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STIELTJES DERIVATIVES AND ITS APPLICATIONS TO INTEGRAL INEQUALITIES OF STIELTJES TYPE

YUNG JIN KIM

ABSTRACT. In the present paper, we obtain integral inequalities involving the Kurzweil-Stieltjes integrals which generalize Gronwall-Bellman inequality and we use the inequalities to verify existence of solutions of a certain integral equation. Such inequalities will play an important role in the study of impulsively perturbed systems [9].

1. INTRODUCTION

Differential and integral inequalities have become a major tool in the analysis of the differential and integral equations that occur in nature or are constructed by people. A good deal of information on this subject may be found, e.g., in [8].

One reason for much of the successful mathematical development in the theory of ordinary and partial differential equations is the availability of some kinds of inequalities and variational principles involving functions and derivatives.

Most of the inequalities developed so far in the literature, which provide explicit known bounds on the functions appearing in differential, integral and other equations, perform quite well in practice and hence have found wide spread acceptance in a variety of applications. Because of this, it is not surprising that numerous studies of new types of inequalities have been made in order to achieve many new developments in various branches of mathematical science and engineering practice.

In the present paper, we obtain integral inequalities involving the Kurzweil-Stieltjes integrals which generalize Gronwall-Bellman inequality. Such inequalities will play an important role in the study of impulsively perturbed systems [9].

For Stieltjes type integral equations, see [5, 7, 13, 14], and for integral inequalities involving Stieltjes type integrals, see [1, 2, 4].

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2. NOTATIONS AND PRELIMINARIES

Assume that $[a,b],[c,d] \subset \mathbf{R}$ are bounded intervals, where $\mathbf{R}$ is the set of all real numbers.

A function $f:[a,b] \to \mathbf{R}$ is called *regulated* on $[a,b]$ if both

$$f(s+) = \lim_{\eta \to 0+} f(s+\eta), \text{ and } f(s-) = \lim_{\eta \to 0+} f(s-\eta)$$

exist for every point $s \in [a,b]$. As a convention we define $f(a-) = f(a)$ and $f(b+) = f(b)$. In this case we denote $f \in G[a,b]$. If we let for $f \in G[a,b]$, $\|f\|_{[a,b],\infty} = \sup_{s\in[a,b]} |f(s)|$, then $(G[a,b], \|\cdot\|_{[a,b],\infty})$ becomes a Banach space.

Let $g:[a,b] \longrightarrow \mathbf{R}$. For a closed interval $I=[c,d]$, we define $g(I) = g(d) - g(c)$. A function $f:[a,b] \longrightarrow \mathbf{R}$ is *of bounded variation* on $[a,b]$ if

$$V_a^b(f) \equiv \sup\{\sum_{i=1}^{n} |f([t_{i-1},t_i])|\} < \infty, \tag{2.1}$$

where the supremum is taken over all partitions

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Then we denote $f \in BV[a,b]$. We use the following notations for the convenience:

$$\Delta^+ f(s) = f(s+) - f(s), \Delta^- f(s) = f(s) - f(s-) \text{ and } \Delta f(s) = f(s+) - f(s-).$$

A *tagged* interval $(\tau,[c,d])$ consists of an interval $[c,d] \subset [a,b]$ and a point $\tau \in [c,d]$. Let $I_i = [c_i,d_i] \subset [a,b]$. A finite collection $\{(\tau_i,[c_i,d_i]) : i = 1,2,...,n\}$ of pairwise non-overlapping tagged intervals is called a *tagged partition of* $[a,b]$ if $\cup_{i=1}^n I_i = [a,b]$. A positive function δ on $[a,b]$ is called a *gauge* on $[a,b]$.

Definition 2.1 ([6,13]). Let δ be a gauge on $[a,b]$. A tagged partition

$$P = \{(\tau_i,[t_{i-1},t_i]) : i = 1,2,...,m\}$$

of $[a,b]$ is said to be $\delta-$*fine* if for every $i = 1,...,m$ we have

$$\tau_i \in [t_{i-1},t_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

If moreover a $\delta-$fine partition P satisfies the implications

$$\tau_i = t_{i-1} \Rightarrow i = 1, \quad \tau_i = t_i \Rightarrow i = m,$$

then it is called a δ^*-*fine partition.*

The following lemma implies that for a gauge δ on $[a,b]$ there exists a δ^*-fine partition of $[a,b]$. This also implies the existence of $\delta-$fine partition of $[a,b]$.

Lemma 2.2 ([6, Lemma 1.2]). *Let δ be a gauge on $[a,b]$ and a dense subset $\Omega \subset (a,b)$ be given. Then there exists δ^*–fine partition $P = \{(\tau_i, [t_{i-1}, t_i]) : i = 1, 2, ..., m\}$ of $[a,b]$ such that $t_i \in \Omega$ for $i = 1, ..., m-1$.*

We are now ready to give a formal definition of both types of the Kurzweil integral.

Definition 2.3 ([6,13]). Assume that $f, g : [a,b] \longrightarrow \mathbf{R}$ are given. We say that fdg is *Kurzweil integrable* (or shortly, *K-integrable*) on $[a.b]$ and $v \in \mathbf{R}$ is its integral if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for

$$S(fdg, P) \equiv \sum_{i=1}^{n} f(\tau_i) g(I_i),$$

we have

$$|S(fdg, P) - v| \leq \varepsilon,$$

provided $P = \{(\tau_i, I_i) : i = 1, ..., n\}$ is a δ–fine tagged partition of $[a,b]$. In this case we denote $v = \int_a^b f(s)dg(s)$ (or, shortly,$v = \int_a^b fdg$).

If, in the above definition, δ–fine is replaced by δ^*–fine, then we say that fdg is *Kurzweil* integrable*(or, shortly, *K^*-integrable*) on $[a,b]$ and we denote $v = (K^*) \int_a^b fdg$.

The integrals have all usual properties as integrals need to have. For the proofs, see, e.g., [13, 14].

Theorem 2.4. *Assume that $f, f_1, f_2, g : [a,b] \longrightarrow \boldsymbol{R}$ and that $f_1 dg$ and $f_2 dg$ are integrable in the sense of Kurzweil or Kurzweil* on $[a,b]$. Let $k_1, k_2 \in \boldsymbol{R}$. Then we have*

$$\int_a^b (k_1 f_1 + k_2 f_2) dg = k_1 \int_a^b f_1 dg + k_2 \int_a^b f_2 dg.$$

If for $c \in [a,b]$, integrals $\int_a^c fdg, \int_c^b fdg$ exist, then $\int_a^b fdg$ exists and we have

$$\int_a^b fdg = \int_a^c fdg + \int_c^b fdg.$$

Kurzweil integrals have the following particular property.

Theorem 2.5. *Assume that $f, g : [a,b] \longrightarrow \boldsymbol{R}$ and that fdg is K-integrable. If g is a regulated function on $[a,b]$, then we have*

$$\lim_{\eta \to 0+} \int_a^{s \pm \eta} fdg = \int_a^s fdg + f(s)(g(s\pm) - g(s)).$$

For the integrability we have the following fundamental result.

Theorem 2.6. *Assume that $f \in G[a,b]$ and $g \in BV[a,b]$. Then fdg is K-integrable on $[a,b]$.*

The following result is used frequently in our proofs.

Theorem 2.7 ([5]). *If $f \in G[a,b]$, then f is bounded on $[a,b]$ and f is continuous at every $t \in [a,b] - C$, where C is a countable set.*

3. The Stieltjes Derivatives

Throughout this section, we assume that $f \in G[a,b]$ and g is a nondecreasing function on $[a,b]$.

A *neighborhood* of $t \in [a,b]$ is an open interval containing t. We say that the function g is *not locally constant* at $t \in (a,b)$ if there exists $\eta > 0$ such that g is not constant on $(t-\varepsilon, t+\varepsilon)$ for every $\varepsilon < \eta$. We also say that the function g is *not locally constant* at a and b, respectively if there exists $\eta > 0$ such that g is not constant on $[a, a+\varepsilon), (b-\varepsilon, b]$, respectively for every $\varepsilon < \eta$.

Definition 3.1. If g is not locally constant at $t \in (a,b)$, we define

$$\frac{df(t)}{dg(t)} = \lim_{\eta,\delta \to 0+} \frac{f(t+\eta) - f(t-\delta)}{g(t+\eta) - g(t-\delta)},$$

provided that the limit exists. If g is not locally constant at $t = a$ and $t = b$ respectively, we define

$$\frac{df(a)}{dg(a)} = \lim_{\eta \to 0+} \frac{f(a+\eta) - f(a)}{g(a+\eta) - g(a)}, \quad \frac{df(b)}{dg(b)} = \lim_{\delta \to 0+} \frac{f(b) - f(b-\delta)}{g(b) - g(b-\delta)},$$

respectively. Sometimes we denote $\frac{df(t)}{dg(t)} = f'_g(t)$.

If both f and g are constant on some neighborhood of t, we define $\frac{df(t)}{dg(t)} = 0$.

Remark 3.2. It is obvious that if g is not continuous at t then $f'_g(t)$ exists. Thus if $f'_g(t)$ does not exist then g is continuous at t. $f'_g(t)$ is called the Stieltjes derivative.

Throughout this section we only prove for $t \in (a,b)$ because the proofs for $t = a, t = b$ are very similar to the proof for $t \in (a,b)$.

We have the following differentiation rule.

Theorem 3.3. *Assume that if g is constant on some neighborhood of t then both f_1 and f_2 are also constant there. If $\frac{df_1(t)}{dg(t)}$ and $\frac{df_2(t)}{dg(t)}$ exist and if $f_1, f_2 \in G[a,b]$, then we have*

$$\frac{d[f_1(t)f_2(t)]}{dg(t)} = \frac{df_1(t)}{dg(t)} f_2(t+) + f_1(t-) \frac{df_2(t)}{dg(t)}, \quad t \in [a,b].$$

Proof. First assume that g is not locally constant at t, then we have

$$\begin{aligned} & \frac{f_1(t+\eta)f_2(t+\eta)-f_1(t-\delta)f_2(t-\delta)}{g(t+\eta)-g(t-\delta)} \\ = \quad & \frac{f_1(t+\eta)-f_1(t-\delta)}{g(t+\eta)-g(t-\delta)}f_2(t+\eta)+f_1(t-\delta)\frac{f_2(t+\eta)-f_2(t-\delta)}{g(t+\eta)-g(t-\delta)} \\ \longrightarrow \quad & \frac{df_1(t)}{dg(t)}f_2(t+)+f_1(t-)\frac{df_2(t)}{dg(t)}, \text{ as } \eta,\delta \longrightarrow 0+. \end{aligned}$$

If g is constant on some neighborhood of t, then $\frac{df_1(t)}{dg(t)} = 0 = \frac{df_2(t)}{dg(t)}$ by definition. In this case the proof is obvious. □

K^*-integrals recover Stieltjes derivatives.

Theorem 3.4. *Assume that if g is constant on some neighborhood of t then f is also constant there. Suppose that $f'_g(t)$ exists at every $t \in [a,b] - \{c_1, c_2, ...\}$, where f is continuous at every $t \in \{c_1, c_2, ...\}$. Then we have*

$$(K^*)\int_a^b f'_g(s)dg(s) = f(b) - f(a).$$

Proof. Let $C = \{c_1, c_2, ...\}$. We put $f'_g(c_i) = 0, i = 1, 2, ...$ Let $\varepsilon > 0$ and define a gauge δ as follows:

if $\tau \in [a,b] - C$, use the existence of $f'_g(\tau)$ to choose a gauge δ so that if (τ, I) is δ^*–fine, then

$$|f(I) - f'_g(\tau)g(I)| \leq \varepsilon|g(I)|;$$

if $\tau = c_i$, use the continuity of f at c_i to choose a gauge δ so that if (τ, I) is δ^*–fine, then

$$|f(I)| < \frac{\varepsilon}{2^i}.$$

Note that if g is constant for some neighborhood of t, then f is also a constant there, by assumption. So, in this case, $f(I) = 0 = g(I)$. Keeping this in mind, we proceed to the proof.

Now suppose that $P = \{(\tau_i, I_i) : i = 1, ..., n\}$ is a δ^*–fine tagged partition of $[a,b]$. Let P_c be the subset of P that has tags in C and $P_1 = P - P_c$. If $(\tau_i, I_i) \in P_1$, then

$$|f(I_i) - f'_g(\tau_i)g(I_i)| \leq \varepsilon g(I_i).$$

If $(\tau_i, I_i) \in P_c$, then $|f(I_i)| < \frac{\varepsilon}{2^j}$ for some $j = 1, 2, \ldots$. Let π be the set of integers i such that c_i is a tag of P_c. Then we have

$$\begin{aligned}
&\left|\sum_{i=1}^{n} f'_g(\tau_i)g(I_i) - f([a,b])\right| \\
&\leq \sum_{P_1} |f'_g(\tau_i)g(I_i) - f(I_i)| + \sum_{P_c} |f(I_i)| \\
&\leq \varepsilon g([a,b]) + \sum_{i\in\pi} \frac{\varepsilon}{2^i} = \varepsilon(g([a,b]) + 1).
\end{aligned}$$

Hence $f'_g dg$ is K^*-integrable and

$$(K^*)\int_a^b f'_g dg = f(b) - f(a).$$

□

Lemma 3.5. *Assume that g is not locally constant at $t \in [a,b]$. If f is continuous at t or if g is not continuous at t, then*

$$\frac{d}{dg(t)} \int_a^t f(s)dg(s) = f(t)$$

Proof. First assume that f is continuous at t, then

$$g([t-\delta, t+\eta]) \inf_{s\in[t-\delta,t+\eta]} f(s) \leq \int_{t-\delta}^{t+\eta} f dg \leq g([t-\delta, t+\eta]) \sup_{s\in[t-\delta,t+\eta]} f(s).$$

This implies

$$\inf_{s\in[t-\delta,t+\eta]} f(s) \leq \frac{\int_{t-\delta}^{t+\eta} f dg}{g([t-\delta, t+\eta])} \leq \sup_{s\in[t-\delta,t+\eta]} f(s).$$

Since f is continuous at t,

$$\lim_{\eta,\delta\to 0+} \inf_{s\in[t-\delta,t+\eta]} f(s) = f(t) = \lim_{\eta,\delta\to 0+} \sup_{s\in[t-\delta,t+\eta]} f(s).$$

Thus we have

$$\frac{d}{dg(t)} \int_a^t f(s)dg(s) = f(t).$$

Next suppose that g is not continuous at t, Then by Theorem 2.5 we have

$$\lim_{\eta,\delta\to 0+} \frac{\int_{t-\delta}^{t+\eta} f dg}{g([t-\delta, t+\eta])} = \frac{f(t)\Delta g(t)}{\Delta g(t)} = f(t).$$

Now the proof is complete. □

Remark 3.6. We see that $\frac{d}{dg(t)}\int_a^t f(s)dg(s)$ may not exist at points where g is continuous since if g is not continuous at t then $\frac{d}{dg(t)}\int_a^t f(s)dg(s)$ exists. If g is constant on some neighborhood of t, then $\int_a^t fdg$ is also constant there. So in this case we have $\frac{d}{dg(t)}\int_a^t f(s)dg(s) = 0$ by definition.

Lemma 3.7. *Suppose that g is not locally constant at t. Then*

$$\frac{d}{dg(t)}\left[e^{-\int_a^t f(s)dg(s)}\right] = -f(t)e^{-\int_a^t f(s)dg(s)},$$

if both f and g are continuous at t.

And suppose that the function f is nonnegative on $[a,b]$. Then

$$\frac{d}{dg(t)}\left[e^{-\int_a^t f(s)dg(s)}\right] = -f(t)e^{-\int_a^t f(s)dg(s)}e^k,$$

if g is not continuous at t, where $-f(t)\Delta^+g(t) \leq k \leq f(t)\Delta^-g(t)$.

And suppose that g is constant on some neighborhood of t. Then we have

$$\frac{d}{dg(t)}\left[e^{-\int_a^t f(s)dg(s)}\right] = 0.$$

Proof. First suppose that g is not locally constant at t. Let $F(t) = \int_a^t fdg$. Assume that both f and g are continuous at t. Since F is continuous at t and e^{-t} is differentiable on $[a,b]$, we have

$$e^{-F(t+\eta)} - e^{-F(t-\delta)} = [F(t+\eta) - F(t-\delta)] \cdot [-e^{-F(t)} + v(\eta,\delta)],$$

where $v(\eta,\delta) \to 0$ as $\eta,\delta \to 0+$. And since $F_g'(t) = f(t)$, we have

$$F(t+\eta) - F(t-\delta) = [g(t+\eta) - g(t-\delta)] \cdot [f(t) + u(\eta,\delta)],$$

where $u(\eta,\delta) \to 0$ as $\eta,\delta \to 0+$. So we have

$$e^{-F(t+\eta)} - e^{-F(t-\delta)} = [g(t+\eta) - g(t-\delta)] \cdot [f(t) + u(\eta,\delta)] \cdot [-e^{-F(t)} + v(\eta,\delta)].$$

Thus we have

$$\frac{e^{-F(t+\eta)} - e^{-F(t-\delta)}}{g(t+\eta) - g(t-\delta)} = [f(t) + u(\eta,\delta)] \cdot [-e^{-F(t)} + v(\eta,\delta)].$$

Taking $\lim_{\eta,\delta\to 0+}$ to both sides, we obtain

$$\frac{d}{dg(t)}\left[e^{-\int_a^t f(s)dg(s)}\right] = -f(t)e^{-\int_a^t f(s)dg(s)}.$$

Next suppose that g is not continuous at t. Then, if $f(t) \neq 0$,

$$\lim_{\eta,\delta\to 0+} \frac{e^{-\int_a^{t+\eta} fdg} - e^{-\int_a^{t-\delta} fdg}}{g([t-\delta, t+\eta])} = \frac{e^{-\int_a^t fdg}[e^{-f(t)\Delta^+ g(t)} - e^{f(t)\Delta^- g(t)}]}{\Delta g(t)}$$

$$= -e^{-\int_a^t fdg} \frac{f(t)\Delta g(t) e^k}{\Delta g(t)} = -f(t)e^{-\int_a^t fdg} e^k,$$

by Theorem 2.5 and the Mean Value Theorem, where

$$-f(t)\Delta^+ g(t) \leq k \leq f(t)\Delta^- g(t).$$

The proof in case that $f(t) = 0$, is obvious since $e^{-f(t)\Delta^+ g(t)} - e^{f(t)\Delta^- g(t)} = 0$.

The last statement is obvious by definition since whenever g is constant on some neighborhood at t, $\int_a^t fdg$ is also constant there. □

4. Integral Inequalities of Gronwall-Bellman Type Involving Stieltjes Integrals

Throughout this section we assume that $f \in G[a,b]$ is nonnegative on $[a,b]$ and a function g is nondecreasing and left-continuous on $[a,b]$. Also we only prove for $t \in (a,b)$ because the proofs for $t = a, t = b$ are very similar to the proof for $t \in (a,b)$.

In this section we obtain our main result by using the previous results.

Theorem 4.1. *Assume that $u \in G[a,b]$ and that $u \geq 0$ on $[a,b]$. Then the integral inequality*

$$u(t) \leq c + \int_a^t f(s)u(s)dg(s), \quad c \geq 0, \quad t \in [a,b].$$

implies

$$u(t) \leq c \cdot e^{\int_a^t f(s)dg(s)}, \quad t \in [a,b].$$

Proof. Define a function $z(t) = c + \int_a^t f(s)u(s)dg(s)$, $t \in [a,b]$, then $u(t) \leq z(t)$ and $z(a) = c$. First suppose that g is not locally constant at t. Then by Lemma 3.5

$$z_g'(t) = f(t)u(t) \leq f(t)z(t),$$

if f and u are continuous at t, or if g is not continuous at t. Then by Lemma 3.5, Lemma 3.7 and Theorem 3.3 we have, if f, g and u are continuous at t,

$$\frac{d}{dg(t)}\left[z(t)e^{-\int_a^t fdg}\right] = z_g'(t)e^{-\int_a^t fdg} - f(t)z(t)e^{-\int_a^t fdg} \leq 0.$$

Now suppose that g is not continuous at t. Then by Lemma 3.7, Theorem 2.5 and remembering that $\Delta^- g(t) = 0$, we have, since $-f(t)\Delta^+ g(t) \leq k \leq f(t)\Delta^- g(t)$,

$$\begin{aligned}
&\frac{d}{dg(t)}\left[z(t)e^{-\int_a^t fdg}\right] \\
&= z_g'(t)\lim_{\eta\to 0+} e^{-\int_a^{t+\eta} fdg} + z(t-)\frac{d}{dg(t)}\left[e^{-\int_a^t fdg}\right] \\
&= z_g'(t)e^{-\int_a^t fdg}e^{-f(t)\Delta^+ g(t)} - [z(t) - f(t)u(t)\Delta^- g(t)]f(t)e^{-\int_a^t fdg}e^k \\
&\leq f(t)z(t)e^{-\int_a^t fdg}e^{-f(t)\Delta^+ g(t)} - f(t)z(t)e^{-\int_a^t fdg}e^k \\
&\leq f(t)z(t)e^{-\int_a^t fdg}\left[e^{-f(t)\Delta^+ g(t)} - e^k\right] \leq 0.
\end{aligned}$$

Finally assume that g is constant on some neighborhood of t, then $z(t)e^{-\int_a^t fdg}$ is also constant there, we have $\frac{d}{dg(t)}\left[z(t)e^{-\int_a^t fdg}\right] = 0$ by definition.

Thus in any case we have

$$\frac{d}{dg(s)}\left[z(s)e^{-\int_a^s fdg}\right] \leq 0 \tag{4.1}$$

for every $s \in [a,b] - \{c_1, c_2, ...\}$, where g is continuous at c_i for every $i = 1, 2, ...$ Hence if we integrate both sides of (4.1) from a to t and using Theorem 3.4 we obtain

$$z(t)e^{-\int_a^t fdg} - z(a) \leq 0.$$

and, since $z(a) = c$, we conclude that

$$u(t) \leq z(t) \leq c \cdot e^{\int_a^t fdg}.$$

This completes the proof. □

Corollary 4.2 ([14, Theorem 4.30]). *Suppose that $u \in G[a,b]$ is nonnegative on $[a,b]$. And assume that $K_1, K_2 > 0$. Then, for every $t \in [a,b]$,*

$$u(t) \leq K_1 + K_2\int_a^t u(s)dg(s)$$

implies

$$u(t) \leq K_1 e^{K_2(g(t)-g(a))}.$$

Proof. The proof is obvious from Theorem 4.1. □

Remark 4.3. In [2], the authors obtained some results on the following integral inequality

$$u(t) \leq c + \int_a^t f(s)u(s)dg(s).$$

But the authors require that $f(s)g'(s) \geq 0$ for every $s \in [a,b]$. This condition is more restrictive than ours.

Theorem 4.4. *Assume that $a, b, q \in G[\alpha, \beta]$ are nonnegative on $[\alpha, \beta]$. Suppose that $u \in G[\alpha, \beta]$ is nonnegative on $[\alpha, \beta]$. Then*

$$u(t) \le f(t) + q(t) \int_\alpha^t [a(s)u(s) + b(s)]dg(s), \quad t \in [\alpha, \beta]$$

implies

$$u(t) \le f(t) + q(t) \int_\alpha^t [a(s)f(s) + b(s)]e^{\int_s^t aqdg} dg(s).$$

Proof. Define a function $z(t) = \int_\alpha^t [a(s)u(s) + b(s)]dg(s)$, $t \in [\alpha, \beta]$, then $u(t) \le f(t) + q(t)z(t)$ and $z(\alpha) = 0$. First suppose that g is not locally constant at t. Then

$$z_g'(t) = a(t)u(t) + b(t) \le a(t)f(t) + a(t)q(t)z(t) + b(t),$$

if $au + b$ is continuous at t, or if g is not continuous at t. Then by Lemma 3.5, Lemma 3.7 and Theorem 3.3 we have, if $au + b, aq$ and g are continuous at t,

$$\begin{aligned}
&\frac{d}{dg(t)}\left[z(t)e^{-\int_\alpha^t aqdg}\right] = z_g'(t)e^{-\int_\alpha^t aqdg} - a(t)q(t)z(t)e^{-\int_\alpha^t aqdg} \\
&= [z_g'(t) - z(t)a(t)q(t)]e^{-\int_\alpha^t aqdg} \\
&\le [a(t)f(t) + a(t)q(t)z(t) + b(t) - a(t)q(t)z(t)]e^{-\int_\alpha^t aqdg} \\
&= [a(t)f(t) + b(t)]e^{-\int_\alpha^t aqdg}.
\end{aligned}$$

Now suppose that g is not continuous at t. Then by Lemma 3.7, Theorem 2.5 and remembering that $\Delta^- g(t) = 0$, we have, since

$$-a(t)q(t)\Delta^+ g(t) \le k \le a(t)q(t)\Delta^- g(t),$$

$$\begin{aligned}
&\frac{d}{dg(t)}\left[z(t)e^{-\int_\alpha^t aqdg}\right] = z_g'(t) \lim_{\eta \to 0+} e^{-\int_\alpha^{t+\eta} aqdg} + z(t-)\frac{d}{dg(t)}\left[e^{-\int_\alpha^t aqdg}\right] \\
&= z_g'(t)e^{-\int_\alpha^t aqdg}e^{-a(t)q(t)\Delta^+ g(t)} - [z(t) - (a(t)u(t) \\
&\quad + b(t))\Delta^- g(t)]a(t)q(t)e^{-\int_\alpha^t aqdg}e^k \\
&\le [a(t)f(t) + a(t)q(t)z(t) + b(t)]e^{-\int_\alpha^t aqdg}e^{-a(t)q(t)\Delta^+ g(t)} \\
&\quad - a(t)q(t)z(t)e^{-\int_\alpha^t aqdg}e^k \\
&\le [a(t)f(t) + b(t)]e^{-\int_\alpha^t aqdg}e^{-a(t)q(t)\Delta^+ g(t)} \\
&\quad + a(t)q(t)z(t)e^{-\int_\alpha^t aqdg}\left[e^{-a(t)q(t)\Delta^+ g(t)} - e^k\right] \\
&\le [a(t)f(t) + b(t)]e^{-\int_\alpha^t aqdg}.
\end{aligned}$$

Finally assume that g is constant on some neighborhood of t, then $z(t)e^{-\int_\alpha^t aqdg}$ is also constant there, we have $\frac{d}{dg(t)}\left[z(t)e^{-\int_\alpha^t aqdg}\right] = 0 \le [a(t)f(t) + b(t)]e^{-\int_\alpha^t aqdg}$.

Thus in any case we have

$$\frac{d}{dg(s)}\left[z(s)e^{-\int_\alpha^s aqdg}\right] \le [a(s)f(s)+b(s)]e^{-\int_\alpha^s aqdg} \tag{4.2}$$

for every $s \in [\alpha,\beta] - \{c_1, c_2, ...\}$, where g is continuous at c_i for every $i = 1, 2, ...$ Hence if we integrate both sides of (4.2) from α to t and using Theorem 3.4 we obtain

$$z(t)e^{-\int_\alpha^t aqdg} - z(\alpha) \le \int_\alpha^t [a(s)f(s)+b(s)]e^{-\int_\alpha^s aqdg}dg(s).$$

and, since $z(\alpha) = 0$, we conclude that

$$u(t) \le f(t) + q(t)z(t) \le f(t) + q(t)\int_\alpha^t [a(s)f(s)+b(s)]e^{\int_s^t aqdg}dg(s).$$

This completes the proof. □

5. An Application to an Integral Equation of Stieltjes Type

In this section we apply our results to an integral equation

$$x(t) = f(t) + q(t)\int_0^t K(s, x(s))dg(s), t \in [0,\infty), \tag{5.1}$$

where f, g, q, K are known functions and x is a unknown function.

A set $\mathcal{A} \subset G[a,b]$ has *uniform one-sided limits* at $t_0 \in [a,b]$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in \mathcal{A}$ we have: if $t_0 < t < t_0+\delta$ then $|x(t) - x(t_0+)| < \varepsilon$; if $t_0 - \delta < t < t_0$ then $|x(t) - x(t_0-)| < \varepsilon$.

A set $\mathcal{A} \subset G[a,b]$ is called *equi-regulated* on $[a,b]$ if it has uniform one-sided limits at every point $t_0 \in [a,b]$.

Let X be a linear space, recall that a semi-norm on X is a mapping $|\cdot| : X \longrightarrow [0,\infty)$ having all the properties of a norm except that $|x| = 0$ does not always imply that $x = 0$.

Suppose that we have a countable family of semi-norms on X, $|\cdot|_n$; we say that this family is *sufficient* if and only if for every $x \in X, x \neq 0$ there exists a positive integer n such that $|x|_n \neq 0$.

Every space $(X, |\cdot|_n)$, endowed with a countable and sufficient family of semi-norms can be organized as a metric space by setting the metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{|x-y|_n}{1+|x-y|_n}. \tag{5.2}$$

It is well-known fact that (X, d) forms a locally convex space (see, e.g.,[11]).

Recall that the convergence determined by the metric d can be characterized as follows:

$x_n \longrightarrow x$ if and only if for every positive integer n, $\lim_{m\to\infty} |x_m - x|_n = 0$.

To accomplish our purpose we need the following results.

Theorem 5.1 ([12, Schaefer's fixed point theorem]). *Let X be a linear locally convex space and let $T : X \longrightarrow X$ be a completely continuous map. If the set*

$$\Phi = \{x \in X : x = \lambda T x \quad \textit{for some} \quad \lambda \in (0,1)\}$$

is bounded, then T has a fixed point.

For compactness of a set $\mathcal{A} \subset G[a,b]$, we have the following result.

Theorem 5.2 ([3, Corollary 2.4]). *A set $\mathcal{A} \subset G[a,b]$ is relatively compact if and only if it is equi-regulated on $[a,b]$and for every $t \in [a,b]$ the set $\{x(t) : x \in \mathcal{A}\}$ is bounded in $\boldsymbol{R}$.*

We will use the following hypothesis:

(**H**) the function $K : [0,\infty) \times \mathbf{R} \longrightarrow \mathbf{R}$ is continuous on $[0,\infty) \times \mathbf{R}$ and there is a regulated function $a : [0,\infty) \longrightarrow [0,\infty)$ such that

$$|K(s,x)| \leq a(s)|x|$$

for every $(s,x) \in [0,\infty) \times \mathbf{R}$.

We say that $g : [0,\infty) \longrightarrow \mathbf{R}$ is locally of bounded variation on $[0,\infty)$ if $g \in BV[0,T]$ for every $T > 0$ and we denote $g \in BV_{loc}[0,\infty)$.

We denote $G[0,\infty)$ as the set of all functions defined on $[0,\infty)$ which have both left and right limits at every point $t \in [0,\infty)$.

Using these preliminaries we can obtain the following result for the integral equation (5.1).

Theorem 5.3. *Assume that $f, q \in G[0,\infty)$ and that $K : [0,\infty) \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ satisfies condition (H). Suppose that $g \in BV_{loc}[0,\infty)$ is left-continuous. Then* (5.1) *has a solution in $G[0,\infty)$.*

Proof. We define semi-norms for every positive integer n as follows:

$$|f|_n = \sup_{s\in[0,n]} |f(s)|.$$

We define an operator $T : G[0,\infty) \longrightarrow G[0,\infty)$ as

$$Tx(t) = f(t) + q(t)\int_0^t K(s, x(s))dg(s),\ t \in [0, \infty).$$

We will show that T is completely continuous on $G[0, \infty)$.

Assume that $x_n \to x$ in $G[0, \infty)$. Then for every positive integer n we have

$$\lim_{m\to\infty} |x_m - x|_n = 0$$

and this implies that there is a nonnegative number M_n such that $|x_m|_n, |x|_n \le M_n$.

By hypothesis (H), we get

$$(5.3)\qquad |K(s, x_m(s)) - K(s, x(s))| \le (|x_m|_n + |x|_n)a(s) \le 2M_n a(s),\ s \in [0, n].$$

Now we set $v(s) = V_0^s(g)$(see, (2.1)). Note that if g is left-continuous on $[0, \infty)$ then $v(s)$ is also left-continuous there. By (5.3) and [10, Corollary 2.3.7], we have

$$(5.4)\qquad \lim_{m\to\infty}\int_0^n |K(s, x_m(s)) - K(s, x(s))|dv(s) = 0.$$

And for every $t \in [0, n]$,

$$|Tx_m(t) - Tx(t)| \le |q|_n \int_0^n |K(s, x_m(s)) - K(s, x(s))|dv(s)$$

implies

$$|Tx_m - Tx|_n \le |q|_n \int_0^n |K(s, x_m(s)) - K(s, x(s))|dv(s).$$

So by (5.4) we have $\lim_{m\to\infty} |Tx_m - Tx|_n = 0$. Thus we conclude that the operator T is continuous on $G[0, \infty)$.

Let $\mathcal{A}$ be a bounded subset of $G[0, \infty)$. Since $\mathcal{A}$ is a bounded set in $G[0, \infty)$, for every positive integer n there is a nonnegative number M_n such that $|x|_n \le M_n$ for all $x \in \mathcal{A}$. Then for every $x \in \mathcal{A}$ and for every $t \in [0, n]$,

$$\begin{aligned}|Tx(t)| &\le |f(t)| + |q(t)|\int_0^t |K(s, x(s))|dv(s)\\ &\le |f(t)| + |q(t)|\int_0^t a(s)|x(s)|dv(s)\\ &\le |f|_n + M_n|q|_n\int_0^n a(s)dv(s).\end{aligned}$$

Thus we conclude that $\{Tx : x \in \mathcal{A}\}$ is equi-bounded on $[0, n]$.

Let $t_0 \in [0, n)$ and assume that $t_j, t_k \to t_0+$ as $j, k \to \infty$. Then

$$
\begin{aligned}
&|Tx(t_j) - Tx(t_k)| \\
&\le \left| q(t_j)\int_0^{t_j} K(s,x(s))dg(s) - q(t_k)\int_0^{t_k} K(s,x(s))dg(s)\right| + |f(t_j) - f(t_k)| \\
&\le \left| q(t_j)\int_0^{t_j} K(s,x(s))dg(s) - q(t_j)\int_0^{t_k} K(s,x(s))dg(s)\right| \\
&\quad + \left| q(t_j)\int_0^{t_k} K(s,x(s))dg(s) - q(t_k)\int_0^{t_k} K(s,x(s))dg(s)\right| + |f(t_j) - f(t_k)| \\
&\le |q(t_j)| \int_{t_k}^{t_j} |K(s,x(s))|dv(s) \\
&\quad + |q(t_j) - q(t_k)| \int_0^{t_k} |K(s,x(s))|dv(s) + |f(t_j) - f(t_k)| \\
&\le |q(t_j)| \int_{t_k}^{t_j} a(s)|x(s)|dv(s) + |q(t_j) - q(t_k)| \int_0^{n} a(s)|x(s)|dv(s) + |f(t_j) - f(t_k)| \\
&\le M_n |q|_n \int_{t_k}^{t_j} a(s)dv(s) + M_n |q(t_j) - q(t_k)| \int_0^{n} a(s)dv(s) + |f(t_j) - f(t_k)|.
\end{aligned}
$$

Since by Theorem 2.5,

$$\int_{t_k}^{t_j} a(s)dv(s) \to 0+$$

as $j, k \to \infty$, for every $\varepsilon > 0$ there is a positive integer N such that

$$|Tx(t_j) - Tx(t_k)| \le \varepsilon,$$

whenever $j, k \ge N$.

Now let $t_j, t_k \to t_0-$ as $j, k \to \infty$. Then similarly we can show that for every $\varepsilon > 0$ there is a positive integer N^* such that

$$|Tx(t_j) - Tx(t_k)| \le \varepsilon,$$

whenever $j, k \ge N^*$. This implies that $\{Tx : x \in \mathcal{A}\}$ is equi-regulated on $[0, n]$. Thus we have shown that by Theorem 5.2 the operator T is completely continuous on $G[0, \infty)$ for the semi-norm $|\cdot|_n$. Hence we conclude that T is completely continuous on $G[0, \infty)$.

Finally we show that the set

$$\Phi = \{x \in G[0,\infty) : x = \lambda Tx \quad for \quad some \quad \lambda \in (0,1)\}$$

is bounded. Let $x \in \Phi$. Then we have for every $t \in [0, n]$

$$|x(t)| = |\lambda Tx(t)| \leq |f(t)| + |q(t)| \int_0^t |K(s, x(s))| dv(s)$$
$$\leq |f(t)| + |q(t)| \int_0^t a(s)|x(s)| dv(s).$$

By Theorem 4.4, we get for every $t \in [0, n]$

$$|x(t)| \leq |f|_n + |q|_n \int_0^t a(s)|f(s)| e^{\int_s^t a|q| dv} dv(s).$$

Since $\int_0^t a(s)|f(s)|e^{\int_s^t a|q|dv} dv(s)$ is regulated on $[0, n]$ and so bounded on $[0, n]$, x is also bounded on $[0, n]$. Thus there is a nonnegative number M_n such that $|x|_n \leq M_n$ for all $x \in \Phi$ and we conclude that Φ is bounded in $G[0, \infty)$.

Hence by Theorem 5.1, there is an $x \in G[0, \infty)$ such that $x = Tx$. This completes the proof. □

Corollary 5.4. *Assume that $f, q \in G[0, \infty)$ and $g \in BV_{loc}[0, \infty)$ is left-continuous on $[0, \infty)$, and $K : [0, \infty) \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ satisfies condition (H).*

Suppose that $f(t), q(t) \to 0$ as $t \to \infty$ and that

$$\int_0^t a(s)|f(s)| e^{\int_s^t a|q| dv} dv(s)$$

is bounded on $[0, \infty)$. Then every solution of (5.1) approaches to 0 as $t \to \infty$.

Proof. In the proof of Theorem 5.3 we get

$$|x(t)| \leq |f(t)| + |q(t)| \int_0^t a(s)|f(s)| e^{\int_s^t a|q| dv} dv(s).$$

By hypotheses, it is obvious that every solution of (5.1) approaches to 0 as $t \to \infty$. □

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ONYANG YONGHWA MIDDLE SCHOOL, ASAN, CHUNGNAM, 336-050, KOREA
Email address: yd7012@daum.net