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ON SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY KENMOTSU MANIFOLD WITH A QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. We define a quarter symmetric non-metric connection in a nearly Kenmotsu manifold and we study semi-invariant submanifolds of a nearly Kenmotsu manifold endowed with a quarter symmetric non-metric connection. Moreover, we discuss the integrability of the distributions on semi-invariant submanifolds of a nearly Kenmotsu manifold with a quarter symmetric non-metric connection.

1. Introduction

K. Kenmotsu introduced and studied a new class of almost contact manifolds called Kenmotsu manifolds in [7]. The notion of nearly Kenmotsu manifold was introduced by J. S. Kim et al. in [8]. The semi-invariant submanifolds in Kenmotsu manifolds were studied by M. Kobayashi [9] and B. B. Sinha and R. N. Yadav [10]. S. K. Lovejoy Das et al. studied the semi-invariant submanifolds of a nearly Sasakian manifold with a quarter symmetric non-metric connection in [5]. The semi-invariant submanifolds of a nearly Kenmotsu manifolds were studied by M. M. Tripathi and S. S. Shukla in [11]. Semi-invariant submanifolds of a nearly Kenmotsu manifold with a semi-symmetric non-metric connection were studied by the authors in [1]. In this paper we study the semi-invariant submanifolds of a nearly Kenmotsu manifold with a quarter symmetric non-metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

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$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is *symmetric* if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is *metric* if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

In [6], S. Golab introduced the idea of a quarter symmetric linear connection. A linear connection ∇ is said to be *quarter symmetric* if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form. M. Ahmad et al. studied some properties of hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection in [2]. In this paper we study some properties of semi-invariant submanifolds of a nearly Kenmotsu manifold with a quarter symmetric non-metric connection.

The paper is organized as follows: In section 2, we give a brief introduction of nearly Kenmotsu manifold. In section 3, we show that the induced connection on a semi-invariant submanifolds of a nearly Kenmotsu manifold with a quarter symmetric non-metric connection is also quarter symmetric non-metric. In section 4, we established some lemmas on semi-invariant submanifolds and in section 5, we discussed the integrability conditions of the distributions on semi-invariant submanifolds of nearly Kenmotsu manifolds with a quarter symmetric non-metric connection.

2. PRELIMINARIES

Let $\bar{M}$ be an $(2m+1)$-dimensional almost contact metric manifold [4] with a metric tensor g, a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η which satisfies

$$\phi^2 = -I + \eta \otimes \xi, \phi\xi = 0, \eta\phi = 0, \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y on $\bar{M}$. If in addition to the condition for an almost contact metric structure we have $d\eta(X,Y) = g(X, \phi Y)$, then the structure is said to be a *contact metric structure.*

The almost contact metric manifold $\bar{M}$ is called a *nearly Kenmotsu manifold* if it satisfies the condition [9]

$$(\bar{\bar{\nabla}}_X\phi)(Y) + (\bar{\bar{\nabla}}_Y\phi)(X) = -\eta(Y)\phi X - \eta(X)\phi Y,$$

where $\bar{\bar{\nabla}}$ denotes the Riemannian connection with respect to g. If, moreover, M satisfies

$$(\bar{\bar{\nabla}}_X\phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.1}$$

then it is called *Kenmotsu manifold.*

Definition. An n-dimensional Riemannian submanifold M of a nearly Kenmotsu manifold $\bar{M}$ is called a *semi-invariant submanifold* if ξ is tangent to M and there exists on M a pair of orthogonal distribution $(D, D^{\perp})$ such that [3]

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$,

(ii) distribution D is invariant under ϕ, that is, $\phi D_x \subset D_x$ for all $x \in M$,

(iii) distribution $D^{\perp}$ is anti-invariant under ϕ, that is, $\phi D_x^{\perp} \subset T_x^{\perp} M$ for all $x \in M$, where $T_x M$ and $T_x^{\perp} M$ are the tangent space of M at x.

The distribution D(resp. $D^{\perp}$) is called the *horizontal*(resp. *vertical*) distribution. A semi-invariant submanifold M is said to be an *invariant*(resp. *anti-invariant*) submanifold if we have $D_x^{\perp} = \{0\}$ (resp. $D_X = \{0\}$) for each $X \in M$. We also call M is *proper* if neither D nor $D^{\perp}$ is null. It is easy to check that each hypersurface of M which is tangent to ξ inherits a structure of semi-invariant submanifold of $\bar{M}$.

Now, we define a *quarter symmetric non-metric connection* $\bar{\nabla}$ in a Kenmotsu manifold by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)\phi X \tag{2.2}$$

such that $(\bar{\nabla}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y)$ for any $X, Y \in TM$, where $\bar{\bar{\nabla}}$ is the induced connection on M.

From (2.1) and (2.2), we have

$$(\bar{\nabla}_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X + \eta(Y)X - \eta(X)\eta(Y)\xi, \tag{2.3}$$

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = &-\eta(X)\phi Y - \eta(Y)\phi X \\ &+\eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi.\end{aligned} \tag{2.4}$$

We denote by g the metric tensor of $\bar{M}$ as well as that induced on M. Let $\bar{\nabla}$ be the quarter symmetric non-metric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to the unit normal N.

Theorem 2.1. *The connection induced on the semi-invariant submanifolds of a nearly Kenmotsu manifold with quarter symmetric non-metric connection is also a quarter symmetric non-metric connection.*

Proof. Let ∇ be the induced connection with respect to the unit normal N on semi-invariant submanifolds of a nearly Kenmotsu manifold with a quarter symmetric non-metric connection $\bar{\nabla}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + m(X, Y),$$

where m is a tensor field of type (0, 2) on semi-invariant submanifold M. If $\nabla^\star$ be the induced connection on semi-invariant submanifolds from Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \nabla^\star{}_X Y + h(X, Y),$$

where h is a second fundamental tensor.

Now using (2.2), we have

$$\nabla_X Y + m(X, Y) = \nabla^\star{}_X Y + h(X, Y) + \eta(Y)\phi X.$$

Equating the tangential and normal components from the both sides in the above equation, we get

$$h(X, Y) = m(X, Y)$$

and

$$\nabla_X Y = \nabla^\star{}_X Y + \eta(Y)\phi X.$$

Thus ∇ is also a quarter symmetric non-metric connection. □

Now, the Gauss formula for a semi-invariant submanifolds of a nearly Kenmotsu manifold with a quarter symmetric non-metric connection is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

and the Weingarten formula for M is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N + \eta(N)\phi X \tag{2.6}$$

for $X, Y \in TM$, $N \in T^\perp M$, where h(resp. A_N) is the second fundamental form (resp. tensor) of M in $\bar{M}$ and $\nabla^\perp$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X, Y), N) = g(A_N X, Y) = g(A_N Y, X). \tag{2.7}$$

Any vector X tangent to M is given as

$$\begin{aligned} X &= PX + QX + \eta(X)\xi, \\ \phi X &= \phi PX + \phi QX, \end{aligned} \tag{2.8}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively.

For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{2.9}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

The Nijenhuis tensor $N(X,Y)$ for a quarter symmetric non-metric connection is defined as

$$N(X,Y) = (\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X - \phi(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_Y\phi)X \tag{2.10}$$

for any $X, Y \in T\bar{M}$.

From (2.4), we have

$$(\bar{\nabla}_{\phi X}\phi)(Y) = \eta(Y)X - \eta(X)\eta(Y)\xi - (\bar{\nabla}_Y\phi)\phi X + \eta(Y)\phi X. \tag{2.11}$$

Also, we have

$$(\bar{\nabla}_Y\phi)(\phi X) = ((\bar{\nabla}_Y\eta)X)\xi + \eta(X)\bar{\nabla}_Y\xi - \phi(\bar{\nabla}_Y\phi)X. \tag{2.12}$$

Now, using (2.12) in (2.11), we have

$$\begin{aligned} (\bar{\nabla}_{\phi X}\phi)Y &= \eta(Y)X - \eta(X)\eta(Y)\xi - ((\bar{\nabla}_Y\eta)X)\xi \\ &\quad -\eta(X)\bar{\nabla}_Y\xi + \phi(\bar{\nabla}_Y\phi)X + \eta(Y)\phi X. \end{aligned} \tag{2.13}$$

By virtue of (2.13) and (2.10), we get

$$\begin{aligned} N(X,Y) &= -2\eta(Y)X - 2\eta(X)Y + 8\eta(X)\eta(Y)\xi + \eta(Y)\bar{\nabla}_X\xi - \eta(X)\bar{\nabla}_Y\xi \\ &\quad +2g(\phi X, Y)\xi + 4\phi(\bar{\nabla}_Y\phi X) - \eta(Y)\phi X - \eta(X)\phi Y \end{aligned} \tag{2.14}$$

for any $X, Y \in T\bar{M}$.

3. Basic Lemmas

Lemma 3.1. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then we have*

$$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y].$$

Proof. By the Gauss formula we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X). \tag{3.1}$$

Also by use of (2.5), the covariant differentiation yields

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X, Y]. \tag{3.2}$$

From (3.1) and (3.2) we get

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y]. \tag{3.3}$$

Using $\eta(X) = 0$ for each $X \in D$ in (2.4), we get

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 0. \tag{3.4}$$

Adding (3.3) and (3.4) we get the result. □

Similar computations also yields the following.

Lemma 3.2. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then we have*

$$2(\bar{\nabla}_X \phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]$$

for any $X \in D$ and $Y \in D^{\perp}$.

Lemma 3.3. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then we have*

$$\begin{aligned} &Q\nabla_X(\phi PY) + Q\nabla_Y(\phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y \\ &\quad = -\eta(Y)\phi QX - \eta(X)\phi QY + \eta(Y)QX + \eta(X)QY + 2Bh(X, Y), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &h(X, \phi PY) + h(Y, \phi PX) + \nabla^{\perp}_X \phi QY + \nabla^{\perp}_Y \phi QX \\ &\quad = 2Ch(X, Y) + \phi Q\nabla_X Y + \phi Q\nabla_Y X, \end{aligned} \tag{3.6}$$

$$\eta(\nabla_X \phi PY + \nabla_Y \phi PX - A_{\phi QY}X - A_{\phi QX}Y) = 0 \tag{3.7}$$

for all $X, Y \in TM$.

Proof. Differentiating (2.8) covariantly and using (2.5) and (2.6), we have

$$\begin{aligned} &(\bar{\nabla}_X \phi)Y + \phi(\nabla_X Y) + \phi h(X, Y) \\ &\quad = \nabla_X(\phi PY) + h(X, \phi PY) - A_{\phi QY}X + \nabla^{\perp}_X \phi QY. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned} &(\bar{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(Y, X) \\ &\quad = \nabla_Y(\phi PX) + h(Y, \phi PX) - A_{\phi QX}Y + \nabla^{\perp}_Y \phi QX. \end{aligned} \tag{3.9}$$

Adding (3.8) and (3.9) and using (2.4) and (2.9) we have

$$\begin{aligned}(3.10)\quad & -\eta(Y)\phi PX-\eta(Y)\phi QX-\eta(X)\phi PY-\eta(X)\phi QY+\eta(Y)PX+\eta(Y)QX \\ & +\eta(X)PY+\eta(X)QY+\phi P\nabla_X Y+\phi Q\nabla_X Y+\phi P\nabla_Y X+\phi Q\nabla_Y X \\ & +2Bh(Y,X)+2Ch(Y,X)=P\nabla_X(\phi PY)+P\nabla_Y(\phi PX) \\ & +Q\nabla_Y(\phi PX)-PA_{\phi QY}X+Q\nabla_X(\phi PY)+\nabla_X^{\perp}\phi QY-PA_{\phi QX}Y \\ & -QA_{\phi QY}X-QA_{\phi QX}Y+\nabla_Y^{\perp}\phi QX+h(Y,\phi PX)+h(X,\phi PY) \\ & +\eta(\nabla_X\phi PY)\xi+\eta(\nabla_Y\phi PX)\xi-\eta(A_{\phi QX}Y)\xi-\eta(A_{\phi QY}X)\xi.\end{aligned}$$

Equations (3.1)–(3.4) follows the results by the comparison of the tangential, normal and vertical components of (3.10). □

Definition. The horizontal distribution D is said to be *parallel* with respect to the connection ∇ on M if $\nabla_X Y\in D$ for all vector fields $X, Y\in D$.

Proposition 3.4. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. If the horizontal distribution D is parallel, then $h(X,\phi Y)=h(Y,\phi X)$ for all $X,Y\in D$.*

Proof. Since D is parallel, $\nabla_X\phi Y\in D$ and $\nabla_Y\phi X\in D$ for each $X, Y\in D$. Now from (3.5) and (3.6), we get

$$(3.11)\qquad h(X,\phi Y)+h(Y,\phi X)=2\phi h(X,Y).$$

Replacing X by ϕX in the above equation, we have

$$(3.12)\qquad h(\phi X,\phi Y)-h(Y,X)=2\phi h(\phi X,Y).$$

Replacing Y by ϕY in (3.11), we have

$$(3.13)\qquad -h(X,Y)+h(\phi X,\phi Y)=2\phi h(X,\phi Y).$$

Comparing (3.12) and (3.13), we have

$$h(X,\phi Y)=h(\phi X,Y)$$

for all $X, Y\in D$. □

Definition. A semi-invariant submanifold is said to be *mixed totally geodesic* if $h(X,Z)=0$ for all $X\in D$ and $Z\in D^{\perp}$.

Lemma 3.5. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then M is mixed totally geodesic if and only if $A_N X\in D$ for all $X\in D$.*

Proof. If $A_N X \in D$, then $g(h(X,Y),N) = g(A_N X, Y) = 0$, which gives $h(X,Y) = 0$ for $Y \in D^{\perp}$. Hence M is mixed totally geodesic. □

4. Integrability Conditions for Distributions

Theorem 4.1. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then the distribution $D \oplus \xi$ is integrable if the following conditions are satisfied:*

$$S(X,Y) \in D \oplus \xi,$$

$$h(X,\phi Y) = h(\phi X, Y)$$

for $X, Y \in D \oplus \xi$.

Proof. The torsion tensor $S(X,Y)$ of the almost contact structure is given by

$$S(X,Y) = N(X,Y) + 2d\eta(X,Y)\xi,$$

where $N(X,Y)$ is the Nijenhuis tensor. Thus we have

$$S(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X,Y)\xi. \tag{4.1}$$

Suppose that $D \oplus \xi$ is integrable, so $N[X,Y] = 0$ for $X, Y \in D \oplus \xi$. Then it reduces to $S(X,Y) = 2d\eta(X,Y)\xi \in D \oplus \xi$.

Using the Gauss formula in (2.14), we get

$$N(X,Y) = 2g(\phi X, Y)\xi + 4\nabla_Y X - 4\phi(\nabla_Y \phi X) + 4\phi h(Y, \phi X) + 4h(Y,X) \tag{4.2}$$

for all $X, Y \in D$. From (4.1) and (4.2), we get

$$-\phi Q(\nabla_Y \phi X) + Ch(Y, \phi X) + h(Y,X) = 0$$

for all $X, Y \in D$. Replacing Y by ϕZ, we have

$$-\phi Q(\nabla_{\phi Z} \phi X) + Ch(\phi Z, \phi X) + h(\phi Z, X) = 0, \tag{4.3}$$

where $Z \in D$. Interchanging X and Z, we have

$$-\phi Q(\nabla_{\phi X} \phi Z) + Ch(\phi X, \phi Z) + h(\phi X, Z) = 0. \tag{4.4}$$

Subtracting (4.4) from (4.3), we have

$$-\phi Q[\phi X, \phi Z] + h(Z, \phi X) - h(\phi X, Z) = 0,$$

from which the assertion follows. □

Lemma 4.2. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then we have*

$$2(\bar{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y - \phi[Y,Z].$$

Proof. From the Weingarten formula, we have

$$\bar{\nabla}_Y\phi Z - \bar{\nabla}_Z\phi Y = -A_{\phi Z}Y + A_{\phi Y}Z + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y. \tag{4.5}$$

Also by the covariant differentiation, we get

$$\bar{\nabla}_Y\phi Z - \bar{\nabla}_Z\phi Y = (\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y + \phi[Y,Z]. \tag{4.6}$$

From (4.5) and (4.6) we have

$$(\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y - \phi[Y,Z]. \tag{4.7}$$

From (2.4) we obtain

$$(\bar{\nabla}_Y\phi)Z + (\bar{\nabla}_Z\phi)Y = 0 \tag{4.8}$$

for any $X, Y \in D$. Adding (4.7) and (4.8), we get the result. □

Proposition 4.3. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then we have*

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y,Z] + \frac{2}{3}g(\phi Y, Z)\xi.$$

Proof. Let $Y, Z \in D^{\perp}$ and $X \in x(M)$. Then from (2.5) and (2.7), we have

$$2g(A_{\phi Z}Y, X) = -g(\bar{\nabla}_Y\phi X, Z) - g(\bar{\nabla}_X\phi Y, Z) - \eta(X)g(\phi Y, Z) + \eta(X)g(Y,Z).$$

By use of (2.4) and $\eta(Y) = 0$ for $Y \in D^{\perp}$, we have

$$2g(A_{\phi Z}Y, X) = -g(\phi\bar{\nabla}_Y Z, X) + g(A_{\phi Y}Z, X) - \eta(X)g(\phi Y, Z) + \eta(X)g(Y,Z).$$

Transvecting X from the both sides, we get

$$2A_{\phi Z}Y = -\phi\bar{\nabla}_Y Z + A_{\phi Y}Z - g(\phi Y, Z)\xi + g(Y,Z)\xi.$$

Interchanging Y and Z, we have

$$2A_{\phi Y}Z = -\phi\bar{\nabla}_Z Y + A_{\phi Z}Y - g(\phi Z, Y)\xi + g(Z,Y)\xi.$$

Subtracting the above two equations, we get

$$(A_{\phi Y}Z - A_{\phi Z}Y) = \frac{1}{3}\phi P[Y,Z] + \frac{2}{3}g(\phi Y, Z)\xi, \tag{4.9}$$

where $[Y, Z]$ is the Lie bracket for $\bar{\nabla}$. □

Theorem 4.4. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with a quarter symmetric non-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{2}{3}g(\phi Y, Z)\xi$$

for all $Y, Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable. Then $[Y, Z] \in D^{\perp}$ for any $Y, Z \in D^{\perp}$. Therefore, $P[Y, Z] = 0$ and from (4.9), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{2}{3}g(\phi Y, Z)\xi. \tag{4.10}$$

Conversely, let (4.10) holds. Then by virtue of (4.9) we have $\phi P[Y, Z] = 0$ for all $Y, Z \in D^{\perp}$. Since rank $\phi = 2n$, therefore we have either $P[Y, Z] = 0$ or $P[Y, Z] = k\xi$. But $P[Y, Z] = k\xi$ is not possible as P being a projection operator on D. Hence $P[Y, Z] = 0$, which is equivalent to $[Y, Z] \in D^{\perp}$ for all $Z \in D^{\perp}$ and thus $D^{\perp}$ is integrable. □

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