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A GENERALIZATION OF STONE'S THEOREM IN HILBERT C^*-MODULES

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Abstract. Stone's theorem states that "A bounded linear operator A is infinitesimal generator of a C_0-group of unitary operators on a Hilbert space H if and only if iA is self adjoint". In this paper we establish a generalization of Stone's theorem in the framework of Hilbert C^*-modules.

1. Introduction

Stone's theorem is a basic theorem of functional analysis, which establishes a one-to-one correspondence between the self-adjoint operators on a Hilbert space and the one-parameter families of strongly continuous unitary operators. The theorem is named after Marshall Stone [5], who formulated and proved this theorem in 1932. The Hille-Yosida theorem generalizes Stone's theorem to strongly continuous one-parameter semigroups of contractions on Banach spaces [6].

The notion of a Hilbert C^*-module is a generalization of that of a Hilbert space. It is a significant tool for studying Morita equivalence of C^*-algebras, C^*-algebra quantum group, operator K-theory and the theory of operator spaces [1, 4]. The paper organized as follows:

The first and second sections are devoted to a description of the essential properties of Hilbert C^*-modules and one parameter semigroups. In the third section we study conditions under which the adjoint of a C_0-semigroup is a C_0-semigroup. In the forth section, we investigate some properties of C_0-groups of unitary operators on Hilbert C^*-modules and generalize Stone's theorem in the setting of Hilbert C^*-modules. Recently, the authors of [7] presented a Stone type theorem in the setting of Hilbert C^*-modules.

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2. PRELIMINARIES

Suppose $\mathcal{A}$ is a C^*-algebra. A complex linear space $\mathcal{X}$ is a right inner product $\mathcal{A}$-module if $\mathcal{X}$ is a right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ and there exists an inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying the following conditions:
(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$,
(ii) $\langle x, \lambda y + \mu z >= \lambda \langle x, y > + \mu \langle x, z >$,
(iii)$\langle x, ya \rangle = \langle x, y \rangle a$,
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$,
for all $x, y, z \in \mathcal{X}$, $\quad a \in \mathcal{A}$ and $\mu, \lambda \in \mathbb{C}$. A Hilbert $\mathcal{A}$-module (Hilbert C^*-module) is an inner product $\mathcal{A}$-module $\mathcal{X}$ which is complete in the norm given by $\|x\| = \|\langle x, x \rangle\|^{1/2}$. The notion of left Hilbert $\mathcal{A}$-module is similarly defined. Every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to the inner product $\langle x, y \rangle = x^* y$ and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{X} \to \mathcal{Y}$ is adjointable if there is a map $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$. We denote by $L(\mathcal{X}, \mathcal{Y})$ the set of all adjointable operators from $\mathcal{X}$ into $\mathcal{Y}$. It is known that T is then a bounded $\mathcal{A}$-linear map. If $\mathcal{X} = \mathcal{Y}$, then $L(\mathcal{X})$ is a C^*-algebra with respect to the operator norm and $\|T\| = \|T^*\|$ for every $T \in L(\mathcal{X})$ ([4, Proposition 2.21]) and $\langle T(x), T(x) \rangle \leq \|T\|^2 \langle x, x \rangle$ ([4, corollary 2.22]).

Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module. Recall that a one parameter family $\mathscr{T} = \{T(t)\}_{t \geq 0}$ of adjointable $\mathcal{A}$-linear operators on $\mathcal{X}$ is called a semigroup if

(i) $T(0) = I \quad$ (I is the identity operator on $\mathcal{X}$),
(ii) $T(s + t) = T(t)T(s)$ for every $t, s \geq 0$.

Furthermore $\mathscr{T}$ is uniformly continuous if

$$\lim_{t \to 0^+} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \left\{ x \in \mathcal{X} : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\},$$

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt} \Big|_{t=0} \quad (x \in D(A))$$

is called the infinitesimal generator of the semigroup $\mathscr{T}$. The domain $D(A)$ of A is a submodule of $\mathcal{X}$, since $xa \in D(A)$ for any $x \in D(A)$ and any $a \in \mathcal{A}$.

A semigroup $\mathscr{T} = \{T(t)\}_{t\geq 0}$ on $\mathcal{X}$ is a strongly continuous semigroup (C_0-semigroup) if $\lim_{t\to 0^+} T(t)x = x$ for each $x \in \mathcal{X}$.
By the same reasoning as [3, corollary 2.5] one can easily prove the following theorem.

Theorem 2.1. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module. If A is the infinitesimal generator of a C_0-semigroup $\mathscr{T} = \{T(t)\}_{t\geq 0}$ on $\mathcal{X}$, then $D(A)$ is dense in $\mathcal{X}$ and A is an $\mathcal{A}$-linear closed operator.*

Example 2.2. Consider the C^*-algebra

$$\mathcal{X} = C[0,1] = \{f : [0,1] \to \mathbb{C}\ ;\ f \text{ is continuous}\}$$

equipped with the supremum norm. In fact, $\mathcal{X}$ is a Hilbert C^*-module over itself. For $f \in C[0,1]$, we define $\mathscr{T} : \mathbb{R}^+ \to B(C[0,1])$ by $(T(t)f)(x) = f(x+t)$ for $t \in \mathbb{R}^+, x \in [0,1]$. It is easy to check that $T(t)$ is a C_0-semigroup. The infinitesimal generator of $\mathscr{T}$ is defined on

$$D(A) = \{f \in \mathcal{X} : \quad f' \text{ exists}, \quad f' \in \mathcal{X} \quad \text{and} \quad f'(0) = 0\}$$

and

$$(Af)(x) = \lim_{t\to 0} \frac{(T(t)f)(x) - f(x)}{t} = \lim_{t\to 0} \frac{f(x+t) - f(x)}{t} = f'(x)$$

for each $x \in [0,1]$. Hence $Af = f'$.

The notion of a C_0-group and the infinitesimal generator are defined similarly, when $-\infty < t < \infty$.
Let $\mathcal{X}$ be a Hilbert C^*-module and let $\mathscr{T} = \{T(t)\}_{-\infty<t<\infty}$ be a C_0-group in $L(\mathcal{X})$, with the infinitesimal generator A. Then $\mathscr{T} = \{T(t)\}_{t\geq 0}$ is a C_0-semigroup in $L(\mathcal{X})$, whose infinitesimal generator is also A. Moreover, if $S(t) = T(-t)$ for $t \geq 0$, then $\mathscr{S} = \{S(t)\}_{t\geq 0}$ is also a C_0-semigroup in $L(\mathcal{X})$ with the infinitesimal generator $-A$. Thus if $\mathscr{T} = \{T(t)\}_{-\infty<t<\infty}$ is a C_0-group of adjointable operators in $L(\mathcal{X})$ then both A and $-A$ are the infinitesimal generators of some C_0-semigroups which are denoted by $\{T_+(t)\}_{t\geq 0}$ and $\{T_-(t)\}_{t\geq 0}$, respectively.
Conversely, if A and $-A$ are the infinitesimal generators of two C_0-semigroups $\{T_+(t)\}_{t\geq 0}$ and $\{T_-(t)\}_{t\geq 0}$, then A is the infinitesimal generator of the C_0-group $\mathscr{T} = \{T(t)\}_{-\infty<t<\infty}$ given by

$$T(t) = \begin{cases} T_+(t) & t \geq 0 \\ T_-(-t) & t \leq 0 \end{cases}$$

It is easy to check that $\overline{D(A)} = \mathcal{X}$ and A is closed. We need the next result later.

Lemma 2.3. *Let $\mathcal{X}$ be a Hilbert C^*-module and let $\mathscr{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on $\mathcal{X}$ with the infinitesimal generator A. If $T(t)^{-1} \in L(\mathcal{X})$ and $S(t) = T(t)^{-1}$ for every $t > 0$, then $\mathscr{S} = \{S(t)\}$ is a C_0-semigroup on $\mathcal{X}$, whose infinitesimal generator is $-A$. Moreover if*

$$U(t) = \begin{cases} T(t) & t \geq 0 \\ T(-t)^{-1} & t \leq 0. \end{cases}$$

then $\mathscr{U} = \{U(t)\}$ is a C_0-group on $\mathcal{X}$.

Proof. [3, Lemma 6.4] □

3. The Adjoint of a Semigroup

Let $\mathcal{X}$ be a Hilbert C^*-module and $\mathscr{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on $\mathcal{X}$. The family $\mathscr{T}^* = \{T(t)^*\}_{t\geq 0}$ is clearly a semigroup that is called the adjoint of the semigroup $\mathscr{T}$.

We now present an important condition on a C_0-semigroup $\mathscr{T}$ under which $\mathscr{T}^*$ is a C_0-semigroup on $\mathcal{X}$.

Theorem 3.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and let $\mathscr{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup of contractions on $\mathcal{X}$. Then $\mathscr{T}^* = \{T(t)^*\}_{t\geq 0}$ is a C_0-semigroup of contractions on $\mathcal{X}$.*

Proof. Since $\|T(t)\| = \|T(t)^*\|$. It is enough to show that $\lim_{t\to 0}\|T(t)^*x - x\| = 0$ for each $x \in \mathcal{X}$. We have

$$\begin{aligned} \langle T(t)^*x - x, T(t)^*x - x\rangle &= \langle T(t)^*x - x, T(t)^*x\rangle - \langle T(t)^*x - x, x\rangle \\ &= \langle T(t)^*x, T(t)^*x\rangle - \langle x, T(t)^*x\rangle - \langle T(t)^*x, x\rangle + \langle x, x\rangle \\ &\leq \|T(t)\|^2\langle x, x\rangle - \langle T(t)x, x\rangle - \langle x, T(t)x\rangle + \langle x, x\rangle \\ &\leq \langle x, x\rangle - \langle T(t)x, x\rangle - \langle x, T(t)x\rangle + \langle x, x\rangle. \end{aligned}$$

We know that in a C^*-algebra $\mathcal{A}$ if $0 \leq a \leq b$, then $\|a\| \leq \|b\|$. Letting $t \to 0$ we obtain,

$$\begin{aligned} \lim_{t\to 0}\|T(t)^*x - x\|^2 &= \lim_{t\to 0}\|\langle T(t)^*x - x, T(t)^*x - x\rangle\| \\ &\leq \lim_{t\to 0}\|\langle x, x\rangle - \langle T(t)x, x\rangle - \langle x, T(t)x\rangle + \langle x, x\rangle\| \\ &= \|\langle x, x\rangle - \langle x, x\rangle - \langle x, x\rangle + \langle x, x\rangle\| \end{aligned}$$

Then $\lim_{t\to 0}\|T(t)^*x - x\| = 0$. Hence $\mathscr{T}^*$ is a C_0-semigroup of contractions. □

The following theorem states a relationship between the infinitesimal generator of a C_0-semigroup of contractions and its adjoint.

Theorem 3.2. *Let $\mathcal{X}$ be a Hilbert C^*-module and let $\mathscr{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup of contractions on $\mathcal{X}$ with the infinitesimal generator A. Then A^* is the infinitesimal generator of the C_0-semigroup $\mathscr{T}^*$ and $D(A^*)$ is dense in $\mathcal{X}$.*

Proof. First we show that $(T(t) - I)^* = T(t)^* - I$. For all $x, y \in \mathcal{X}$, we have

$$\langle (T(t) - I)x, y\rangle = \langle T(t)x, y\rangle - \langle x, y\rangle = \langle x, T(t)^*y\rangle - \langle x, y\rangle = \langle x, (T(t)^* - I)y\rangle.$$

Hence $\langle x, ((T(t) - I)^* - (T(t)^* - I))y\rangle = 0$. It follows that

$$\langle \frac{(T(t) - I)x}{t}, y\rangle = \langle x, \frac{(T(t) - I)^*}{t} y\rangle = \langle x, \frac{T(t)^* - I}{t} y\rangle.$$

Letting $t \to 0$ we get

$$\langle Ax, y\rangle = \langle x, \lim_{t\to 0} \frac{T(t)^* - I}{t} y\rangle \quad (x \in D(A), y \in D(A^*))$$

$$\langle x, A^*y\rangle = \langle x, \lim_{t\to 0} \frac{T(t)^* - I}{t} y\rangle \quad (x \in D(A), y \in D(A^*))$$

$$\langle x, A^*y - \lim_{t\to 0} \frac{T(t)^* - I}{t} y\rangle = 0 \quad (x \in D(A), y \in D(A^*))$$

$$A^*y = \lim_{t\to 0} \frac{T(t)^*y - y}{t} \quad (y \in D(A^*)).$$

Thus A^* is the infinitesimal generator of the C_0-semigroup $\mathscr{T}^*$ and by Theorem [3, corollary 2.5], $D(A^*)$ is dense in $\mathcal{X}$. □

4. A Generalization of Stone's Theorem

Let $\mathcal{X}$ be a Hilbert C^*-module. Recall that an operator $U \in L(\mathcal{X})$ is normal if $UU^* = U^*U$, is unitary if $UU^* = U^*U = I$ and is self-adjoint if $U = U^*$.

As a consequence of Theorem 3.1 and Theorem 3.2 we conclude that if $\mathscr{T}$ is a C_0-semigroup of unitary operators on $\mathcal{X}$ with the infinitesimal generator A, then A^* is the infinitesimal generator of the C_0-semigroup $\mathscr{T}^*$.

Recall that U is a unitary element of C^*-algebra $L(\mathcal{X})$ if and only if U is isometric and surjective [1]. It is clear that the unitary operator U is invertible and $U^* = U^{-1}$.

The following theorem presents the necessity part of Stone's theorem generalized to Hilbert C^*-modules, see [7].

Theorem 4.1. *If A is the infinitesimal generator of a C_0-group of unitary operators $\mathscr{U} = \{U(t)\}_{-\infty<t<\infty}$ on a Hilbert C^*-module $\mathcal{X}$, then iA is a self-adjoint operator.*

Proof. If A is the infinitesimal generator of a C_0-group of unitary operators $\mathscr{U} = \{U(t)\}_{-\infty<t<\infty}$ on a Hilbert C^*-module $\mathcal{X}$, then A is densely defined and

$$-Ax = \lim_{t\to 0}\frac{U(-t)x - x}{t} = \lim_{t\to 0}\frac{U(t)^{-1}x - x}{t} = \lim_{t\to 0}\frac{U(t)^*x - x}{t} = A^*x$$

for every $x \in D(A)$. Thus $A = -A^*$. Hence $iA = (iA)^*$ and iA is self-adjoint. □

Theorem 4.2. *Let $\mathscr{T} = \{T(t)\}_{-\infty<t<\infty}$ be a C_0-group of normal contractions on a Hilbert C^*-module with the infinitesimal generator A, then A is a normal operator.*

Proof. Since $T(t)$ is normal, $T(t)T(t)^* = T(t)^*T(t)$ for all $t \in \mathbb{R}$. For $s, t \in \mathbb{Q}$ there exist positive integers m, n, k, r such that $s = \frac{n}{m}, t = \frac{k}{r}$. Thus $T(t)T(s)^* = T(\frac{k}{r})(T(\frac{m}{n}))^* = T(\frac{1}{r})^k(T(\frac{1}{n})^m)^* = (T(\frac{1}{n})^m)^*T(\frac{1}{r})^k = T(s)^*T(t)$. The density of $\mathbb{Q}$ in $\mathbb{R}$ yields that $T(t)T(s)^* = T(s)^*T(t)$ for all $s, t \in \mathbb{R}$. For all $x \in D(A)$, we have

$$\begin{aligned}
\langle A^*Ax, x\rangle &= \langle Ax, Ax\rangle \\
&= \langle \lim_{t\to 0}\frac{T(t)x - x}{t}, \lim_{s\to 0}\frac{T(s)x - x}{s}\rangle \\
&= \lim_{t,s\to 0}\frac{1}{ts}[\langle T(t)x, T(s)x\rangle - \langle T(t)x, x\rangle - \langle x, T(s)x\rangle - \langle x, x\rangle] \\
&= \lim_{t,s\to 0}\frac{1}{ts}[\langle T(s)^*x, T(t)^*x\rangle - \langle x, T(t)^*x\rangle - \langle T(s)^*x, x\rangle - \langle x, x\rangle] \\
&= \lim_{t,s\to 0}\frac{1}{ts}[\langle T(s)^*x - x, T(t)^*x - x\rangle] \\
&= \langle \lim_{s\to 0}\frac{T(s)^*x - x}{s}, \lim_{t\to 0}\frac{T(t)^*x - x}{t}\rangle \\
&= \langle A^*x, A^*x\rangle = \langle AA^*x, x\rangle.
\end{aligned}$$

Hence $A^*A = AA^*$. □

Corollary 4.3. *Let $\mathscr{T} = \{T(t)\}_{-\infty<t<\infty}$ be a C_0-group of self-adjoint contractions on a Hilbert C^*-module with the infinitesimal generator A. Then A is self-adjoint.*

5. Dissipative Operators

A linear operator A is called dissipative if $\|(\lambda I - A)x\| \geq \lambda\|x\|$ for all $x \in D(A)$ and $\lambda > 0$.

Theorem 5.1. *Let $\mathcal{X}$ be a Hilbert C^*-module and A be a linear operator such that $\overline{D(A)} = \mathcal{X}$. If A is dissipative and there is a $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = \mathcal{X}$, then A is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{X}$.*

Proof. The proof is similar to that of [3, Theorem 4.3], since every Hilbert C^*-module is a Banach space. □

The next result was stated in [7] as well via a different proof.

Theorem 5.2. *Suppose $\mathcal{X}$ is a Hilbert C^*-module. If iA is self-adjoint and there exists λ_1 such that $R(\lambda_1 I - A) = \mathcal{X}$, then the operator A is the infinitesimal generator of a C_0-group of unitaries.*

Proof. If iA is self-adjoint, then A is densely defined and $A = -A^*$.
For every $x \in D(A)$ and $\lambda > 0$ we therefore have

$$\begin{aligned} \|(\lambda I - A)x\|^2 &= \|\langle(\lambda I - A)x, (\lambda I - A)x\rangle\| \\ &= \|\langle\lambda x, \lambda x\rangle - \langle\lambda x, Ax\rangle - \langle Ax, \lambda x\rangle + \langle Ax, Ax\rangle\| \\ &= \|\lambda^2\langle x, x\rangle - \lambda\langle x, Ax\rangle - \lambda\langle Ax, x\rangle + \langle Ax, Ax\rangle\| \\ &= \|\lambda^2\langle x, x\rangle - \lambda\langle A^*x, x\rangle - \lambda\langle Ax, x\rangle + \langle Ax, Ax\rangle\| \\ &= \|\lambda^2\langle x, x\rangle + \lambda\langle Ax, x\rangle - \lambda\langle Ax, x\rangle + \langle Ax, Ax\rangle\| \\ &= \|\lambda^2\langle x, x\rangle + \langle Ax, Ax\rangle\|. \end{aligned}$$

Since $\lambda^2\langle x, x\rangle$ and $\langle Ax, Ax\rangle$ are positive elements in the C^*-algebra $\mathcal{A}$ and $\lambda^2\langle x, x\rangle + \langle Ax, Ax\rangle \geq \lambda 2\langle x, x\rangle$, by [2, Theorem 2.2.5], we get

$$\|(\lambda I - A)x\|^2 \geq \|\lambda^2\langle x\ ,\ x\rangle\|.$$

Thus $\|(\lambda I - A)x\| \geq \lambda\|x\|$ for every $x \in D(A)$ and $\lambda > 0$. Hence A is dissipative. Replacing λ_1 by $-\lambda_2$ in $R(\lambda_1 I - A) = \mathcal{X}$ we obtain $R(\lambda_2 I + A) = R(\lambda_2 I - A^*) = \mathcal{X}$. For every $x \in D(A^*)$ and $\lambda > 0$ we have

$$\begin{aligned} \|(\lambda I - A^*)x\|^2 &= \|\langle(\lambda I - A^*)x\ ,\ (\lambda I - A^*)x\rangle\| \\ &= \|\langle\lambda x, \lambda x\rangle - \langle\lambda x, A^*x\rangle - \langle A^*x, \lambda x\rangle + \langle A^*x, A^*x\rangle\| \\ &= \|\lambda^2\langle x, x\rangle - \lambda\langle x, A^*x\rangle - \lambda\langle A^*x, x\rangle + \langle A^*x, A^*x\rangle\| \\ &= \|\lambda^2\langle x, x\rangle - \lambda\langle Ax, x\rangle - \lambda\langle A^*x, x\rangle + \langle A^*x, A^*x\rangle\| \\ &= \|\lambda^2\langle x, x\rangle + \lambda\langle A^*x, x\rangle - \lambda\langle A^*x, x\rangle + \langle A^*x, A^*x\rangle\| \\ &= \|\lambda^2\langle x, x\rangle + \langle A^*x, A^*x\rangle\| \\ &\geq \|\lambda^2\langle x, x\rangle\|. \end{aligned}$$

Thus $\|(\lambda I - A^*)x\| \geq \lambda\|x\|$. Hence $A^* = -A$ is dissipative.
By Theorem 5.1, A and A^* are infinitesimal generators of some C_0-semigroups of contractions, say $\{U_+(t)\}_{t\geq 0}$ and $\{U_-(t)\}_{t\geq 0}$, respectively. Define

$$U(t) = \begin{cases} U_+(t) & t \geq 0 \\ U_-(-t) & t \leq 0. \end{cases}$$

Then $\mathscr{U} = \{U(t)\}_{-\infty<t<\infty}$ is a C_0-group. Do to $I = U(t-t) = U(t)U(-t) = U(-t)U(t)$, we have $U(-t) = U(t)^{-1}$, $\|U(t)\| \leq 1$ and $\|U(-t)\| \leq 1$.

Since $\|x\| = \|U(t)^{-1}U(t)x\| \leq \|U(t)^{-1}\| \, \|U(t)x\| = \|U(-t)\| \, \|U(t)x\| \leq \|U(t)x\| \leq \|x\|$ for every $t \in \mathbb{R}$, $U(t)$ is an isometry.

On the other hand $U(t)U(t)^{-1} = I$, so that $R(U(t)) = \mathcal{X}$. Hence $U(t)$ is a unitary for every $t \in \mathbb{R}$. Thus $\mathscr{U} = \{U(t)\}_{-\infty<t<\infty}$ is a C_0-group of unitary operators on the Hilbert $\mathcal{A}$-module $\mathcal{X}$. □

Note that if $\mathscr{T} = \{T(t)\}_{t\geq 0}$ is a uniformly continuous semigroup with the infinitesimal generator A in $L(\mathcal{X})$, then $\mathscr{T}^* = \{T(t)^*\}_{t\geq 0}$ is a uniformly continuous semigroup with the infinitesimal generator A^* in $L(\mathcal{X})$. Indeed, $\lim_{t\to 0}\|T(t)^* - I\| = \lim_{t\to 0}\|T(t) - I\| = 0$. We now can state Stone's theorem for uniformly continuous semigroups of unitary operators on Hilbert C^*-modules.

Corollary 5.3. *A linear operator A is the infinitesimal generator of a uniformly continuous group of unitary operators on a Hilbert C^*-module $\mathcal{X}$ if and only if iA is self-adjoint.*

Proof. If A is the infinitesimal generator of a uniformly continuous group of unitary operators $\mathscr{U} = \{U(t)\}_{-\infty<t<\infty}$ on a Hilbert C^*-module $\mathcal{X}$, then A is bounded and

$$-A = \lim_{t\to 0}\frac{U(-t) - I}{t} = \lim_{t\to 0}\frac{U(t)^{-1} - I}{t} = \lim_{t\to 0}\frac{U(t)^* - I}{t} = A^*.$$

Conversely, if iA is self-adjoint, then the semigroup $\mathscr{T} = \{T(t)\}_{t\geq 0}$ with $T(t) = e^{-it(iA)} = e^{tA}$ is a unitary in $L(\mathcal{X})$ and its infinitesimal generator is A. If $T(t)^* = e^{-tA}$, then $\mathscr{T}^* = \{T(t)^*\}_{t\geq 0}$ is the adjoint $\mathscr{T}$. It follows that $\mathscr{U} = \{U(t)\}_{-\infty<t<\infty}$, where

$$U(t) = \begin{cases} T(t) & t \geq 0 \\ T^*(-t) & t \leq 0. \end{cases}$$

is a uniformly continuous group of unitary operators with the infinitesimal generator A. Hence $\lim_{t\to 0}\|e^{tA} - I\| = 0$. □

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