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TRANSVERSAL HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE SASAKIAN MANIFOLD

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ABSTRACT. In this paper, we study the geometry of transversal half lightlike submanifolds of an indefinite Sasakian manifold. The main result is to prove three characterization theorems for such a transversal half lightlike submanifold. In addition to these main theorems, we study the geometry of totally umbilical transversal half lightlike submanifolds of an indefinite Sasakian manifold.

INTRODUCTION

Recently many authors studied the geometry of lightlike submanifolds of semi-Riemannian or indefinite Kaehlerian manifolds [1, 2, 3, 4, 8, 9, 10]. Several authors have studied the geometry of lightlike submanifolds M of indefinite Sasakian manifolds $\bar{M}$ [6, 7, 11, 12]. The authors in above papers principally assumed that M is totally umbilical [3, 6, 9, 12] or screen conformal [1, 5, 8], or the screen distribution $S(TM)$ of M is totally umbilical in M [3, 10]. Because these conditions are very useful tools to study of (non-degenerate and lightlike) submanifolds.

The purpose of this paper is to prove the following three characterization theorems for transversal half lightlike submanifolds of an indefinite Sasakian manifold: (1) There exists no screen conformal transversal half lightlike submanifolds of an indefinite Sasakian manifold. (2) There exists no transversal half lightlike submanifolds M of an indefinite Sasakian manifold such that the screen distribution $S(TM)$ of M is totally umbilical in M. (3) There exists no proper totally umbilical transversal half lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$. In addition to these main theorems, we study the geometry of totally umbilical transversal half lightlike submanifolds M of an indefinite Sasakian manifold $\bar{M}$.

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1. Half Lightlike Submanifolds

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite contact metric manifold* [6, 11, 12] if there exists a $(1,1)$-type tensor field J, a vector field ζ, called the characteristic vector field, and its 1-form θ satisfying

$$\begin{aligned} &J^2X = -X + \theta(X)\zeta, \ J\zeta = 0, \ \theta\circ J = 0, \ \theta(\zeta) = 1, \\ &\bar{g}(\zeta,\zeta) = \varepsilon, \quad \bar{g}(JX, JY) = \bar{g}(X,Y) - \varepsilon\,\theta(X)\theta(Y), \\ &\theta(X) = \varepsilon\bar{g}(\zeta, X), \quad d\theta(X,Y) = \bar{g}(JX,Y), \ \varepsilon = \pm 1, \end{aligned} \tag{1.1}$$

for any vector fields X, Y on $\bar{M}$. Then the set $(J, \theta, \zeta, \bar{g})$ is called an *indefinite contact metric structure* on $\bar{M}$. We say that $\bar{M}$ has a *normal contact structure* if $N_J + d\theta\otimes\zeta = 0$, where N_J is the Nijenhuis tensor field of J [6, 7]. A normal contact metric manifold is called an *indefinite Sasakian manifold* [11, 12] for which we have

$$\bar{\nabla}_X\zeta = JX, \tag{1.2}$$

$$(\bar{\nabla}_X J)Y = \varepsilon\,\theta(Y)X - \bar{g}(X,Y)\zeta. \tag{1.3}$$

For any indefinite Sasakian manifold $\bar{M} = (\bar{M}, J, \zeta, \theta, \bar{g})$, It is known that *the characteristic vector field ζ on $\bar{M}$ is a spacelike vector field*, i.e., $\varepsilon = 1$ [11].

An indefinite Sasakian manifold $\bar{M}$ is called an *indefinite Sasakian space form*, denoted by $\bar{M}(c)$, if it has the constant J-sectional curvature c [11, 12]. The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ is given by

$$\begin{aligned} 4\bar{R}(X,Y)Z = &(c+3)\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ &+(c-1)\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \bar{g}(X,Z)\theta(Y)\zeta \\ &-\bar{g}(Y,Z)\theta(X)\zeta + \bar{g}(JY,Z)JX + \bar{g}(JZ,X)JY - 2\bar{g}(JX,Y)JZ\}, \end{aligned} \tag{1.4}$$

for any vector fields X, Y and Z in $\bar{M}$.

A submanifold M of codimension 2 is called a *half lightlike submanifold* if the radical distribution $Rad(TM) = TM\cap TM^\perp$ of M is a vector subbundle of both the tangent bundle TM and the normal bundle $TM^\perp$ of rank 1. Then there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and $TM^\perp$ respectively, called the *screen* and *co-screen distribution* on M;

$$TM = Rad(TM)\oplus_{orth} S(TM), \ TM^\perp = Rad(TM)\oplus_{orth} S(TM^\perp), \tag{1.5}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\bar{g}(L,L) = \epsilon(=\pm 1)$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and L belong to $S(TM)^{\perp}$. Hence we have the following orthogonal decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltr(TM))$ satisfying

$$\bar{g}(\xi, N) = 1, \; \bar{g}(N,N) = \bar{g}(N,X) = \bar{g}(N,L) = 0, \; \forall\, X \in \Gamma(S(TM)). \tag{1.6}$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Then the tangent bundle $T\bar{M}$ of the ambient manifold $\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{1.7}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.5). Then the local Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{1.8}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{1.9}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N, \tag{1.10}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{1.11}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{1.12}$$

for all X, $Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections of M and on $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_{ξ} and A_L are linear operators on TM and τ, ρ and ϕ are 1-forms on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both B and D are symmetric bilinear forms. From the facts $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X,Y) = \epsilon\bar{g}(\bar{\nabla}_X Y, L)$ for all X, $Y \in \Gamma(TM)$, we know that B and D are independent of the choice of a screen distribution and

$$B(X,\xi) = 0, \quad D(X,\xi) = -\epsilon\phi(X), \; \forall\, X \in \Gamma(TM). \tag{1.13}$$

We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental tensor* of M. The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{1.14}$$

for all X, Y, $Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N), \ \forall X \in \Gamma(TM). \tag{1.15}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.16}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{1.17}$$

$$\epsilon D(X,PY) = g(A_L X, PY), \qquad \bar{g}(A_L X, N) = \epsilon\rho(X), \tag{1.18}$$

$$\epsilon D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \ \forall X,\, Y \in \Gamma(TM). \tag{1.19}$$

By (1.16) and (1.17), we show that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C respectively and

$$A^*_\xi \xi = 0. \tag{1.20}$$

2. Transversal Half Lightlike Submanifolds

Definition 1. A half lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is said to be a *transversal half lightlike submanifold* [14] of $\bar{M}$ if the characteristic vector field ζ of $\bar{M}$ belongs to the transversal vector bundle $tr(TM)$ of M.

For any transversal half lightlike submanifold M, ζ is decomposed by $\zeta = aL + fN$, where $a = \epsilon\theta(L)$ and $f = \theta(\xi)$ are smooth functions on $\bar{M}$. Since $1 = \bar{g}(\zeta,\zeta) = a^2\bar{g}(L,L) = \epsilon a^2$, we see that $\epsilon = 1$, that is, L is a unit spacelike vector field and $a = \pm 1$. We may assume that $a = 1$ without loss of generality. In this case we have

$$\zeta = L + fN. \tag{2.1}$$

Proposition 2.1. *Let M be a transversal half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then there exist a screen distribution $S(TM)$ such that $J(Rad(TM))$ and $J(ltr(TM)) = J(S(TM^\perp))$ are subbundles of $S(TM)$ of rank 1.*

Proof. Using (1.1), we have $\bar{g}(J\xi, \xi) = 0$. Thus $J\xi$ belongs to $TM \oplus_{orth} S(TM^\perp)$. If $Rad(TM) \cap J(Rad(TM)) \neq \{0\}$, then there exists a non-vanishing smooth real valued function h such that $J\xi = h\xi$. Applying J to this equation and using (1.1), we have $(h^2+1)\xi = f\zeta$. Taking the scalar product with ξ and N in this equation by turns, we get $f = 0$ and $h^2+1 = 0$ respectively. It is an impossible case for real M. Thus $Rad(TM) \cap J(Rad(TM)) = \{0\}$. Moreover, if $S(TM^{\perp}) \cap J(Rad(TM)) \neq \{0\}$, then there exists a non-vanishing smooth real valued function b such that $J\xi = bL$. In this case we have $-f^2 = \bar{g}(J\xi, J\xi) = b^2\bar{g}(L, L) = b^2$. Thus $b = 0$. It is a contradiction to $b \neq 0$. This implies $S(TM^{\perp}) \cap J(Rad(TM)) = \{0\}$. This enables one to choose a screen distribution $S(TM)$ such that it contains $J(Rad(TM))$ as a vector subbundle. From the facts $\bar{g}(JN, N) = 0$ and $\bar{g}(JN, \xi) = -\bar{g}(N, J\xi) = 0$, using the above method, we also show that $J(ltr(TM))$ is a vector subbundle of $S(TM)$ of rank 1. On the other hand, from the facts $\bar{g}(JL, L) = 0$, $\bar{g}(JL, \xi) = -\bar{g}(L, J\xi) = 0$ and $\bar{g}(JL, N) = -\bar{g}(L, JN) = 0$, we show that $J(S(TM^{\perp}))$ is also a vector subbundle of $S(TM)$ of rank 1. Applying J to (2.1) and using $J\zeta = 0$, we have $JL = -fJN$. As $\text{rank}\, J(ltr(TM)) = \text{rank}\, J(S(TM^{\perp})) = 1$, we get $J(ltr(TM)) = J(S(TM^{\perp}))$. □

Note 1. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^* = TM/Rad(TM)$ considered by Kupeli [13]. Thus all screen distributions are mutually isomorphic. For this reason, in this paper, we consider only transversal half lightlike submanifold M of $\bar{M}$ equipped with a screen distribution $S(TM)$ such that $J(Rad(TM))$ and $J(ltr(TM)) = J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$ of rank 1.

Definition 2. (1) M is said to be *screen conformal* [1] if there exist a non-vanishing smooth function φ on a neighborhood $\mathcal{U}$ such that $A_N = \varphi A^*_{\xi}$, that is,

$$C(X, PY) = \varphi B(X, Y), \ \forall X, Y \in \Gamma(TM).$$

(2) $S(TM)$ is said to be *totally umbilical* [3] in M if there exist a smooth function γ on any coordinate neighborhood $\mathcal{U}$ in M such that

$$C(X, PY) = \gamma\, g(X, Y), \ \forall X, Y \in \Gamma(TM).$$

(3) M is said to be *totally umbilical* [3] if there is a smooth transversal vector field $\mathcal{H} \in \Gamma(tr(TM))$ on any coordinate neighborhood $\mathcal{U}$ in M such that

$$h(X, Y) = \mathcal{H}\, g(X, Y), \ \forall X, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0\,(\mathcal{H} \neq 0)$, we say that M is *totally geodesic* (*proper totally umbilical*).

It is easy to see that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions β and δ such that

$$B(X,Y) = \beta g(X,Y),\ D(X,Y) = \delta g(X,Y),\ \forall X,\ Y \in \Gamma(TM).$$

Theorem 2.2. (1) *There exists no screen conformal transversal half lightlike submanifold of an indefinite Sasakian manifold.*

(2) *There exists no transversal half lightlike submanifold of an indefinite Sasakian manifold such that its screen distribution is totally umbilical.*

(3) *There exists no proper totally umbilical transversal half lightlike submanifold of an indefinite Sasakian manifold.*

Proof. First of all, we prove that the function f, defined by (2.1), satisfies $f \neq 0$: Assume that $f = 0$. Then we get $\zeta = L$ by (2.1). Using (1.2) and (1.10), we have

$$JX = -A_L X + \phi(X)N, \quad \forall X \in \Gamma(TM).$$

Taking the scalar product with ξ and JN in this equation by turns, we have

$$\phi(X) = -g(X, J\xi), \quad \eta(X) = -D(X, JN), \quad \forall X \in \Gamma(TM),$$

respectively. From this two equations and (1.13) with $\epsilon = 1$, we have

$$1 = \eta(\xi) = -D(\xi, JN) = \phi(JN) = -g(JN, J\xi) = -1.$$

It is a contradiction. Thus we have $f \neq 0$.

Next, we prove $f = C(\xi, J\xi)$ and $fB(X, JN) = -D(X, J\xi)$ for all $X \in \Gamma(TM)$: Consider the local null and unit timelike vector fields U and V on $S(TM)$ respectively such that $g(U,V) = 1$ and the 1-form v defined by

$$U = JL = -fJN, \quad V = -f^{-1}J\xi, \quad v(X) = -g(X,V), \quad \forall X \in \Gamma(TM). \tag{2.2}$$

Applying the operator $\bar{\nabla}_X$ to $fV = -J\xi$ and using (1.3), (1.8) and (1.12), we have

$$f\bar{\nabla}_X V = -fX - \{Xf + f\tau(X)\}V + \phi(X)U + J(A^*_\xi X). \tag{2.3}$$

Taking the scalar product with L, V and N in this equation by turns and using (1.1), (2.2) and the facts $g(V,V) = -1$, $g(V,U) = 1$ and $\bar{g}(V,N) = 0$, we have

$$fB(X, JN) = -D(X, J\xi), \tag{2.4}$$

$$Xf + f\tau(X) + \phi(X) = -fv(X), \tag{2.5}$$

$$f\eta(X) + B(X, JN) = C(X, J\xi), \tag{2.6}$$

respectively. Replace X by ξ in (2.6) and using (1.13), we have $f = C(\xi, J\xi)$.

(1) If M is screen conformal, then we have

$$f = C(\xi, J\xi) = \varphi B(\xi, J\xi) = 0.$$

It is a contradiction to $f \neq 0$. Thus M is not screen conformal.

(2) If $S(TM)$ is totally umbilical in M, then we have

$$f = C(X, J\xi) = \gamma g(\xi, J\xi) = 0.$$

It is a contradiction to $f \neq 0$. Thus $S(TM)$ is not totally umbilical in M.

(3) If M is totally umbilical, then, from (2.4), we have

$$f\beta g(X, JN) = -\delta g(X, J\xi), \ \forall X \in \Gamma(TM). \tag{2.7}$$

Replace X by JN in (2.7) and using $g(J\xi, JN) = 1$ and $g(JN, JN) = 0$, we have $\delta = 0$. Replace X by $J\xi/f$ in (2.7) with $\delta = 0$, we have $\beta = 0$. This results imply that M is totally geodesic. Thus there exist no proper totally umbilical transversal half lightlike submanifold M of $\bar{M}$. □

Corollary 1. *Let M be a totally umbilical transversal half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is totally geodesic.*

Theorem 2.3. *Let M be a transversal half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $S(TM)$ is integrable. Furthermore, M is locally a product manifold $L \times M^*$, where L is a null curve on M and M^* is a leaf of $S(TM)$.*

Proof. For all $Y \in \Gamma(TM)$, we show that $\bar{g}(\zeta, PY) = 0$. Applying the operator $\bar{\nabla}_X$ to this equation and using (1.2), (1.8) and (1.11), we have

$$\bar{g}(JX, PY) + fC(X, PY) + D(X, PY) = 0, \ \forall X, \ Y \in \Gamma(TM), \text{ i.e.,}$$
$$f\{C(X,Y) - C(Y,X)\} = \bar{g}(X, JY) - \bar{g}(JX, Y), \ \forall X, \ Y \in \Gamma(S(TM)).$$

Thus we see that C is symmetric on $S(TM)$ if and only if $\bar{g}(X, JY) = \bar{g}(JX, Y)$ for all $X, \ Y \in \Gamma(S(TM))$. From (1.1) we show that $\bar{g}(X, JY) = \bar{g}(JX, Y)$ is equivalent to $\bar{g}(JX, JY) = -g(X, Y) + \theta(X)\theta(Y)$. Thus C is symmetric on $S(TM)$. From (1.11), we have $C(X, Y) = g(\nabla_X Y, N)$. Using this equation, we obtain

$$0 = C(X, Y) - C(Y, X) = g(\nabla_X Y - \nabla_Y X, N) = \eta([X, Y]),$$

for all $X, \ Y \in \Gamma(S(TM))$. Thus we have $[X, Y] \in \Gamma(S(TM))$. This implies that $S(TM)$ is integrable. Due to Duggal and Bejancu [3], M is locally a product manifold $L \times M^*$, where L is a null curve on M and M^* is a leaf of $S(TM)$. □

Definition 3. We say that M is *locally symmetric* [11] if its curvature tensor R be parallel, i.e., have vanishing covariant differential, $\nabla_X R = 0$ for all $X \in \Gamma(TM)$.

Theorem 2.4. *Let M be a totally umbilical transversal half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If M is locally symmetric, then M is a space of constant curvature* 1.

Proof. As M is totally umbilical, by Theorem 2.2(3) and (1.13), we get $B = D = \phi = 0$. From (1.16) and (1.18), we have $A^*_\xi X = 0$ and $A_L X = \rho(X)\xi$ for any $X \in \Gamma(TM)$. Using this resuls, (2.3) and (2.5) and the fact $f \neq 0$, we have

$$\nabla_X V = -X + v(X)V, \ \forall X \in \Gamma(TM). \tag{2.8}$$

Using (2.8) and the fact that ∇ is torsion free connection, we show that

$$R(X,Y)V = 2dv(X,Y)V + v(X)Y - v(Y)X, \ \forall X,\, Y \in \Gamma(TM).$$

Taking the scalar product with V in this equation and using the third equation of (2.2){denote (2.2)-3}, we have $dv = 0$. Therefore we obtain

$$R(X,Y)V = v(X)Y - v(Y)X, \ \forall X,\, Y \in \Gamma(TM). \tag{2.9}$$

Differentiating (2.2)-3 with $Y \in \Gamma(TM)$ and using (2.2) and (2.8), we have

$$(\nabla_X v)(Y) = g(X,Y) + v(X)v(Y), \quad \forall X,\, Y \in \Gamma(TM). \tag{2.10}$$

Differentiating (2.9) with $Z \in \Gamma(TM)$ and using (2.9) and the fact that M is locally symmetric, i.e., $\nabla_X R = 0$ for any $X \in \Gamma(TM)$, we have

$$R(X,Y)\nabla_Z V = (\nabla_Z v)(X)Y - (\nabla_Z v)(Y)X, \ \forall X,\, Y \in \Gamma(TM).$$

Substituting (2.8) and (2.10) in this equation and using (2.9), we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad \forall X,\, Y \in \Gamma(TM). \tag{2.11}$$

Thus M is a space of constant curvature 1. □

Denote by $\bar{R}$ and R the curvature tensors of the connections $\bar{\nabla}$ and ∇ of $\bar{M}$ and M respectively. Using the local Gauss-Weingarten formulas (1.8)~(1.10) for M, for all X, Y, $Z \in \Gamma(TM)$, we obtain the Gauss equation for M:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z \\ &+ B(X,Z)A_N Y - B(Y,Z)A_N X + D(X,Z)A_L Y - D(Y,Z)A_L X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &\qquad + \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N, \\ &+ \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}L.\end{aligned} \tag{2.12}$$

Definition 4. A half lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is said to be *irrotational*[13] if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

Note 2. For any irrotational M, since $B(X, \xi) = 0$ due to the first equation of (1.13), we have $D(X, \xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

Theorem 2.5. *Let M be an irrotational transversal half lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$. Then we have $c = 1$.*

Proof. Replacing Z by ξ in (2.12) and using (1.12), (1.13) and the facts that $\phi = 0$ and $B(Y, A^*_\xi X) = B(X, A^*_\xi Y)$ for all X, $Y \in \Gamma(TM)$, we have

$$\bar{R}(X,Y)\xi = R(X,Y)\xi + \{D(Y, A^*_\xi X) - D(X, A^*_\xi Y)\}L. \tag{2.13}$$

Using (2.13) and the fact $R(X,Y)Z \in \Gamma(TM)$ for X, Y, $Z \in \Gamma(TM)$, we get

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,\, \xi) &= -\bar{g}(\bar{R}(X,Y)\xi,\, Z) = -g(R(X,Y)\xi,\, Z) \\ &= g(R(X,Y)Z,\, \xi) = 0, \ \forall\, X,\, Y,\, Z \in \Gamma(TM). \end{aligned} \tag{2.14}$$

Taking the scalar product with ξ in (1.4) and using (1.1), (2.1) and (2.14), we get

$$\begin{aligned} (c-1)\{ & fg(X,Z)\theta(Y) - fg(Y,Z)\theta(X) - \bar{g}(JY,Z)g(X,J\xi) \\ & - \bar{g}(JZ,X)g(Y,J\xi) + 2\bar{g}(JX,Y)g(Z,J\xi)\} = 0. \end{aligned}$$

Replacing Z by $J\xi$ and Y by ξ in this equation and using (1.1), we have

$$4f^2(c-1)g(X, J\xi) = 0, \ \forall\, X \in \Gamma(TM). \tag{2.15}$$

Replace X by JN in (2.15), we get $f^2(c-1) = 0$. As $f \neq 0$, we have $c = 1$. □

Corollary 2. *There exist no irrotational transversal half lightlike submanifolds M of indefinite Sasakian space form $\bar{M}(c)$ with $c \neq 1$.*

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \ \forall\, X,\, Y \in \Gamma(TM).$$

In general, $R^{(0,2)}$ is not symmetric [3, 4, 5]. A tensor field $R^{(0,2)}$ is called the *induced Ricci tensor*[5], denoted by Ric, of M if it is symmetric.

Theorem 2.6 ([4]). *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. The Ricci type tensor $R^{(0,2)}$ is symmetric, if and only if, the 1-form τ given by (1.9) is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*

The local Weingarten formula (1.9) becomes

$$\bar{\nabla}_X N = -A_N X + \nabla^\ell_X N + E^s(X,N), \ \forall\, X \in \Gamma(TM), \ \ N \in \Gamma(ltr(TM)). \tag{2.16}$$

The symbol ∇^ℓ is a linear connection on $ltr(TM)$. We call ∇^ℓ the *lightlike transversal connection* on M. In this case we have $\nabla^\ell_X N = \tau(X)N$ and $E^s(X, N) = \rho(X)L$.

Definition 5. We define the curvature tensor R^ℓ of the lightlike transversal vector bundle $ltr(TM)$ of M by

$$R^\ell(X,Y)N = \nabla^\ell_X\nabla^\ell_Y N - \nabla^\ell_Y\nabla^\ell_X N - \nabla^\ell_{[X,\,Y]}N, \tag{2.17}$$

for all $X,\, Y \in \Gamma(TM)$. If R^ℓ vanishes identically, then the lightlike transversal connection ∇^ℓ of M is said to be *flat* (or *trivial*) [11].

Theorem 2.7. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. The following assertions are equivalent:*

(1) *The lightlike transversal connection of M is flat, i.e., $R^\ell = 0$.*
(2) *The 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*
(3) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*

Proof. Applying the operator ∇^ℓ_X to $\nabla^\ell_Y N = \tau(Y)N$, we have

$$\nabla^\ell_X\nabla^\ell_Y N = \{X(\tau(Y)) + \tau(Y)\tau(X)\}N.$$

By straightforward calculations from this equation and (2.17), we have

$$R^\ell(X,Y)N = 2d\tau(X,Y)N, \quad \forall\, X,\, Y \in \Gamma(TM).$$

From this result and Theorem 2.6, we have our assertions. □

Theorem 2.8. *Let M be a totally umbilical transversal half lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If M is locally symmetric, then the lightlike transversal connection of M is flat and the Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor Ric of M.*

Proof. Using (1.8), (1.9) and (1.10), we have

$$\begin{aligned} \bar{R}(X,Y)N = {} & -\nabla_X(A_NY) + \nabla_Y(A_NX) + A_N[X,Y] \\ & + \tau(X)A_NY - \tau(Y)A_NX + \rho(X)A_LY - \rho(Y)A_LX \\ & + \{B(Y,A_NX) - B(X,A_NY) + 2d\tau(X,Y) + \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N \\ & + \{D(Y,A_NX) - D(X,A_NY) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L. \end{aligned} \tag{2.18}$$

Taking the scalar product with ξ in (2.18) with $\phi = 0$, we have

$$\bar{g}(\bar{R}(X,Y)N,\, \xi) = 2d\tau(X,Y), \;\forall\, X,\, Y \in \Gamma(TM).$$

Using this, (2.11), (2.13) and the fact M is locally symmetric, we have

$$2d\tau(X,Y) = \bar{g}(\bar{R}(X,Y)N,\, \xi) = -\bar{g}(\bar{R}(X,Y)\xi,\, N)$$
$$= -\bar{g}(R(X,Y)\xi,\, N) = g(X,\xi)\eta(Y) - g(Y,\xi)\eta(X) = 0,$$

for all X, $Y \in \Gamma(TM)$. Thus, from Theorem 2.7, we have our theorem. □

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