

SPATIAL BEHAVIOR OF SOLUTION FOR THE STOKES FLOW EQUATION

YAN LIU, WENHUI LIAO, AND CHANGHAO LIN

ABSTRACT. In this paper, the equation of the transient Stokes flow of an incompressible viscous fluid is studied. Growth and decay estimates are established associating some appropriate cross sectional line and area integral measures. The method of the proof is based on a first-order differential inequality leading to an alternative of Phragmén-Lindelöf type in terms of an area measure of the amplitude in question. In the case of decay, we also indicate how to bound the total energy.

1. Introduction

In 1960-1980's, principle of Saint-Venant type used to be one of the most popular subjects of applied mathematics and mechanics. A great number of investigation results have expanded enormously the classical Saint-Venant principle. A review of recent work on Saint-Venant's principle is given in the work of Horgan and Knowles [6], and has been periodically updated by Horgan [3], [4]. A common feature of Saint-Venant type theorems is to establish the exponential decay estimates of energy with axial distance from the near end of a semi-infinite strip or cylinder.

Knowles [9] established exponential decay estimates for solution of the biharmonic equation in his study of *Saint-Venant's* principle in plane isotropic elastostatics for bounded, simply connected domain of general shape. Since then, many authors have investigated the same problems in a semi-infinite strip (see [5], [3] and the references cited therein). Their common goal has been to try to establish an energy decay results. It is well known that another interpretation for the biharmonic equation in the plane is that of the stream function

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in two dimensional Stokes flow, hence the results of the *Saint-Venant's* principle in plane elastostatics are also relevant to the study of the spatial evolution of stationary stokes flows in a semi-infinite parallel plate channel. Numerous authors have dealt with Saint-Venant type decay estimate for solutions of the biharmonic equations in a semi-infinite channel in \mathbb{R}^2 , we mention in particular the papers of Flavin [1], Horgan [5], Knowles [8] and Flavin et al. [2]. In a paper [10], Lin has established energy decay estimates for solutions of the plane Stokes flow in a semi-infinite channel, subject to nonzero boundary conditions on the end only. Such estimates yielding an exponential decay of energy with axial distance from the end are based on differential inequality techniques developed by Knowles [9], and Horgan and Knowles [6] in their investigations of Saint-Venant's principle in classical elasticity theory. Recently, Song [13] investigated the same time-dependent Stokes flow problem and obtained an analogous result with an improved decay rate. The papers concerning with biharmonic equations may be viewed as a version of *Saint-Venant's* principle in steady state or transient stokes flow, but common to all was the assumption that the solution must satisfy some a priori decay criterion at infinity. One would like to remove this a priori restriction. One way to do this is to derive an appropriate *Phragmén-Lindelöf* alternative.

The classical Phragmén-Lindelöf theorem states that harmonic function which vanish on the cylindrical surface must either grow exponential or decay exponential with distance from the finite end of the cylinder. Phragmén-Lindelöf type alternative results for harmonic functions were obtained by Horgan et al. [7] under non-linear boundary conditions on the lateral surface of a semi-infinite cylinder. Particularly, Payne et al. [12] established the Phragmén-Lindelöf type results in three type special domains in \mathbb{R}^2 . Additional references for Phragmén-Lindelöf type results may be found in [11] and [4] therein.

In this paper, we investigate the spatial behaviour for the equation of the transient Stokes flow of an incompressible viscous fluid. We use a stream function to transform the equation to a biharmonic equation, unlike many other papers deal with the biharmonic equation, our method of the proof is based on a first-order differential inequality not the second-order differential inequality. In the next section we will formulate the initial and boundary value problem that serves as the basis for our analysis. Section 3 is devoted to deriving the Phragmén-Lindelöf type alternative results. In the final section we will indicate how to bound the total energy.

In the present paper, the comma is used to indicate partial differentiation and the differentiation with respect to the direction x_k is denoted as $_{,k}$, thus $u_{,\alpha}$ denotes $\frac{\partial u}{\partial x_\alpha}$, and $u_{,t}$ denotes $\frac{\partial u}{\partial t}$. The usual summation convention is employed with repeated Greek subscripts α summed from 1 to 2. Hence, $u_{\alpha,\alpha} = \sum_{\alpha=1}^2 \frac{\partial u_\alpha}{\partial x_\alpha}$.

2. Basic equation

We consider the time-dependent Stokes equations governing the transient slow flow of an incompressible slow viscous fluid on an unbounded region Ω_0 defined by

$$(2.1) \quad \Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 < x_2 < h\},$$

where h is a fixed constant, and we introduce the notation

$$(2.2) \quad L_z = \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\}.$$

The velocity field $v_\alpha(x_1, x_2, t)$ and the pressure $p(x_1, x_2, t)$ ($\alpha=1,2$) for the transient Stokes flow of an incompressible viscous fluid are to be classical solutions of the initial-boundary value problem:

$$(2.3) \quad v_{\alpha,t} = \nu \Delta v_\alpha - p_{,\alpha} \quad \text{in } \Omega_0 \times [0, \infty),$$

$$(2.4) \quad v_{\alpha,\alpha} = 0 \quad \text{in } \Omega_0 \times [0, \infty),$$

$$(2.5) \quad v_\alpha(x_1, 0, t) = v_\alpha(x_1, h, t) = 0 \quad \alpha = 1, 2,$$

$$(2.6) \quad v_\alpha(0, x_2, t) = f_\alpha(x_2, t) \quad \alpha = 1, 2.$$

$$(2.7) \quad v_\alpha(x_1, x_2, 0) = 0 \quad (x_1, x_2) \in \Omega_0,$$

where Δ is the two dimensional Laplace operator and ν is the constant kinematic viscosity. The functions $f_\alpha(x_2, t)$ are assumed to satisfy the compatibility $f_\alpha(0, t) = f_\alpha(h, t) = 0$. For simplicity, it is assumed that $\int_{L_0} f_1 dx_2 = 0$ for all $t > 0$.

In order to eliminate the pressure term $p_{,\alpha}$, we introduce the stream function $u(x_1, x_2, t)$ such that

$$(2.8) \quad v_1 = u_{,2}, v_2 = -u_{,1}.$$

So the equations (2.2)-(2.6) are transformed into the following fourth order initial-boundary value problem:

$$(2.9) \quad \Delta^2 u = (\Delta u)_{,t} \quad \text{in } \Omega_0,$$

$$(2.10) \quad u(x_1, 0, t) = u_n(x_1, 0, t) = 0 \quad x_1 > 0, t > 0,$$

$$(2.11) \quad u(x_1, h, t) = u_n(x_1, h, t) = 0 \quad x_1 > 0, t > 0,$$

$$(2.12) \quad u(0, x_2, t) = g_1(x_2, t) = \int_0^{x_2} f_1(s, t) ds \quad 0 \leq x_2 \leq h, t > 0,$$

$$(2.13) \quad u_{,1}(0, x_2, t) = g_2(x_2, t) = -f_2(x_2, t) \quad 0 \leq x_2 \leq h, t > 0,$$

$$(2.14) \quad u_{,\alpha}(x_1, x_2, 0) = 0 \quad x_1 > 0, 0 < x_2 < h,$$

where we make no assumption on u as $x_1 \rightarrow \infty$. Here Δ is the harmonic operator, and Δ^2 is the biharmonic operator, u_n is the outward normal derivative,

and we adapt the standard notations, i.e., $u_{,i} = \frac{\partial u}{\partial x_i}$, $u_{,t} = \frac{\partial u}{\partial t}$. The differentiable functions g_1 and g_2 are prescribed and assumed to satisfy appropriate compatibility conditions: $g_1(0, t) = g_1(h, t) = g_2(0, t) = g_2(h, t) = g'_1(0, t) = g'_1(h, t) = 0$, where g'_1 denotes the partial differentiation with respect to x_2 .

We define energy expressions of the forms:

$$(2.15) \quad E_1(z, t) = 2 \int_0^t \int_{z_1}^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \int_{z_1}^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} \\ + 2 \int_0^t \int_{z_1}^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \int_{z_1}^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t},$$

$$(2.16) \quad E_2(z, t) = \frac{1}{2} \int_0^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} + \frac{1}{2} \int_0^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t},$$

where we use the summation convention on repeated indices and the Greek subscripts range over 1, 2. Our purpose is to determine the alternative that either $E_1(z, t)$ grows exponentially or $E_2(z, t)$ decays exponentially as $z \rightarrow \infty$.

We start our analysis by the following identity. On using the divergence theorem and the initial-boundary conditions (2.10)-(2.14), it leads to

$$(2.17) \quad 0 = \int_0^t \int_0^z \int_{L_\xi} u_{,\eta} (\Delta u_{,\eta} - \Delta^2 u) dAd\eta \\ = - \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \int_0^t \int_{L_z} u_{,\eta} u_{,1\eta} dx_2 d\eta \\ - \int_0^t \int_{L_0} u_{,\eta} u_{,1\eta} dx_2 d\eta + \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\beta\beta} dAd\eta \\ - \int_0^t \int_{L_z} u_{,\eta} u_{,1\beta\beta} dx_2 d\eta + \int_0^t \int_{L_0} u_{,\eta} u_{,1\beta\beta} dx_2 d\eta \\ = - \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \int_0^t \int_{L_z} u_{,\eta} u_{,1\eta} dx_2 d\eta \\ - \int_0^t \int_{L_0} u_{,\eta} u_{,1\eta} dx_2 d\eta - \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\beta\eta} u_{,\alpha\beta} dAd\eta \\ + \int_0^t \int_{L_z} u_{,\alpha\eta} u_{,1\alpha} dx_2 d\eta - \int_0^t \int_{L_0} u_{,\alpha\eta} u_{,1\alpha} dx_2 d\eta \\ - \int_0^t \int_{L_z} u_{,\eta} u_{,1\beta\beta} dx_2 d\eta + \int_0^t \int_{L_0} u_{,\eta} u_{,1\beta\beta} dx_2 d\eta.$$

From (2.17), we define

$$\phi_1(z, t) = - \int_0^t \int_{L_z} (u_{,\eta} u_{,1\eta} + u_{,\alpha\eta} u_{,1\alpha} - u_{,\eta} u_{,1\beta\beta}) dx_2 d\eta$$

$$(2.18) \quad = - \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta - \frac{1}{2} \int_0^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} + \phi_1(0, t).$$

Deriving as (2.17), we obtain

$$(2.19) \quad \begin{aligned} 0 &= \int_0^t \int_0^z \int_{L_\xi} u_{,11} (\Delta u_{,\eta} - \Delta^2 u) dAd\eta \\ &= - \int_0^t \int_0^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta - \frac{1}{2} \int_0^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t} \\ &\quad + \int_0^t \int_{L_z} (u_{,\alpha\alpha} u_{,\alpha\alpha 1} + u_{,12} u_{,2\eta}) dx_2 d\eta \\ &\quad - \int_0^t \int_{L_0} (u_{,\alpha\alpha} u_{,\alpha\alpha 1} + u_{,12} u_{,2\eta}) dx_2 d\eta. \end{aligned}$$

From (2.19), we define

$$(2.20) \quad \begin{aligned} \phi_2(z, t) &= - \int_0^t \int_{L_z} (u_{,\alpha\alpha} u_{,1\alpha\alpha} + u_{,12} u_{,2\eta}) dx_2 d\eta \\ &= - \int_0^t \int_0^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta - \frac{1}{2} \int_0^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t} + \phi_2(0, t). \end{aligned}$$

We now introduce the function

$$(2.21) \quad \begin{aligned} \phi(z, t) &= \phi_1(z, t) + \phi_2(z, t) \\ &= - \int_0^t \int_{L_z} (u_{,\eta} u_{,1\eta} + u_{,\alpha\eta} u_{,1\alpha} - u_{,\eta} u_{,1\beta\beta}) dx_2 d\eta \\ &\quad - \int_0^t \int_{L_z} (u_{,\alpha\alpha} u_{,1\alpha\alpha} + u_{,12} u_{,2\eta}) dx_2 d\eta \\ &= - \int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta - \frac{1}{2} \int_0^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} + \phi_1(0, t) \\ &\quad - \int_0^t \int_0^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta - \frac{1}{2} \int_0^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t} + \phi_2(0, t). \end{aligned}$$

Obviously, from the definition of $\phi(z, t)$, we obtain

$$(2.22) \quad \frac{\partial}{\partial z} \phi(z, t) \leq 0.$$

From the definition of (2.21), we also have for arbitrary $z > z_0 \geq 0$,

$$\phi(z, t) - \phi(z_0, t) = - \int_0^t \int_{z_0}^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta - \frac{1}{2} \int_{z_0}^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t}$$

$$(2.23) \quad - \int_0^t \int_{z_0}^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta - \frac{1}{2} \int_{z_0}^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t}.$$

From (2.23), we can see that if $\phi(z, t) \rightarrow 0$ as $z \rightarrow \infty$, then

$$(2.24) \quad \begin{aligned} \phi(z, t) &= \int_0^t \int_z^\infty \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} \\ &+ \int_0^t \int_z^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t}. \end{aligned}$$

On the other hand, if $-\phi(z, t)$ is unbounded below by a positive number $\chi(z, t)$, then

$$(2.25) \quad \begin{aligned} &\int_0^t \int_0^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_0^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t} \\ &+ \int_0^t \int_0^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \frac{1}{2} \int_0^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t} \\ &\geq \chi(z, t) + \phi(0, t). \end{aligned}$$

In the following part, we will use the following inequality: Clearly, we know, for $u(z, 0, t) = 0$, we have

$$u(z, x_2, t) = \int_0^{x_2} u_{,s}(z, s, t) ds,$$

thus, using Schwarz's inequality, we easily find that

$$(2.26) \quad \int_{L_z} u^2 dx_2 \leq h^2 \int_{L_z} u_{,2}^2 dx_2.$$

3. Phragmén-Lindelöf type alternative results

In this part, we'll use the technique which was used by Lin and Payne in [11]. For $t > 0$, let \tilde{t} be a value of time between 0 and t , at which function

$$m(z, t) = \frac{1}{2} \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2|_{\eta=t} + \frac{1}{2} \int_{L_z} u_{,1\alpha} u_{,1\alpha} dx_2|_{\eta=t}$$

gains its maximum value, i.e.,

$$(3.1) \quad \begin{aligned} m(z, \tilde{t}) &= \frac{1}{2} \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2|_{\eta=\tilde{t}} + \frac{1}{2} \int_{L_z} u_{,1\alpha} u_{,1\alpha} dx_2|_{\eta=\tilde{t}} \\ &= \max_{0 \leq s \leq t} \left\{ \frac{1}{2} \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2|_{\eta=s} + \frac{1}{2} \int_{L_z} u_{,1\alpha} u_{,1\alpha} dx_2|_{\eta=s} \right\}. \end{aligned}$$

We now define

$$(3.2) \quad t^* = \lim_{Z \rightarrow \infty} \left\{ \sup_{0 \leq z \leq Z} \tilde{t}(z) \right\}.$$

Following the same procedure as in producing t^* , we can also define $0 \leq t_1^* \leq t$, $0 \leq t_2^* \leq t$ such that

$$(3.3) \quad m_1(z, \tilde{t}_1) = \frac{1}{2} \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2 |_{\eta=\tilde{t}_1} = \max_{0 \leq s \leq t} \left\{ \frac{1}{2} \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2 |_{\eta=s} \right\},$$

$$(3.4) \quad m_2(z, \tilde{t}_2) = \frac{1}{2} \int_{L_z} u_{,1\alpha} u_{,1\alpha} dx_2 |_{\eta=\tilde{t}_2} = \max_{0 \leq s \leq t} \left\{ \frac{1}{2} \int_{L_z} u_{,1\alpha} u_{,1\alpha} dx_2 |_{\eta=s} \right\}.$$

We also define

$$t_1^* = \lim_{Z \rightarrow \infty} \left\{ \sup_{0 \leq z \leq Z} \tilde{t}_1(z) \right\},$$

and

$$t_2^* = \lim_{Z \rightarrow \infty} \left\{ \sup_{0 \leq z \leq Z} \tilde{t}_2(z) \right\}.$$

From the definitions of $m_1(z, \tilde{t}_1)$, $m_2(z, \tilde{t}_2)$, $m(z, \tilde{t})$ and t_1^*, t_2^* , we can easily get

$$(3.5) \quad m_2(z, t_2^*) \leq m_1(z, t_1^*) \leq m(z, t^*).$$

Now, we write

$$(3.6) \quad \begin{aligned} \phi(z, t^*) = & - \int_0^{t^*} \int_{z_0}^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta - \frac{1}{2} \int_{z_0}^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA |_{\eta=t^*} \\ & - \int_0^{t^*} \int_{z_0}^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta - \frac{1}{2} \int_{z_0}^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA |_{\eta=t^*} + \phi(z_0, t^*). \end{aligned}$$

From (2.21), we can get

$$(3.7) \quad \begin{aligned} |\phi(z, t^*)| \leq & \left| \int_0^{t^*} \int_{L_z} (u_{,\eta} u_{,1\eta} + u_{,\alpha\eta} u_{,1\alpha} - u_{,\eta} u_{,1\beta\beta}) dx_2 d\eta \right| \\ & + \left| \int_0^{t^*} \int_{L_z} (u_{,\alpha\alpha} u_{,1\alpha\alpha} + u_{,12} u_{,2\eta}) dx_2 d\eta \right|. \end{aligned}$$

Making use of Schwarz's inequality, inequality (2.26), and Young's inequality, it yields

$$(3.8) \quad \begin{aligned} & \left| \int_0^{t^*} \int_{L_z} u_{,\eta} u_{,1\eta} dx_2 d\eta \right| \\ & \leq h \left[\frac{\varepsilon_1}{2} \int_0^{t^*} \int_{L_z} u_{,2\eta} u_{,2\eta} dx_2 d\eta + \frac{1}{2\varepsilon_1} \int_0^{t^*} \int_{L_z} u_{,1\eta} u_{,1\eta} dx_2 d\eta \right], \end{aligned}$$

$$(3.9) \quad \left| \int_0^{t^*} \int_{L_z} u_{,\alpha\eta} u_{,1\alpha} dx_2 d\eta \right|$$

$$\begin{aligned}
&\leq \frac{\varepsilon_2}{2} \int_0^{t^*} \int_{L_z} u_{,\alpha\eta} u_{,\alpha\eta} dx_2 d\eta + \frac{1}{2\varepsilon_2} \int_0^{t^*} \int_{L_z} u_{,1\alpha} u_{,1\alpha} dx_2 d\eta \\
&\leq \frac{\varepsilon_2}{2} \int_0^{t^*} \int_{L_z} u_{,\alpha\eta} u_{,\alpha\eta} dx_2 d\eta + \frac{1}{2\varepsilon_2} t^* \int_{L_z} u_{,1\alpha} u_{,1\alpha} dx_2 |_{\eta=t_2^*} \\
&\leq \frac{\varepsilon_2}{2} \int_0^{t^*} \int_{L_z} u_{,\alpha\eta} u_{,\alpha\eta} dx_2 d\eta + \frac{1}{2\varepsilon_2} t^* \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2 |_{\eta=t_1^*}, \\
(3.10) \quad &\left| \int_0^{t^*} \int_{L_z} u_{,\eta} u_{,1\alpha\alpha} dx_2 d\eta \right| \\
&\leq h \left[\frac{\varepsilon_3}{2} \int_0^{t^*} \int_{L_z} u_{,2\eta} u_{,2\eta} dx_2 d\eta + \frac{1}{2\varepsilon_3} \int_0^{t^*} \int_{L_z} u_{,1\alpha\alpha} u_{,1\beta\beta} dx_2 d\eta \right] \\
&\leq h \left[\frac{\varepsilon_3}{2} \int_0^{t^*} \int_{L_z} u_{,2\eta} u_{,2\eta} dx_2 d\eta + \frac{1}{2\varepsilon_3} \int_0^{t^*} \int_{L_z} u_{,1\alpha\beta} u_{,1\alpha\beta} dx_2 d\eta \right], \\
(3.11) \quad &\left| \int_0^{t^*} \int_{L_z} u_{,\alpha\alpha} u_{,1\alpha\alpha} dx_2 d\eta \right| \\
&\leq \frac{\varepsilon_4}{2} \int_0^{t^*} \int_{L_z} u_{,\alpha\alpha} u_{,\beta\beta} dx_2 d\eta + \frac{1}{2\varepsilon_4} \int_0^{t^*} \int_{L_z} u_{,\alpha\alpha 1} u_{,\alpha\alpha 1} dx_2 d\eta \\
&\leq \frac{\varepsilon_4}{2} t^* \int_{L_z} u_{,\alpha\alpha} u_{,\beta\beta} dx_2 |_{\eta=t_1^*} + \frac{1}{2\varepsilon_4} \int_0^{t^*} \int_{L_z} u_{,\alpha\alpha 1} u_{,\beta\beta 1} dx_2 d\eta \\
&\leq \frac{\varepsilon_4}{2} t^* \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2 |_{\eta=t_1^*} + \frac{1}{2\varepsilon_4} \int_0^{t^*} \int_{L_z} u_{,1\alpha\beta} u_{,1\alpha\beta} dx_2 d\eta, \\
(3.12) \quad &\left| \int_0^{t^*} \int_{L_z} u_{,12} u_{,2\eta} dx_2 d\eta \right| \\
&\leq \frac{\varepsilon_5}{2} \int_0^{t^*} \int_{L_z} u_{,12} u_{,12} dx_2 d\eta + \frac{1}{2\varepsilon_5} \int_0^{t^*} \int_{L_z} u_{,2\eta} u_{,2\eta} dx_2 d\eta \\
&\leq \frac{\varepsilon_5}{2} t^* \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2 |_{\eta=t_1^*} + \frac{1}{2\varepsilon_5} \int_0^{t^*} \int_{L_z} u_{,2\eta} u_{,2\eta} dx_2 d\eta \\
&\leq \frac{\varepsilon_5}{2} t^* \int_{L_z} u_{,\alpha\beta} u_{,\alpha\beta} dx_2 |_{\eta=t_1^*} + \frac{1}{2\varepsilon_5} \int_0^{t^*} \int_{L_z} u_{,2\eta} u_{,2\eta} dx_2 d\eta.
\end{aligned}$$

If we choose $\varepsilon_1 = \frac{\sqrt{2}}{2}$, $\varepsilon_2 = 1$, $\varepsilon_3 = \frac{\sqrt{2}}{2}$, $\varepsilon_4 = 1$, $\varepsilon_5 = 1$, by combining (3.5) and (3.7)-(3.12), we obtain

$$(3.13) \quad |\phi(z, t^*)| \leq \left(\frac{3}{2} t^* + k \right) \left(-\frac{\partial \phi(z, t^*)}{\partial z} \right),$$

where $k = \frac{\sqrt{2}}{2} h + 1$.

We may write (3.13) as two inequalities:

$$(3.14) \quad -\frac{\partial\phi(z, t^*)}{\partial z} \geq \frac{-\phi(z, t^*)}{\frac{3}{2}t^* + k},$$

and

$$(3.15) \quad -\frac{\partial\phi(z, t^*)}{\partial z} \geq \frac{\phi(z, t^*)}{\frac{3}{2}t^* + k}.$$

The following discussion will be divided into two cases.

Case 1: If there exists a point (z_1, t^*) such that $-\phi(z_1, t^*) > 0$, then for all $z > z_1$, $-\phi(z, t^*) > 0$. An integration of (3.14) leads to

$$(3.16) \quad -\phi(z, t^*) \geq -\phi(z_1, t^*) \exp\left(\frac{z - z_1}{\frac{3}{2}t^* + k}\right).$$

Case 2: If there doesn't exist a point (z_1, t^*) such that $-\phi(z_1, t^*) > 0$, then for all $z \geq 0$, $\phi(z, t^*) \geq 0$. We have, from (3.15),

$$(3.17) \quad \phi(z, t^*) \leq \phi(0, t^*) \exp\left(\frac{-z}{\frac{3}{2}t^* + k}\right).$$

From (3.16), we can get $\lim_{z \rightarrow \infty} (-\phi(z, t^*)) = \infty$, that is to say for z large enough, we can get

$$-\phi(z, t^*) \geq -2\phi(z_1, t^*).$$

From (3.6), we get

$$(3.18) \quad \begin{aligned} -\phi(z, t^*) &\leq 2 \int_0^{t^*} \int_{z_1}^z \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \int_{z_1}^z \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} \\ &\quad + 2 \int_0^{t^*} \int_{z_1}^z \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \int_{z_1}^z \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*} \\ &= E_1(z, t^*). \end{aligned}$$

Combining (3.16) and (3.18), we obtain

$$(3.19) \quad \lim_{z \rightarrow \infty} \exp\left(\frac{z_1 - z}{\frac{3}{2}t^* + k}\right) E_1(z, t^*) \geq -\phi(z, t^*).$$

Clearly, if $\lim_{z \rightarrow \infty} \phi(z, t^*) = 0$, we can easily get from (3.6) that

$$(3.20) \quad \begin{aligned} \phi(z, t^*) &= \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} \\ &\quad + \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*}. \end{aligned}$$

We have

$$(3.21) \quad \begin{aligned} E_2(z, t^*) &= \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*} \\ &\leq \phi(z, t^*). \end{aligned}$$

Summarizing all the results above, we conclude the following theorem:

Theorem 1. *If u is a solution of the initial boundary value problem (2.9)-(2.14), then, either*

$$(3.22) \quad \lim_{z \rightarrow \infty} \exp\left(\frac{z_1 - z}{\frac{3}{2}t^* + k} E_1(z, t^*)\right) \geq -\phi(z_1, t^*),$$

is satisfied, or the “energy” function $E_2(z, t)$ satisfies the estimates

$$(3.23) \quad \begin{aligned} E_2(z, t) \leq E_2(z, t^*) \leq \phi(z, t^*) \leq \phi(0, t^*) \exp\left[\left(-\frac{1}{\frac{3}{2}t^* + k}\right)z\right] \\ \leq \phi(0, t^*) \exp\left[\left(-\frac{1}{\frac{3}{2}t + k}\right)z\right], \end{aligned}$$

where t^ is a value (see (3.2)) belonging to $(0, t]$.*

4. Upper bound for the total energy

We now show how a explicit bound can be determined for $\phi(0, t^*)$ which occurred in (3.23).

Under the case of decay, we define

$$(4.1) \quad \begin{aligned} F(z, t^*) &= \phi(z, t^*) \\ &= \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} \\ &\quad + \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*}. \end{aligned}$$

If we define

$$(4.2) \quad F(z, t^*) = F_1(z, t^*) + F_2(z, t^*),$$

where

$$\begin{aligned} F_1(z, t^*) &= \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,\alpha\eta} u_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*}, \\ F_2(z, t^*) &= \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*}, \end{aligned}$$

and

$$(4.3) \quad F_u(z, t^*) = F_1(z, t^*),$$

$$(4.4) \quad F_w(z, t^*) = \int_0^{t^*} \int_z^\infty \int_{L_\xi} w_{,\alpha\eta} w_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t^*},$$

$$(4.5) \quad F_{uw}(z, t^*) = \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,\alpha\eta} w_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t^*},$$

where $w(x_1, x_2, t)$ is an arbitrary smooth function defined on Ω_0 that satisfies the same boundary conditions as u .

By using the Schwarz's inequality, we get

$$(4.6) \quad \sqrt{F_u(z, t^*) F_w(z, t^*)} \geq F_{uw}(z, t^*).$$

It is clear that

$$(4.7) \quad \begin{aligned} F_{uw}(0, t^*) &= F_u(0, t^*) + \int_0^{t^*} \int_0^\infty \int_{L_\xi} u_{,\alpha\eta} (w_{,\alpha\eta} - u_{,\alpha\eta}) dAd\eta \\ &\quad + \frac{1}{2} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} (w_{,\alpha\beta} - u_{,\alpha\beta}) dA|_{\eta=t^*} \\ &= F_u(0, t^*) + I_1 + I_2. \end{aligned}$$

Making use of the divergence theorem, and (2.9)-(2.14), we obtain

$$(4.8) \quad \begin{aligned} I_1 &= - \int_0^{t^*} \int_0^\infty \int_{L_\xi} u_{,\alpha\alpha\eta} (w_{,\eta} - u_{,\eta}) dAd\eta \\ &= \int_0^{t^*} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta\beta} (w_{,\alpha\eta} - u_{,\alpha\eta}) dAd\eta \\ &= - \int_0^{t^*} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} (w_{,\alpha\beta\eta} - u_{,\alpha\beta\eta}) dAd\eta \\ &\geq \frac{1}{2} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} - \frac{\varepsilon_1}{2} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} \\ &\quad - \frac{1}{2\varepsilon_1} t^* \int_0^{t^*} \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dAd\eta, \end{aligned}$$

$$(4.9) \quad \begin{aligned} I_2 &= \frac{1}{2} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t^*} - \frac{1}{2} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} \\ &\geq - \frac{\varepsilon_2}{4} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} - \frac{1}{4\varepsilon_2} \int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t^*} \\ &\quad - \frac{1}{2} \int_0^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*}. \end{aligned}$$

Setting $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{1}{2}$, then combining (4.6)-(4.8) and (4.9), we are led to

$$(4.10) \quad \sqrt{F_u(0, t^*) F_w(0, t^*)} \geq \frac{3}{4} F_u(0, t^*) - Q(w(0, t^*)),$$

where we have

$$Q(w(0, t^*)) = \frac{t^*}{2} \int_0^{t^*} \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dAd\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t^*}.$$

From (4.10), we can conclude

$$(4.11) \quad \frac{3}{4}(\sqrt{F_u(0, t^*)} - \frac{2}{3}(\sqrt{F_w(0, t^*)})^2)^2 \leq \frac{1}{3}(F_w(0, t^*) + 3Q(w(0, t^*))),$$

which implies that

$$(4.12) \quad \sqrt{F_u(0, t^*)} \leq \frac{2}{3}\sqrt{F_w(0, t^*)} + \frac{2}{3}(\sqrt{F_w(0, t^*)} + 3Q(w(0, t^*)))^{\frac{1}{2}}.$$

Squaring and making use of the arithmetic-geometric mean inequality, we obtain

$$(4.13) \quad F_u(0, t^*) \leq \frac{16}{9}F_w(0, t^*) + \frac{8}{3}Q(w(0, t^*)).$$

Since we don't know the value of t^* , the bound for $F_u(0, t^*)$ can be made explicit only if we can eliminate the dependent on t^* . Clearly, we know, for any function φ such that $\varphi(x, 0) = 0$,

$$(4.14) \quad \int_{L_z} \varphi^2 dx_2|_{\eta=t^*} = 2 \int_0^{t^*} \int_{L_z} \varphi \varphi_{,\eta} dx_2 d\eta \leq \int_0^t \int_{L_z} \varphi^2 dx_2 d\eta + \int_0^t \int_{L_z} \varphi_{,\eta}^2 dx_2 d\eta.$$

Using (4.14), we obtain

$$\begin{aligned} F_w(0, t^*) &= \int_0^{t^*} \int_0^\infty \int_{L_\xi} w_{,\alpha\eta} w_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t^*} \\ &\leq \int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\eta} w_{,\alpha\eta} dAd\eta + \frac{1}{2} \int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dAd\eta \\ &\quad + \frac{1}{2} \int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dAd\eta \\ &= \bar{F}_w(0, t), \end{aligned}$$

and

$$\begin{aligned} Q(w(0, t^*)) &= \frac{t^*}{2} \int_0^{t^*} \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dAd\eta + \frac{1}{2} \int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t^*} \\ &\leq \frac{t}{2} \int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dAd\eta + \frac{1}{2} \int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dAd\eta \\ &\quad + \frac{1}{2} \int_0^t \int_0^\infty \int_{L_\xi} w_{,\alpha\beta\eta} w_{,\alpha\beta\eta} dAd\eta \\ &= \bar{Q}(w(0, t)). \end{aligned}$$

We can get

$$(4.15) \quad F_u(0, t^*) \leq \frac{16}{9} \bar{F}_w(0, t) + \frac{8}{3} \bar{Q}(w(0, t)).$$

We now define $w(x_1, x_2, t)$ as

$$(4.16) \quad w(x_1, x_2, t) = \left[\int_0^{x_2} \frac{\partial g_1}{\partial \xi}(\xi, t) d\xi + x_1 \left(g_2(x_2, t) + \int_0^{x_2} \frac{\partial g_1}{\partial \xi}(\xi, t) d\xi \right) \right] e^{-x_1},$$

Obviously, it is readily to verify $w(x_1, x_2, t)$ satisfying the same initial-boundary value conditions as $u(x_1, x_2, t)$.

Combining (4.15) and (4.16), we can say we have bounded $F_u(0, t^*)$.

The next step is to bound $F_2(0, t^*)$.

We will use the following lemma:

Lemma 1. *If $\phi(z, t^*)$ decays to zero as $z \rightarrow \infty$, then for arbitrary $z \geq 0$, $z_0 > 0$,*

$$(4.17) \quad z_0^2 F_2(z + z_0, t^*) \leq 8t F_1(z, t^*).$$

Proof. We adapt the notation

$$(4.18) \quad \begin{aligned} \tilde{F}_2(z, t^*) &= \int_0^{t^*} \int_z^\infty \int_{L_\xi} (\xi - z)^2 u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta \\ &\quad + \frac{1}{2} \int_z^\infty \int_{L_\xi} (\xi - z)^2 u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*}. \end{aligned}$$

Obviously, we have the following identity

$$(4.19) \quad 0 = \int_0^{t^*} \int_z^\infty \int_{L_\xi} (\xi - z)^2 u_{,11} (\Delta^2 u - \Delta u_{, \eta}) dAd\eta.$$

By applying the divergence theorem repeatedly and using the initial-boundary value conditions, we obtain

$$(4.20) \quad \begin{aligned} 0 &= - \int_0^{t^*} \int_z^\infty \int_{L_\xi} (\xi - z)^2 u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta + \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{, \alpha\beta} u_{, \alpha\beta} dAd\eta \\ &\quad - 2 \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,12} u_{,12} dAd\eta - \frac{1}{2} \int_z^\infty \int_{L_\xi} (\xi - z)^2 u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*} \\ &\quad + 2 \int_0^{t^*} \int_z^\infty \int_{L_\xi} (\xi - z) u_{,12} u_{,2\eta} dAd\eta. \end{aligned}$$

Inserting (4.20) into (4.18), we find

$$(4.21) \quad \tilde{F}_2(z, t^*) \leq \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{, \alpha\beta} u_{, \alpha\beta} dAd\eta + 2 \left| \int_0^{t^*} \int_z^\infty \int_{L_\xi} (\xi - z) u_{,12} u_{,2\eta} dAd\eta \right|.$$

We have

$$\begin{aligned}
& \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dAd\eta \leq t^* \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} \\
& \leq t \left(\int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*} + \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*} \right) \\
(4.22) \quad & \leq 4tF_1(z, t^*).
\end{aligned}$$

Using the Schwarz's inequality, we have

$$\begin{aligned}
2 \left| \int_0^{t^*} \int_z^\infty \int_{L_\xi} (\xi - z) u_{,12} u_{,2\eta} dAd\eta \right| & \leq \varepsilon \int_z^\infty \int_{L_\xi} (\xi - z)^2 u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*} \\
(4.23) \quad & + \frac{1}{\varepsilon} t^* \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,2\eta}^2 dAd\eta.
\end{aligned}$$

If we choose $\varepsilon = \frac{1}{4}$, combining (4.22) and (4.23), we obtain

$$(4.24) \quad \tilde{F}_2(z, t^*) \leq 8tF_1(z, t^*).$$

We note that for arbitrary $z_0 > 0$,

$$(4.25) \quad \tilde{F}_2(z, t^*) \geq \int_0^{t^*} \int_{z+z_0}^\infty \int_{L_\xi} (\xi - z)^2 u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta \geq z_0^2 F_2(z + z_0, t^*).$$

In view of (4.25), we have proved Lemma 1. \square

For we know,

$$(4.26) \quad \int_z^\infty \int_{L_\xi} u_{,1\alpha} u_{,1\alpha} dA|_{\eta=t^*} \leq \int_z^\infty \int_{L_\xi} u_{,\alpha\beta} u_{,\alpha\beta} dA|_{\eta=t^*}.$$

Thus, we only need to bound $F_3(0, t^*) = \int_0^{t^*} \int_z^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta$.

Now, we begin to bound

$$\begin{aligned}
& \int_0^{t^*} \int_0^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta = \int_0^{t^*} \int_0^1 \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta \\
(4.27) \quad & + \int_0^{t^*} \int_1^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta.
\end{aligned}$$

Setting $z = 0$, $z_0 = 1$ in (4.17) leads to

$$(4.28) \quad \int_0^{t^*} \int_1^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta \leq 8tF_1(0, t^*).$$

Using (4.24), and setting $z = 0$, we obtain

$$(4.29) \quad \int_0^{t^*} \int_0^\infty \int_{L_\xi} \xi^2 u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta \leq 8tF_1(0, t^*).$$

Then (4.29) is equivalent to

$$(4.30) \quad \int_0^{t^*} \int_0^\infty \int_{L_\xi} \psi_{,\xi} \psi_{,\xi} dAd\eta \leq 8tF_1(0, t^*),$$

where $\psi = u_{,\alpha\beta}$.

For the first integral on the right hand side of (4.25), making a variational transformation $x_1 = \frac{1}{\xi}$, and using the inequality (4.30), we deduce that

$$(4.31) \quad \begin{aligned} \int_0^{t^*} \int_0^1 \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta &= \int_0^{t^*} \int_1^\infty \int_{L_\xi} \xi^2 u_{,\xi\alpha\beta} u_{,\xi\alpha\beta} dAd\eta \\ &\leq \int_0^{t^*} \int_0^\infty \int_{L_\xi} \xi^2 u_{,\xi\alpha\beta} u_{,\xi\alpha\beta} dAd\eta \leq 8tF_1(0, t^*). \end{aligned}$$

Combining (4.28) and (4.31), we finally obtain

$$(4.32) \quad \int_0^{t^*} \int_0^\infty \int_{L_\xi} u_{,1\alpha\beta} u_{,1\alpha\beta} dAd\eta \leq 16tF_1(0, t^*).$$

In view of (4.15), (4.26) and (4.32), we can conclude that we have bounded $F_2(0, t^*)$.

Thus, we have bounded $\phi(0, t^*)$ in terms of data.

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YAN LIU
DEPARTMENT OF APPLIED MATHEMATICS
GUANGDONG UNIVERSITY OF FINANCE
GUANGZHOU 510521, P. R. CHINA
E-mail address: liuyan99021324@tom.com

WENHUI LIAO
DEPARTMENT OF APPLIED MATHEMATICS
GUANGDONG UNIVERSITY OF FINANCE
GUANGZHOU 510521, P. R. CHINA
E-mail address: liaowenhui526@sina.com

CHANGHAO LIN
SCHOOL OF MATHEMATICAL SCIENCES
SOUTH CHINA NORMAL UNIVERSITY
GUANGZHOU 510631, P. R. CHINA
E-mail address: linchh@scnu.edu.cn