Bull. Korean Math. Soc. **48** (2011), No. 2, pp. 427–444
DOI 10.4134/BKMS.2011.48.2.427

HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH LIE PARALLEL NORMAL JACOBI OPERATOR

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Abstract. In this paper we give some non-existence theorems for Hopf hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator $\bar{R}_N$ and totally geodesic $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb flow.

0. Introduction

The Jacobi fields along geodesics of a given Riemannian manifold $(\bar{M}, \bar{g})$ play an important role in the study of differential geometry. It satisfies a very well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operators. That is, if $\bar{R}$ is the curvature operator of $\bar{M}$ and X is any vector field tangent to $\bar{M}$, the Jacobi operator with respect to X at $x \in \bar{M}$, $\bar{R}_X \in \mathrm{End}(T_x\bar{M})$, is defined as $\bar{R}_X(Y)(x) = (\bar{R}(Y,X)X)(x)$ for all $Y \in T_x\bar{M}$, being a self-adjoint endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$. Clearly, each vector field X tangent to $\bar{M}$ provides a Jacobi operator with respect to X (See [7] and [9]).

If the structure vector field $\xi = -JN$ of a real hypersurface M in complex projective space $P_n(\mathbb{C})$ is invariant under the shape operator, ξ is said to be *Hopf*, where J denotes a Kähler structure of $P_n(\mathbb{C})$, and N is a unit normal vector field of M in $P_n(\mathbb{C})$.

In the quaternionic projective space $\mathbb{H}P^m$ Pérez and Suh [10] classified the real hypersurfaces in $\mathbb{H}P^m$ with $\mathfrak{D}^\perp$-parallel curvature tensor $\nabla_{\xi_\nu} R = 0$ for $\nu = 1, 2, 3$, where R denotes the curvature tensor of M in $\mathbb{H}P^m$ and $\mathfrak{D}^\perp$ is a distribution defined by $\mathfrak{D}^\perp = \mathrm{Span}\,\{\xi_1, \xi_2, \xi_3\}$. In this case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic quaternionic submanifold $\mathbb{H}P^k$ in $\mathbb{H}P^m$, $2 \le k \le m-2$.

Received September 22, 2009; Revised July 2, 2010.
2010 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C15.
Key words and phrases. complex two-plane Grassmannians, Hopf hypersurfaces, Reeb vector field, normal Jacobi operator, Lie derivative.
This work was supported by grant Proj. No. BSRP-2010-0020931 from National Research Foundation.

The vector fields $\{\xi_1, \xi_2, \xi_3\}$ mentioned above, which are said to be *almost contact structure*, are defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a local basis of a quaternionic Kähler structure of $\mathbb{H}P^m$ and N is a unit normal vector field of M in $\mathbb{H}P^m$.

In quaternionic space forms, Berndt [1] introduced the notion of *normal Jacobi operator*

$$\bar{R}_N X = \bar{R}(X, N)N \in \mathrm{End}\,(T_x M)\,, \quad x \in M$$

for real hypersurfaces M in a quaternionic projective space $\mathbb{H}P^m$ or in a quaternionic hyperbolic space $\mathbb{H}H^m$, where $\bar{R}$ denotes the curvature tensor of $\mathbb{H}P^m$ and $\mathbb{H}H^m$ respectively. Berndt [1] also showed that "*the curvature adaptedness*", when the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator A, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M, where $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The situation for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+1})$ with parallel normal Jacobi operator $\bar{R}_N$ is not so simple and will be quite different from the cases in $\mathbb{H}P^m$.

In this paper the present authors consider a real hypersurface M in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, that is, $\mathcal{L}_X \bar{R}_N = 0$ for any $X \in T_x M$, $x \in M$, where $\bar{R}$ and N respectively denote the curvature tensor of the ambient space $G_2(\mathbb{C}^{m+2})$ and a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The curvature tensor $\bar{R}(X, Y)Z$ for any vector fields X, Y and Z on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in Section 1. Then the normal Jacobi operator $\bar{R}_N$ for the unit normal vector field N can be defined from the curvature tensor $\bar{R}(X, N)N$ by putting $Y = Z = N$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (See Berndt [2]). From these two structures J and $\mathfrak{J}$, we have geometric conditions naturally induced on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ such that $[\xi] = \mathrm{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\ \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By these two conditions, Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The structure vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Reeb* vector field. Moreover, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in section 2 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. The flow generated by the integral curves of the Reeb vector field is said to be a *geodesic* Reeb flow if M becomes a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$.

We say that the Reeb vector field is *Killing* if the Lie derivative of the Riemannian metric g for M in $G_2(\mathbb{C}^{m+2})$ along the Reeb direction vanishes, that is, $\mathcal{L}_\xi g = 0$. This means that the Reeb flow is isometric. Using such a notion, Berndt and Suh [4] proved that a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with isometric Reeb flow becomes an open part of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In [15], Suh also gave a characterization for this kind of hypersurfaces in terms of another geometric *Lie invariant*. Namely, he characterized them as the hypersurfaces in $G_2(\mathbb{C}^{m+2})$ such that the shape operator A is invariant under the Reeb flow.

Now by putting a unit normal vector field N into the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, the normal Jacobi operator $\bar{R}_N$ can be defined in such a way that

$$\begin{aligned}\bar{R}_N X &= \bar{R}(X,N)N \\ &= X + 3\eta(X)\xi + 3\sum\nolimits_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum\nolimits_{\nu=1}^{3}\big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\big\}\end{aligned}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

In the paper [8] due to Jeong, Pérez and Suh, we classified real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting normal Jacobi operator, that is, $\bar{R}_N \circ \phi = \phi \circ \bar{R}_N$ or $\bar{R}_N \circ A = A \circ \bar{R}_N$. The fact that the normal Jacobi operator $\bar{R}_N$ *commutes* with the shape operator A (or the structure tensor ϕ) of M in $G_2(\mathbb{C}^{m+2})$ means that the eigenspaces of the normal Jacobi operator are invariant under the shape operator A (or the structure tensor ϕ). Also, in [5], Jeong, Kim and Suh introduced the notion of *parallel* normal Jacobi operator for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. Such an operator is said to be parallel if $\nabla_X \bar{R}_N = 0$ for any tangent vector field X on M. This means that the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are parallel along any curve γ in M. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be *parallel* along γ if they are invariant with respect to any parallel displacement along γ. Using this notion, they gave a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator.

Related to such a parallel normal Jacobi operator, in this paper the authors give a theorem for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with *Lie parallel* normal Jacobi operator, that is, $\mathcal{L}_X \bar{R}_N = 0$ for any $X \in T_xM$, $x \in M$. This means that all the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are invariant under any parallel displacement ϕ_t^* generated from the flow ϕ_t such that $\phi_t(x) = \gamma(t)$ and $\gamma(0) = x$ for the integral curve γ of X in T_xM, $x \in M$.

Then the authors prove the following for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operators:

Theorem 1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator. If the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field ξ are totally geodesic, then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

On the other hand, in the paper [6] of Jeong and Suh, we gave non-existence theorems for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with *Lie ξ-parallel* normal Jacobi operator, that is, $\mathcal{L}_\xi \bar{R}_N = 0$ as follows:

Theorem B. *There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$ if the Reeb vector field $\xi \in \mathfrak{D}^\perp$.*

Theorem C. *There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$ if the Reeb vector field $\xi \in \mathfrak{D}$.*

Then as an application of Theorem 1 to Theorems B and C the authors can assert the following:

Theorem 2. *There does not exist any Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator if the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are totally geodesic.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details refer to [2], [3], and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. The space $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, negative B restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximum sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. From now on, in this paper we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition, that is, a Cartan decomposition

$$\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}\,,$$

where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν, $\nu = 1, 2, 3$, is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar\nabla$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar R$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive some fundamental formulas which will be used in the proof of our theorems and the equation of Codazzi for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (See [3], [4], [12], [13], and [14]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). More explicitly, we can define a tensor field ϕ of type $(1,1)$, a vector field ξ and its dual 1-form η on M by $g(\phi X, Y) = g(JX, Y)$

and $\eta(X)=g(X,\xi)$ for any tangent vector fields X and Y on M. Then they satisfy the following

$$\phi^2X=-X+\eta(X)\xi\,,\quad \phi\xi=0\,,\quad \eta(\phi X)=0\quad\text{and}\quad \eta(\xi)=1$$

for any tangent vector field X.

Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M in such a way that a tensor filed ϕ_ν of type $(1,1)$, a vector field ξ_ν and its dual 1-form η_ν on M defined by $g(\phi_\nu X,Y)=g(J_\nu X,Y)$ and $\eta_\nu(X)=g(\xi_\nu,X)$ for any tangent vector fields X and Y on M. Then they also satisfy the following

$$\phi_\nu^2X=-X+\eta_\nu(X)\xi\,,\quad \phi_\nu\xi_\nu=0\,,\quad \eta_\nu(\phi_\nu X)=0\quad\text{and}\quad \eta_\nu(\xi_\nu)=1$$

for any vector field X tangent to M and $\nu=1,2,3$.

Using the above expression (1.2) for the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes

$$\begin{aligned}(\nabla_XA)Y-(\nabla_YA)X&=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\\&+\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\big\}\\&+\sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\big\}\\&+\sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu\,.\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\&\phi\xi_\nu=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\&\phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\&\phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.\end{aligned}\tag{2.1}$$

Now let us note that

$$JX=\phi X+\eta(X)N\,,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N\tag{2.2}$$

for any vector field X tangent to M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX,\tag{2.3}$$

$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX,\tag{2.4}$$

$$(2.5)\qquad \begin{aligned}(\nabla_X\phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX\\ &-g(AX,Y)\xi_\nu.\end{aligned}$$

Summing up these formulas, we find the following

$$(2.6)\qquad \begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu)\\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu)\\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX\\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$(2.7)\qquad \phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu.$$

3. Lie parallel normal Jacobi operator

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator, that is, $\mathcal{L}_X\bar{R}_N = 0$ for any vector field X tangent to M. Then first of all, we write the normal Jacobi operator $\bar{R}_N$, which is given by

(3.1)

$$\begin{aligned}\bar{R}_N(X) = \bar{R}(X,N)N &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu\\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu\xi + \eta_\nu(\xi)N)\Big\}\\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu\\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\Big\}\end{aligned}$$

where we have used the following

$$\begin{aligned}g(J_\nu JN,N) &= -g(JN,J_\nu N) = -g(\xi,\xi_\nu) = -\eta_\nu(\xi),\\ g(J_\nu JX,N) &= g(X,JJ_\nu N) = -g(X,J\xi_\nu)\\ &= -g(X,\phi\xi_\nu + \eta(\xi_\nu)N) = -g(X,\phi\xi_\nu),\end{aligned}$$

and

$$J_\nu JN = -J_\nu\xi = -\phi_\nu\xi - \eta_\nu(\xi)N.$$

Of course, by (2.7) we know that the normal Jacobi operator $\bar{R}_N$ is a symmetric endomorphism of T_xM, $x\in M$.

Now let us consider the Lie derivative of the normal Jacobi operator along any direction. Then for any vector fields X and Y tangent to M it is given by

$$(3.2)\qquad \begin{aligned}(\mathcal{L}_X\bar{R}_N)Y &= \mathcal{L}_X(\bar{R}_NY) - \bar{R}_N(\mathcal{L}_XY)\\ &= [X,\bar{R}_NY] - \bar{R}_N[X,Y]\\ &= (\nabla_X\bar{R}_N)Y - \nabla_{\bar{R}_NY}X + \bar{R}_N(\nabla_YX)\end{aligned}$$

where the terms in the right side can be given respectively as follows:

$$(\nabla_X\bar{R}_N)Y = 3(\nabla_X\eta)(Y)\xi + 3\eta(Y)\nabla_X\xi + 3\sum_{\nu=1}^{3}(\nabla_X\eta_\nu)(Y)\xi_\nu$$

$$
\begin{aligned}
&+3\sum_{\nu=1}^{3}\eta_\nu(Y)\nabla_X\xi_\nu-\sum_{\nu=1}^{3}\Big[X(\eta_\nu(\xi))(\phi_\nu\phi Y-\eta(Y)\xi_\nu)\\
&+\eta_\nu(\xi)\{(\nabla_X\phi_\nu\phi)Y-(\nabla_X\eta)(Y)\xi_\nu-\eta(Y)\nabla_X\xi_\nu\}\\
&-(\nabla_X\eta_\nu)(\phi Y)\phi_\nu\xi-\eta_\nu((\nabla_X\phi)Y)\phi_\nu\xi-\eta_\nu(\phi Y)\nabla_X(\phi_\nu\xi)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\bar{R}_N Y}X=\nabla_Y X+3\eta(Y)\nabla_\xi X+3\sum_{\nu=1}^{3}\eta_\nu(Y)\nabla_{\xi_\nu}X-\sum_{\nu=1}^{3}\eta_\nu(\xi)\nabla_{\phi_\nu\phi Y}X\\
+\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta(Y)\nabla_{\xi_\nu}X+\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\nabla_{\phi_\nu\xi}X
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{R}_N(\nabla_Y X)=\nabla_Y X+3\eta(\nabla_Y X)\xi+3\sum_{\nu=1}^{3}\eta_\nu(\nabla_Y X)\xi_\nu\\
-\sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi\nabla_Y X-\eta(\nabla_Y X)\xi_\nu)-\eta_\nu(\phi\nabla_Y X)\phi_\nu\xi\}.
\end{aligned}
$$

Then by the formulas given in section 2, (3.2) gives the following for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator $\bar{R}_N$:

$$
\begin{aligned}
(\mathcal{L}_X\bar{R}_N)Y=\ &3g(\phi AX,Y)\xi+3\eta(Y)\phi AX+3\sum_{\nu=1}^{3}g(\phi_\nu AX,Y)\xi_\nu\\
&+3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu AX\\
&-\sum_{\nu=1}^{3}\Bigg[X(\eta_\nu(\xi))(\phi_\nu\phi Y-\eta(Y)\xi_\nu)\\
&+\eta_\nu(\xi)\Big\{-q_{\nu+1}(X)\phi_{\nu+2}\phi Y+q_{\nu+2}(X)\phi_{\nu+1}\phi Y\\
&+\eta_\nu(\phi Y)AX-g(AX,\phi Y)\xi_\nu\\
&+\eta(Y)\phi_\nu AX-g(AX,Y)\phi_\nu\xi-g(\phi AX,Y)\xi_\nu\\
&-\eta(Y)(q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX)\Big\}\\
&-g(\phi_\nu AX,\phi Y)\phi_\nu\xi-\eta(Y)\eta_\nu(AX)\phi_\nu\xi+g(AX,Y)\eta_\nu(\xi)\phi_\nu\xi\\
&-\eta_\nu(\phi Y)\Big\{\eta_\nu(\xi)AX-g(AX,\xi)\xi_\nu+\phi_\nu\phi AX\Big\}\Bigg]\\
&-3\eta(Y)\nabla_\xi X-3\sum_{\nu=1}^{3}\eta_\nu(Y)\nabla_{\xi_\nu}X\\
&+\sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\nabla_{\phi_\nu\phi Y}X-\eta(Y)\nabla_{\xi_\nu}X)-\eta_\nu(\phi Y)\nabla_{\phi_\nu\xi}X\}\\
&+3\eta(\nabla_Y X)\xi+3\sum_{\nu=1}^{3}\eta_\nu(\nabla_Y X)\xi_\nu\\
&-\sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi\nabla_Y X-\eta(\nabla_Y X)\xi_\nu)-\eta_\nu(\phi\nabla_Y X)\phi_\nu\xi\Big\}\\
=\ &0,
\end{aligned}
\tag{3.3}
$$

where in the first equality we have used the following formulas

$$3\sum_{\nu=1}^{3} g(q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2}, Y)\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2}\} = 0$$

and

$$\sum_{\nu=1}^{3}\{\eta_{\nu+1}(\phi Y)q_{\nu+2}(X)\phi_\nu\xi - \eta_{\nu+2}(\phi Y)q_{\nu+1}(X)\phi_\nu\xi - \eta_\nu(\phi Y)q_{\nu+1}(X)\phi_{\nu+2}\xi + \eta_\nu(\phi Y)q_{\nu+2}(X)\phi_{\nu+1}\xi\} = 0.$$

In particular by putting $X = \xi$ in (3.3) we have the following

$$\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)Y = {} & 3g(\phi A\xi, Y)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, Y)\xi_\nu \\
& + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu A\xi \\
& - \sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))(\phi_\nu\phi Y - \eta(Y)\xi_\nu) \\
& + \eta_\nu(\xi)\Big\{ - q_{\nu+1}(\xi)\phi_{\nu+2}\phi Y + q_{\nu+2}(\xi)\phi_{\nu+1}\phi Y \\
& + \eta_\nu(\phi Y)A\xi - g(A\xi, \phi Y)\xi_\nu \\
& + \eta(Y)\phi_\nu A\xi - g(A\xi, Y)\phi_\nu\xi - g(\phi A\xi, Y)\xi_\nu \\
& - \eta(Y)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_\nu A\xi)\Big\} \\
& - g(\phi_\nu A\xi, \phi Y)\phi_\nu\xi - \eta(Y)\eta_\nu(A\xi)\phi_\nu\xi + g(A\xi, Y)\eta_\nu(\xi)\phi_\nu\xi \\
& - \eta_\nu(\phi Y)\Big\{\eta_\nu(\xi)A\xi - g(A\xi, \xi)\xi_\nu + \phi_\nu\phi A\xi\Big\}\Big] \\
& - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi A\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu \\
& + \sum_{\nu=1}^{3}\Big[\eta_\nu(\xi)\Big\{\phi A\phi_\nu\phi Y - \eta(Y)\phi A\xi_\nu\Big\} - \eta_\nu(\phi Y)\phi A\phi_\nu\xi\Big] \\
& + \sum_{\nu=1}^{3}\Big[\eta_\nu(\xi)\{\phi_\nu AY - \eta(AY)\phi_\nu\xi\} - \eta_\nu(AY)\phi_\nu\xi \\
& + \eta(AY)\eta_\nu(\xi)\phi_\nu\xi\Big] \\
= {} & 0,
\end{aligned} \tag{3.4}$$

where in the first equality we have used the second formula of (2.3). From this, by putting $Y = \xi$ in (3.4) we have the following

$$(\mathcal{L}_\xi \bar{R}_N)\xi = 6\sum_{\nu=1}^{3} g(\phi_\nu A\xi, \xi)\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu A\xi$$

$$\begin{aligned}(3.5)\qquad &+\sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))\xi_\nu+\eta_\nu(\xi)\big\{q_{\nu+2}(\xi)\xi_{\nu+1}-q_{\nu+1}(\xi)\xi_{\nu+2}\big\}\Big]\\ &-4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi A\xi_\nu\\ =&\ 0.\end{aligned}$$

4. Lie parallel normal Jacobi operator

In this section we want to prove the following:

Proposition 4.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator. If the integral curves of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field ξ are totally geodesic, then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. When the function $\alpha = g(A\xi,\xi)$ identically vanishes, the proposition was proved directly by Pérez and Suh [11]. Thus we consider only the case that the function α is non-vanishing in this proof.
By putting $A\xi = \alpha\xi$ into (3.5) we have

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)(\alpha\phi_\nu\xi-\phi A\xi_\nu)=0, \tag{4.1}$$

where we have used the following formula

$$\sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))\xi_\nu+\eta_\nu(\xi)\{q_{\nu+2}(\xi)\xi_{\nu+1}-q_{\nu+1}(\xi)\xi_{\nu+2}\}\Big]=0.$$

Now let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$. Then naturally we know that $\eta(\xi_2) = \eta(\xi_3) = 0$. Hereafter, unless otherwise stated, let us assume $\eta(X_0)\eta(\xi_1) \neq 0$.
Then (4.1) reduces to

$$\alpha\phi_1\xi-\phi A\xi_1=0.$$

From this, by taking the structure tensor ϕ and also using that ξ is principal, we have

$$A\xi_1=\alpha\xi_1 \quad\text{and}\quad AX_0=\alpha X_0. \tag{4.2}$$

Then putting $X = X_0$ and $Y = \xi$ into (3.3) and using (4.2) gives

$$\begin{aligned}0&=(\mathcal{L}_{X_0}\bar{R}_N)\xi\\ &=3\alpha\phi X_0+3\alpha\sum_{\nu=1}^{3}g(\phi_\nu X_0,\xi)\xi_\nu+3\alpha\eta_1(\xi)\phi_1X_0\\ &\quad+\eta_1(\xi)\{q_3(X_0)\xi_2-q_2(X_0)\xi_3\}-3\nabla_\xi X_0-4\eta_1(\xi)\nabla_{\xi_1}X_0\\ &\quad+3\eta(\nabla_\xi X_0)\xi+3\sum_{\nu=1}^{3}\eta_\nu(\nabla_\xi X_0)\xi_\nu-\eta_1(\xi)\phi_1\phi\nabla_\xi X_0\\ &\quad+\eta_1(\xi)\eta(\nabla_\xi X_0)\xi_1+\sum_{\nu=1}^{3}\eta_\nu(\phi\nabla_\xi X_0)\phi_\nu\xi,\end{aligned}$$

where we have used

$$X_0(\eta_1(\xi))\xi_1=g(\nabla_{X_0}\xi_1,\xi)\xi_1+g(\xi_1,\nabla_{X_0}\xi)\xi_1$$

$$\begin{aligned}
&= g(\phi_1 AX_0, \xi)\xi_1 + g(\xi_1, \phi AX_0)\xi_1 \\
&= -\alpha g(X_0, \phi_1\xi)\xi_1 - \alpha g(\phi_1\xi, X_0)\xi_1 \\
&= -2\alpha g(X_0, \phi_1(\eta(X_0)X_0 + \eta(\xi_1)\xi_1)) \\
&= -2\alpha\eta(X_0)g(X_0, \phi_1 X_0) \\
&= 0\,.
\end{aligned}$$

From this, together with (2.3) and (2.4), and using $\phi X_0 \in \mathfrak{D}$, $\nabla_\xi X_0 \in \mathfrak{D}$ and $\eta(\nabla_\xi X_0) = 0$, we have

$$\begin{aligned}
0 &= (\mathcal{L}_{X_0}\bar{R}_N)\xi \\
&= 3\alpha(\phi X_0 + \eta_1(\xi)\phi_1 X_0) + \eta_1(\xi)\{q_3(X_0)\xi_2 - q_2(X_0)\xi_3\} \\
&\quad - 3\nabla_\xi X_0 - 4\eta_1(\xi)\nabla_{\xi_1}X_0 - \eta_1(\xi)\phi_1\phi\nabla_\xi X_0 \\
&\quad + \sum\nolimits_{\nu=1}^{3}\eta_\nu(\phi\nabla_\xi X_0)\phi_\nu\xi,
\end{aligned} \tag{4.3}$$

because we know the following

$$g(\phi X_0, \xi_\nu) = -g(X_0, \phi\xi_\nu) = -g(X_0, \phi_\nu\xi) = 0\,,$$

$$\eta(\nabla_\xi X_0) = g(\nabla_\xi X_0, \xi) = g(\nabla_\xi X_0, \eta(X_0)X_0 + \eta(\xi_1)\xi_1) = 0$$

and

$$\begin{aligned}
g(\nabla_\xi X_0, \xi_\nu) &= -g(X_0, \nabla_\xi \xi_\nu) \\
&= -\alpha g(X_0, \phi_\nu\xi) \\
&= -\alpha g(X_0, \phi\xi_\nu) \\
&= \alpha g(\phi X_0, \xi_\nu) \\
&= 0
\end{aligned}$$

for any $\nu = 1, 2, 3$.

On the other hand, we know that

$$\nabla_{\xi_1}X_0 \in \mathfrak{D}\,, \tag{4.4}$$

because

$$\begin{aligned}
g(\nabla_{\xi_1}X_0, \xi_\nu) &= -g(X_0, \nabla_{\xi_1}\xi_\nu) \\
&= -g(X_0, q_{\nu+2}(\xi_1)\xi_{\nu+1} - q_{\nu+1}(\xi_1)\xi_{\nu+2} + \phi_\nu A\xi_1) \\
&= -\alpha g(X_0, \phi_\nu\xi_1) \\
&= 0\,.
\end{aligned}$$

Moreover, the following formulas hold

$$g(\phi\nabla_\xi X_0, \xi_2) = 0 \quad \text{and} \quad g(\phi\nabla_\xi X_0, \xi_3) = 0\,. \tag{4.5}$$

In fact, differentiating $g(\phi X_0, \xi_2) = 0$ gives

$$\begin{aligned}
0 &= g((\nabla_\xi\phi)X_0, \xi_2) + g(\phi\nabla_\xi X_0, \xi_2) + g(\phi X_0, \nabla_\xi\xi_2) \\
&= g(\phi\nabla_\xi X_0, \xi_2) + \alpha g(\phi X_0, \phi\xi_2)
\end{aligned}$$

$$= g(\phi\nabla_\xi X_0, \xi_2)$$

and similarly the latter term comes from $g(\phi X_0, \xi_3) = 0$.

By taking the inner product (4.3) with ξ_3, and using the facts that ϕX_0, $\phi_1 X_0$, $\nabla_\xi X_0$ and $\nabla_{\xi_1} X_0$ belong to the distribution $\mathfrak{D}$, we have

$$\begin{aligned} 0 &= -\,\eta_1(\xi)q_2(X_0) - \eta_1(\xi)g(\phi_1\phi\nabla_\xi X_0, \xi_3) + \eta_1(\phi\nabla_\xi X_0)g(\phi_1\xi, \xi_3) \\ &= -\,\eta_1(\xi)q_2(X_0). \end{aligned}$$

Similarly, by taking the inner product with ξ_2 to (4.3), we have the following relations

$$q_2(X_0) = 0 \quad \text{and} \quad q_3(X_0) = 0 \tag{4.6}$$

under the assumption of $\eta_1(\xi)\neq 0$. Then (4.4), (4.5) and (4.6) give

$$\begin{aligned} 0 &= (\mathcal{L}_{X_0}\bar{R}_N)\xi \\ &= 3\alpha(\phi X_0 + \eta_1(\xi)\phi_1 X_0) - 3\nabla_\xi X_0 - 4\eta_1(\xi)\nabla_{\xi_1} X_0 \\ &\quad -\eta_1(\xi)\phi_1\phi\nabla_\xi X_0 + \eta_1(\phi\nabla_\xi X_0)\phi_1\xi. \end{aligned} \tag{4.7}$$

On the other hand, by the assumption of M being Hopf and using (4.2), we have

$$\begin{aligned} \nabla_\xi\xi &= \phi A\xi \\ &= \phi A(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\ &= \alpha(\eta(X_0)\phi X_0 + \eta(\xi_1)\eta(X_0)\phi_1 X_0) \\ &= \alpha\eta(X_0)(\phi X_0 + \eta(\xi_1)\phi_1 X_0) \\ &= 0. \end{aligned}$$

Consequently, we see

$$\phi X_0 + \eta(\xi_1)\phi_1 X_0 = 0. \tag{4.8}$$

from the assumption of $\alpha \neq 0$ and $\eta(X_0) \neq 0$.

Substituting (4.8) into (4.7), we have

$$\begin{aligned} 0 &= (\mathcal{L}_{X_0}\bar{R}_N)\xi \\ &= -3\nabla_\xi X_0 - 4\eta_1(\xi)\nabla_{\xi_1} X_0 - \eta_1(\xi)\phi_1\phi\nabla_\xi X_0 + \eta_1(\phi\nabla_\xi X_0)\phi_1\xi. \end{aligned}$$

Now, by applying the operator ϕ_1 to (4.8) we have

$$\phi_1\phi X_0 = \eta(\xi_1)X_0. \tag{4.9}$$

Then by differentiating (4.9) along the direction of the Reeb vector field ξ and using (2.1), (2.3), (2.4), (2.5) and (4.8), we have

$$q_2(\xi)\eta(\xi_1)\phi_2 X_0 + q_3(\xi)\eta(\xi_1)\phi_3 X_0 + \phi_1\phi\nabla_\xi X_0 = \eta(\xi_1)\nabla_\xi X_0. \tag{4.10}$$

By taking the inner product (4.10) with ξ_2 and ξ_3 respectively and using the fact that $\nabla_\xi X_0$, $\phi_\nu X_0 \in \mathfrak{D}$, $\nu = 1, 2, 3$, we have the following respectively

$$g(\nabla_\xi X_0, \phi_3 X_0) = 0 \quad \text{and} \quad g(\nabla_\xi X_0, \phi_2 X_0) = 0. \tag{4.11}$$

On the other hand, the assumption that $\mathfrak{D}^{\perp}$-component of ξ is totally geodesic and (4.2) give

$$q_2(\xi_1) = q_3(\xi_1) = 0\,. \tag{4.12}$$

Let us differentiate the formula (4.9) along the direction of ξ_1. Then by virtue of the formulas (2.3), (2.4), (2.5) and (4.12), we have

$$\phi_1\phi\nabla_{\xi_1}X_0 = \eta(\xi_1)\nabla_{\xi_1}X_0\,. \tag{4.13}$$

On the other hand, by taking the inner product (4.10) with $\phi_2 X_0$, $\phi_3 X_0$ respectively and using (2.1), (2.7) and (4.11) respectively we have

$$q_2(\xi) = 0 \quad \text{and} \quad q_3(\xi) = 0. \tag{4.14}$$

Then (4.10) implies that

$$\phi_1\phi\nabla_{\xi}X_0 = \eta(\xi_1)\nabla_{\xi}X_0. \tag{4.15}$$

Moreover, by differentiating (4.8) along the direction of ξ and using (2.3), (2.4), (2.5) and (4.14), we have

$$\phi\nabla_{\xi}X_0 = \alpha\eta_1(\xi)\eta(X_0)\xi_1 - \eta_1(\xi)\phi_1\nabla_{\xi}X_0.$$

From this, by applying ϕ and using (4.15) we have

$$\nabla_{\xi}X_0 = -\alpha\eta(\xi_1)\phi_1 X_0. \tag{4.16}$$

Now differentiating (4.8) along the direction ξ_1 and using (2.3) and (2.5), we have

$$\alpha\eta(X_0)\xi_1 + \phi\nabla_{\xi_1}X_0 = -\eta_1(\xi)\phi_1\nabla_{\xi_1}X_0.$$

Similarly, by applying ϕ to above equation and using (4.13) we have

$$\nabla_{\xi_1}X_0 = \alpha\phi_1 X_0. \tag{4.17}$$

Then (4.16) and (4.17) give

$$\nabla_{\xi}X_0 = -\eta(\xi_1)\nabla_{\xi_1}X_0. \tag{4.18}$$

On the other hand, we know that

$$\begin{aligned}\nabla_{\xi}X_0 &= \nabla_{\eta(X_0)X_0+\eta(\xi_1)\xi_1}X_0\\ &= \eta(X_0)\nabla_{X_0}X_0 + \eta(\xi_1)\nabla_{\xi_1}X_0\\ &= \eta(\xi_1)\nabla_{\xi_1}X_0,\end{aligned} \tag{4.19}$$

because the $\mathfrak{D}$-component of the Reeb vector field ξ is totally geodesic. From (4.18) and (4.19) we see that $\eta(\xi_1)\nabla_{\xi_1}X_0 = 0$. This means that $\nabla_{\xi_1}X_0 = 0$. From this together with (4.17), it follows that $\phi_1 X_0 = 0$. This gives a contradiction. So we only have $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^{\perp}$. □

5. Lie parallel normal Jacobi operator for $\xi \in \mathfrak{D}^{\perp}$

In order to give a complete proof of Theorem 2, first we consider the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$. Now in this direction we introduce some lemmas given in Jeong and Suh [6] as follows:

Lemma 5.A. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^{\perp}$. Then $A\xi = \alpha\xi + \beta U$, where U is a unit vector field orthogonal to ξ and belongs to $\mathfrak{D}$.*

Moreover, from Lemma 5.A, they proved the following lemmas:

Lemma 5.B. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^{\perp}$. Then β identically vanishes, that is, the Reeb vector field ξ is principal.*

Lemma 5.C. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^{\perp}$. Then $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.*

From these lemmas we assert:

Lemma 5.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie parallel normal Jacobi operator and $\xi \in \mathfrak{D}^{\perp}$. Then the Reeb vector ξ is principal and $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.*

Before going to give our proof of Theorem 2 in the introduction, let us check "What kind of model hypersurfaces given in Theorem A satisfy Lie parallel normal Jacobi operator." In other words, it will be an interesting problem to know whether there exist real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the condition $\mathcal{L}_X \bar{R}_N = 0$ for $\xi \in \mathfrak{D}^{\perp}$.
Then by virtue of Lemmas 5.1, we are able to recall the proposition given by Berndt and Suh [3] as follows:
For a tube of type (A) in Theorem A we have the following:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X \mid X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X \mid X \perp \mathbb{H}\xi, JX = -J_1 X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ *and* $\mathbb{H}\xi$ *respectively denotes real, complex and quaternionic span of the structure vector* ξ *and* $\mathbb{C}^{\perp}\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$.

In the proof of Lemma 5.C (See Section 4 in [6]) we have asserted that $A\xi_2 = 0$ and $A\xi_3 = 0$. But the principal curvature $\beta = \sqrt{2}\cot(\sqrt{2}r)$ given in Proposition A is never vanishing for any $r \in (0, \frac{\pi}{4})$. So this gives a contradiction. Accordingly, we completed the proof of our Theorem 2 for the case $\xi \in \mathfrak{D}^{\perp}$.

6. Lie parallel normal Jacobi operator for $\xi \in \mathfrak{D}$

Next we consider the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$. Then in this section we introduce the following lemmas due to Jeong and Suh [6] for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator.

Lemma 6.A. *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *satisfying Lie* ξ*-parallel normal Jacobi operator and* $\xi \in \mathfrak{D}$*. Then the Reeb vector* ξ *is principal.*

Then by using Lemma 6.A, Jeong and Suh [6] also verified the following:

Lemma 6.B. *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *satisfying Lie* ξ*-parallel normal Jacobi operator and* $\xi \in \mathfrak{D}$*. Then* $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.

By virtue of these Lemmas 6.A and 6.B we have

Lemma 6.C. *Let* M *be a real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *satisfying Lie parallel normal Jacobi operator and* $\xi \in \mathfrak{D}$*. Then the Reeb vector field* ξ *is principal and* $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.

From this Lemma 6.1, together with Theorem A due to Berndt and Suh [3], we have that M is locally a tube over a totally geodesic and totally real quaternionic projective space $\mathbb{H}P^n$, $m = 2n$. So for the geometrical structure of such a tube we recall the following proposition.

Proposition B. *Let* M *be a connected real hypersurface of* $G_2(\mathbb{C}^{m+2})$*. Suppose that* $A\mathfrak{D} \subset \mathfrak{D}$*,* $A\xi = \alpha\xi$*, and* ξ *is tangent to* $\mathfrak{D}$*. Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$*, and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$*. The corresponding multiplicities are*

$$m(\alpha) = 1,\ m(\beta) = 3 = m(\gamma),\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\ T_\beta = \mathfrak{J}J\xi,\ T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^{\perp},\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu.$$

Now, using the assumption that M is Hopf in (3.4), we have the following

$$\begin{aligned}
&(\mathcal{L}_\xi \bar{R}_N)Y\\
&= 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, Y)\xi_\nu + 4\alpha\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi\\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi A\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu\\
&\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)\big(\phi A\phi_\nu\phi Y - \eta(Y)\phi A\xi_\nu\big) - \eta_\nu(\phi Y)\phi A\phi_\nu\xi\Big\}\\
&\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)\big(\phi_\nu AY - \alpha\eta(Y)\phi_\nu\xi\big) - \eta_\nu(AY)\phi_\nu\xi + \eta(AY)\eta_\nu(\xi)\phi_\nu\xi\Big\}\\
&= 0\,.
\end{aligned}$$

Moreover, using the fact that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$, we have

$$\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)Y &= 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, Y)\xi_\nu + 4\alpha\sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi\\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi A\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu\\
&\quad - \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\phi A\phi_\nu\xi - \sum_{\nu=1}^{3}\eta_\nu(AY)\phi_\nu\xi\\
&= 0
\end{aligned} \tag{6.1}$$

for any $Y \in T_xM$, $x \in M$.

Let us construct a sub-distribution $\mathfrak{D}_0$ of the distribution $\mathfrak{D}$ in such a way that

$$[\xi] \oplus \mathfrak{D}_0 = \mathfrak{D},$$

where $[\xi]$ denotes an one-dimensional vector subspace spanned by the Reeb vector field ξ. Then $\mathfrak{D}_0$ becomes $\mathfrak{D}_0 = \{Y \in \mathfrak{D} \,|\, Y \perp \xi\}$. Here, if we substitute any $Y \in \mathfrak{D}_0$ in (6.1) and use $\xi \in \mathfrak{D}$, the left side of (6.1) becomes

$$\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)Y &= 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, Y)\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AY)\xi_\nu\\
&\quad - \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\phi A\phi_\nu\xi - \sum_{\nu=1}^{3}\eta_\nu(AY)\phi_\nu\xi.
\end{aligned}$$

From this, putting $Y = \phi_\mu\xi \in T_\gamma$, and using $A\phi_\mu\xi = 0$, $\mu = 1, 2, 3$, given in Proposition B, it becomes

$$(\mathcal{L}_\xi \bar{R}_N)\phi\xi_\mu = 4\alpha\xi_\mu.$$

From this, with the assumption of $\mathcal{L}_\xi \bar{R}_N = 0$, we have $\alpha = 0$. But the principal curvature $\alpha = -2\tan(2r)$ in Proposition B never vanishes for $r \in (0, \frac{\pi}{4})$. This gives a contradiction for the case $\xi \in \mathfrak{D}$. Accordingly, we complete the proof of our Theorem 2 for $\xi \in \mathfrak{D}$ in the introduction.

Acknowledgments. The present authors would like to express their deep gratitude to the referee for his careful reading of our manuscript and valuable comments to develop the first version.

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