is a minimal extension of domains such that R is integrally closed in T, then R is a going-down domain (in the sense of [5], [11]) if and only if T is a going-down domain. Note that the context of the second of these results arises naturally from that of the first result because Prüfer domains give the most natural examples of going-down domains [5, page 448] (and of the locally divided domains discussed below). Our main purpose here is to generalize the two motivating results from the context of minimal extensions $R \subset T$ of domains in which R is integrally closed in T to the context of normal pairs (R, T) in which R is a complemented ring. This purpose is accomplished for domains in Proposition 2.5 and Theorem 2.1; and, more generally, for complemented R in Proposition 2.12 (to handle Prüfer rings, in the sense of [19]) and Theorems 2.8 and 2.10. The last two of these results handle going-down rings and extensionally going-down (EGD) rings, in the sense of [8].

It is important to note that a domain is a Prüfer ring if and only if it is a Prüfer domain; and that a domain R is a going-down domain $\Leftrightarrow$ R is a going-down ring $\Leftrightarrow$ R is an EGD ring (cf. [6, Remark 2.11]). However, as shown by examples in [8], neither "going-down ring" nor "EGD ring" implies the other in general.

Along the way, we also obtain analogues of the above results for some other classes of domains that are situated between the class of Prüfer domains and the class of going-down domains, as well as some natural analogues for the "complemented" context. This note is organized so as to give the results on domains first, then give a result (Proposition 2.6) that eases the transition to the "complemented" case, and then give the results for complemented base rings. In this way, the "complemented" results are often shown to follow naturally from their "domain" counterparts via proofs that minimize the need to explicitly recall the definitions of the various types of rings that are involved.

However, the following definitions and facts will be needed. The going-down property of ring extensions is denoted GD, as in [21, page 28]. A domain R is called a *going-down domain* if $R \subseteq T$ satisfies GD for each overring T of R. As in [6] (resp., [1]), a domain (resp., ring) R is said to be a *divided domain* (resp., *divided ring*) if P and Ra are comparable under inclusion for all $P \in \text{Spec}(R)$ and all $a \in R$. As in [6] (resp., [2]), a domain (resp., ring) R is said to be a *locally divided domain* (resp., *locally divided ring*) if R_P is a divided domain (resp., divided ring) for all $P \in \text{Spec}(R)$. A domain (resp., ring) is a quasilocal locally divided domain (resp., quasilocal locally divided ring) if and only if it is a divided domain (resp., divided ring). It is also known (cf. [6]) that valuation domain $\Rightarrow$ divided domain $\Rightarrow$ quasilocal going-down domain; and, similarly, that Prüfer domain $\Rightarrow$ locally divided domain $\Rightarrow$ going-down domain. Moreover (cf. [1], [2]), divided ring $\Rightarrow$ quasilocal going-down ring; and locally divided ring $\Rightarrow$ going-down ring. Known examples show that none of the above implications is reversible.

Besides the above material, note that $\subset$ denotes proper inclusion. Any unexplained material is standard, as in [18] and [21].

2. Results

We begin by generalizing [10, Theorem 2.1 (a)].

Theorem 2.1. *Let (R, T) be a normal pair of domains. Then R is a going-down domain if and only if T is a going-down domain.*

Proof. Suppose first that R is a going-down domain. It suffices to show that T_Q is a going-down domain for each $Q \in \mathrm{Spec}(T)$. Put $P := Q \cap R$. By [4, Proposition 4.1], $T_Q = R_P$. Then the assertion follows because R_P inherits the "going-down domain" property from R, as the class of going-down domains is stable under formation of rings of fractions.

Conversely, suppose that T is a going-down domain. As (R_P, T_P) inherits the "normal pair" property from (R, T) (where, as usual, $T_P := T_{R \setminus P}$), we may assume, without loss of generality, that (R, P) is quasilocal. By Davis' characterization of normal pairs with quasilocal base, there exists $M \in \mathrm{Spec}(R)$ such that $MT = M$, $T = R_M$ and $V := R/M$ is a valuation domain. Note that $T = R_M$ inherits the "going-down domain" property from R; and V, being a valuation domain, is also a going-down domain. Then, by applying [9, Corollary 2.5] to the pullback description $T \times_V R/M = R$, we conclude that R is a going-down domain. □

Given the close relation between going-down domains and locally divided domains (cf. [6, Theorem 2.5], [2, Corollary 3.6]), it is natural to ask if Theorem 2.1 admits analogues for locally divided domains and divided domains. The next two results show that the answers are "Yes" and "No", respectively.

Theorem 2.2. *Let (R, T) be a normal pair of domains. Then:*

(*a*) *If R is a divided domain, then T is a divided domain.*

(*b*) *R is a locally divided domain if and only if T is a locally divided domain.*

Proof. (a) Since R is divided, it is quasilocal, and so Davis' characterization of normal pairs with quasilocal base shows that $T = R_P$ for some $P \in \mathrm{Spec}(R)$. But any localization of a divided domain is a divided domain [6, Lemma 2.2 (a)].

(b) We adapt the proof of Theorem 2.1. Suppose first that R is locally divided. To prove that T is locally divided, we will show that T_Q is a divided domain for each $Q \in \mathrm{Spec}(T)$. Put $P := Q \cap R$. Then by [4, Proposition 4 (1)], $T_Q = R_P$, and so the assertion follows as above.

Conversely, suppose that T is locally divided. To prove that R is locally divided, we will show that R_P is a divided domain for each $P \in \mathrm{Spec}(R)$. As above, $(R_P, T_{R \setminus P})$ is a normal pair such that $T_{R \setminus P}$ is quasilocal. But the class of locally divided domains is stable under formation of rings of fractions [6, Remark 2.7 (b)] (cf. also [2, Proposition 2.1 (a)]). Hence $T_{R \setminus P}$ is a quasilocal locally divided domain, i.e., a divided domain. Thus, replacing R and T with R_P and $T_{R \setminus P}$, we may assume, without loss of generality, that R is quasilocal and T is divided. Using Davis' characterizations of normal pairs with quasilocal base, we have a prime ideal $M = MT$ of R such that $T = R_M$ and $V := R/M$ is a valuation domain. As T and V are each divided domains, it follows by applying [9, Corollary 2.6] to the pullback description $T \times_V R/M = R$ that R is also divided. □

We next show that Theorem 2.2 (a) is best possible, in the sense that its converse fails, even in the archetypical context of minimal ring extensions from the motivating result in [10]. The stipulation "but not a field" in Example 2.3 is included in order to rule out the trivial case of a non-quasilocal Prüfer domain R and its quotient field T.

Example 2.3. There exists a normal pair (R, T) of domains such that T is a divided domain but not a field and R is not a divided domain (although R is a locally divided domain).

For a proof, let V_1 and V_2 be incomparable valuation domains of (Krull) dimension 1 having the same quotient field. (For instance, take V_1 and V_2 to be $\mathbb{Z}_{2\mathbb{Z}}$ and $\mathbb{Z}_{3\mathbb{Z}}$, respectively.) We will show that $R := V_1 \cap V_2$ and $T := V_1$ exhibit the asserted behavior. To see this, note first that R is a one-dimensional Bézout domain with exactly two maximal ideals, say N_1 and N_2, labeled so that $R_{N_1} = V_1$ and $R_{N_2} = V_2$ (cf. [21, Theorem 107]). As $T = R_{N_1}$ is a minimal overring of the Prüfer domain R (by [18, Theorem 26.1 (2)]), (R, T) is the archetypical kind of normal pair that was noted in the Introduction. Moreover, T is a divided domain since it is a valuation domain. But R is not a divided domain, since it is not quasilocal. Finally, the parenthetical assertion follows from Theorem 2.2 (b).

Recall that a quasilocal integrally closed domain R is a going-down domain if and only if it is a divided domain [22, Corollary 11] (cf. also [6, Theorem 2.5]). Thus, in comparing Theorems 2.1 and 2.2, one is led to ask if there is an example of a normal pair (R, T) of distinct domains such that R and T are each quasilocal going-down domains that are not divided domains (and hence are not integrally closed). The next example answers this question affirmatively.

Example 2.4. There exists a normal pair (R, T) of domains such that R and T are distinct quasilocal going-down domains that are not divided domains (and hence neither R nor T is integrally closed).

For a proof, let (T, M) be the quasilocal going-down domain that is not a divided domain which was constructed in [6, Example 2.9] (where it was called "D"). In that construction, use a field F such that there exist a valuation domain V of F that is distinct from F. (For instance, take $F := \mathbb{Q}$.) It is clear from the construction in [6] that $F \subseteq T/M$. Hence, by extension of valuations (cf. [21, Theorem 56]), there is a valuation domain W of T/M that is distinct from T/M. Let R be the pullback $R := T \times_{T/M} W$. By the folklore of pullbacks (cf. [9, Lemma 2.2]), $M \in \operatorname{Spec}(R)$, $R/M \cong W$ and $R_M = T$. The order-theoretic impact of the gluing description of $\operatorname{Spec}(R)$ in [17, Theorem 1.4] that results from the above pullback description of R ensures that R is quasilocal. We now have enough information to be able to apply Davis' characterization of normal pairs with quasilocal base [4, Theorem 1], thus concluding that (R, T) is a normal pair. Of course, $R \neq T$ since $W \neq T/M$. Note that R inherits the "going-down domain" property from T by Theorem 2.1; and R is not a divided domain, by Theorem 2.2 (b). Finally, the parenthetical assertion follows from the above remarks.

Recall that Prüfer domains are the best-known examples of going-down domains. Thus, it is natural to ask if there is a "Prüfer" variant of Theorem 2.1. We give such a result next, thus generalizing the case for minimal ring extensions in [3, Proposition 2.1] (cf. also [10, Corollary 2.5]).

Proposition 2.5. *Let (R, T) be a normal pair of domains. Then R is a Prüfer domain if and only if T is a Prüfer domain. Moreover, if R is a valuation domain, then T is a valuation domain.*

Proof. To prove the first assertion, one can adapt the proof of Theorem 2.2 (b), changing "locally divided domain" (resp., "divided domain") to "Prüfer domain" (resp., "valuation domain") throughout. In doing so, note that the class of Prüfer domains is stable under formation of rings of fractions [18, Proposition 22.5]; and replace the earlier appeal to [9, Corollary 2.6] with the fundamental result on Nagata composition [23, item (11.4)]. One way to prove the "Moreover" assertion is to adapt the proof of Theorem 2.2 (a), noting that any localization of a valuation domain is a valuation domain. □

In view of Example 2.3, the converse of the "Moreover" assertion in Proposition 2.5 is false.

We turn next to generalizations of the above material to a context that involves rings with nontrivial zero-divisors. It will be necessary to consider the rings $R_{[P]}$ and $R_{(P)}$, which were defined in the Introduction. We will often use the fact that these rings are equal in case R is complemented. This fact is a special case of a result on Marot rings [20, Theorem 7.6].

The next result collects some useful technical facts.

Proposition 2.6. *Let R be a ring and $P \in Spec(R)$.*

(*a*) *The canonical R-algebra homomorphism $R \to R_P$ is an injection if and only if $R_{(P)} = R_P$ canonically* (*in the sense that the canonical R-algebra homomorphism $R_{(P)} \to R_P$ is an isomorphism*).

(*b*) *Suppose that $R_{(P)}$ has unique maximal ideal $PR_{(P)}$. Then $R_{(P)} = R_P$ canonically* (*in the sense that the canonical R-algebra homomorphism $R_{(P)} \to R_P$ is an isomorphism*).

(*c*) *If R is a complemented ring and $P \in Spec(R)$, then $PR_{(P)}$ is a maximal ideal of $R_{(P)}$.*

Proof. (a) Recall that $R_{(P)} := R_S$, where $S := \text{reg}(R \setminus P)$. Therefore, it is easy to check that $R_{(P)} = R_P$ canonically if and only if $\text{reg}(R \setminus P) = R \setminus P$. On the other hand, it is also clear that the canonical map $R \to R_P$ is an injection if and only if $\text{reg}(R \setminus P) = R \setminus P$.

(b) This follows from [18, Corollary 5.2].

(c) This follows by combining [15, Lemma 2.7] and [21, Exercise 1, page 41]. □

In [15, Theorem 3.8], we extended Davis' characterization of normal pairs of quasilocal domains to complemented rings where the base ring is almost quasilocal and the former role of R_M is played by $R_{[M]}$. We next show that if the base ring is complemented and quasilocal, then the role of $R_{[M]}$ can indeed be played by R_M itself.

Corollary 2.7. *Let $R \subseteq T$ be rings such that R is a quasilocal complemented ring. Then (R,T) is a normal pair if and only if there exists $M \in Spec(R)$ such that $T = R_M$ canonically, R/M is a valuation domain and $MT = M$.*

Proof. For the "if" assertion, note that R is almost quasilocal since R is quasilocal; and that $R_{(M)} = R_M$ canonically by Lemma 2.6 (a), while $R_{(M)} = R_{[M]}$ since R is complemented, so that $R_{[M]} = R_M = T$ canonically. Therefore, by the characterization in [15, Theorem 3.8], (R,T) is a normal pair.

For the converse, suppose that (R,T) is a normal pair. As R is almost quasilocal, [15, Theorem 3.8] reduces our task to proving that $R_{[M]} = R_M$. Since R is complemented, it suffices to show that $R_{(M)} = R_M$. By Proposition 2.6(c), $MR_{(M)}$ is a maximal ideal of $R_{(M)}$. Since [15, Proposition 3.7 (e)] ensures that T is quasilocal, it follows that $MR_{(M)}$ is the unique maximal ideal of $R_{(M)}$. Hence by Proposition 2.6 (b), $R_{(M)} = R_M$. □

We next generalize Theorem 2.1 to a context with nontrivial zero-divisors. For the next-to-last step in the next proof, one needs to recall from [8] that a ring A is called a *going-down ring* if A/P is a going-down domain for each $P \in \mathrm{Spec}(A)$.

Theorem 2.8. *Let (R,T) be a normal pair such that R is a complemented ring. Then R is a going-down ring if and only if T is a going-down ring.*

Proof. Assume first that R is a going-down ring. Since "going-down ring" is a local property [8, Proposition 2.1 (b)], it suffices to prove that T_Q is a going-down ring for each $Q \in \mathrm{Spec}(T)$. Put $P := Q \cap R$. By [15, Proposition 3.11 (a)], $R_P = T_Q$ canonically. But R_P inherits the "going-down ring" property from R.

Conversely, assume that T is a going-down ring. By [8, Proposition 2.1 (a)], it is enough to show that R/P is a going-down domain for each $P \in \mathrm{Min}(R)$. But minimal prime ideals are lain over in any ring extension [21, Exercise 1, page 41], and so there exists $Q \in \mathrm{Spec}(T)$ such that $Q \cap R = P$. Hence, by [15, Proposition 3.11 (a)], $(R/P, T/Q)$ is a normal pair. However, T/Q is a going-down domain by the hypothesis on T, and so an application of Theorem 2.1 completes the proof. □

At this point it is convenient to recall the notion of a weak Baer ring. This generalization of the notion of a domain is particularly relevant because, in spite of the examples in [8] that were mentioned in the Introduction, it was shown in [12, Proposition 2.1 (b)] that a weak Baer ring is a going-down ring if and only if it is an EGD ring. For our purposes, the most useful characterization of a weak Baer ring is as a complemented ring R such that R_P is a domain for each $P \in \mathrm{Spec}(R)$. For additional background concerning weak Baer rings, see the seventh paragraph of the Introduction of [12]. It follows from [12, Theorem 2.5] that if R is a complemented ring which is also an EGD ring, then R is a going-down ring which is a weak Baer ring. One is thus led to ask for a "complemented EGD ring" analogue of Theorem 2.8. We give such a result in Theorem 2.10. First, we give a lemma of some independent interest.

Lemma 2.9. *Let (R,T) be a normal pair such that R is a complemented ring.*

Then each localization of R at a prime ideal of R is a domain if and only if each localization of T at a prime ideal of T is a domain.

Proof. For the "only if" assertion, let $Q \in \mathrm{Spec}(T)$ and notice by [15, Proposition 3.11 (a)] that $P := Q \cap R$ satisfies $T_Q = R_P$.

Conversely, suppose that each localization of T at a prime ideal of T is a domain. Let $P \in \mathrm{Spec}(R)$. As $R_P \subseteq T_{R\setminus P}$ is a normal pair, [15, Proposition 3.7 (e)] gives that $T_{R\setminus P}$ is a quasilocal ring. It then follows from [18, Corollary 5.2] that $T_{R\setminus P}$ must be the localization of T at a prime ideal of T (and hence a domain). Therefore, its subring R_P is also a domain, as desired. □

Theorem 2.10. *Let (R, T) be a normal pair such that R is a complemented ring. Then R is an EGD ring if and only if T is an EGD ring.*

Proof. The "normal pair" hypothesis gives that T is an overring of R [15, Proposition 3.4], and so T inherits the "complemented" property from R. Therefore, by [12, Theorem 2.5], it suffices to prove that R is a weak Baer going-down ring if and only if T is a weak Baer going-down ring. Hence, by Theorem 2.8, it suffices to show that R is a weak Baer ring if and only if T is a weak Baer ring. As R and T are each complemented, it therefore suffices to prove that each localization of R at a prime ideal of R is a domain if and only if each localization of T at a prime ideal of T is a domain. Accordingly, an application of Lemma 2.9 completes the proof. □

Next, we give zero-divisor analogues of the earlier results for locally divided domains and divided domains. First, we recall that a ring R is said to be *reduced* if it has no nonzero nilpotent elements.

Proposition 2.11. *Let (R, T) be a normal pair such that R is a complemented ring. Then:*

(*a*) *If R is a divided ring, then T is a divided ring.*

(*b*) *R is a locally divided ring if and only if T is a locally divided ring.*

Proof. (a) Adapt the proof of Theorem 2.2 (a), replacing the earlier appeal to [4, Theorem 1] with an appeal to Corollary 2.7, bearing in mind that any localization of a divided ring is a divided ring [8, Remark (c), page 4] (cf. also [1, Proposition 4]).

(b) To prove the "only if" assertion, adapt the proof of the first part of Theorem 2.2 (b), replacing the earlier appeal to [4, Proposition 4 (1)] with an appeal to [15, Proposition 3.11 (a)].

Conversely, suppose that T is a locally divided ring. We must show that R_P is a divided ring for each $P \in \mathrm{Spec}(R)$. Since the class of locally divided rings is stable under formation of rings of fractions [2, Proposition 2.1 (a)], we may argue as in the proof of the second part of Theorem 2.2 (b) to reduce to the case where R is quasilocal and T is a divided ring. By Corollary 2.7, we have a prime ideal $M = MT$ of R such that $T = R_M$ and $V := R/M$ is a valuation domain. Moreover, T is actually a domain, because T is a reduced ring (since R is reduced) whose set of prime ideals is linearly ordered by inclusion [1, Proposition 1 (a)]. Hence R is a domain. As V and T are each divided domains, it follows from [7, Proposition 2.12] that $R = R + MR_M$ is a divided domain and, hence, a locally divided ring. □

Note that Example 2.3 also shows that the converse of Proposition 2.11 (a) is false.

In closing, we give zero-divisor analogues of the earlier results for Prüfer domains and valuation domains. Note that some technical care will be necessary since a Prüfer ring need not be a locally divided ring [2, Example 2.18 (a)].

Proposition 2.12. *Let (R, T) be a normal pair such that R is a complemented ring. Then R is a Prüfer ring if and only if T is a Prüfer ring.*

Proof. A ring A is a Prüfer ring if and only if $(A, \mathrm{tq}(A))$ is a normal pair [19, Theorem 13]. Hence, any overring of a Prüfer ring must also be a Prüfer ring. As T is an overring of R by [15, Proposition 3.4], the "only if" assertion follows.

For the converse, assume that T is a Prüfer ring. Recall that $K := \mathrm{tq}(R)$ is also $\mathrm{tq}(T)$. Hence, (T, K) is a normal pair. Note that it suffices to prove that (R, K) is a normal pair. This, in turn, follows from [15, Proposition 3.9 (a)], since both (R, T) and (T, K) are normal pairs. □

Note that if R and T satisfy the conditions in Proposition 2.12, then both these rings have the property considered in Lemma 2.9, namely, of being locally a domain, in view of the fact that any localization (at a prime ideal) of a complemented Prüfer ring must be a valuation domain [14, Lemma 2.1].

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