LOCI OF RATIONAL CURVES OF SMALL DEGREE ON THE MODULI SPACE OF VECTOR BUNDLES

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ABSTRACT. For a smooth algebraic curve C of genus $g \ge 4$, let $SU_C(r, d)$ be the moduli space of semistable bundles of rank $r \ge 2$ over C with fixed determinant of degree d. When (r, d) = 1, it is known that $SU_C(r, d)$ is a smooth Fano variety of Picard number 1, whose rational curves passing through a general point have degree $\ge r$ with respect to the ample generator of $Pic(SU_C(r, d))$. In this paper, we study the locus swept out by the rational curves on $SU_C(r, d)$ of degree < r. As a by-product, we present another proof of Torelli theorem on $SU_C(r, d)$.

1. Introduction

Let C be a smooth algebraic curve over \mathbb{C} of genus $g \geq 4$. Let $\mathcal{M} := SU_C(r,d)$ be the moduli space of semistable bundles of rank $r \geq 2$ over C with fixed determinant of degree d. Throughout this paper, we assume (r,d) = 1. In this case, it is known that \mathcal{M} is a smooth Fano variety of Picard number 1. Hence it is an important project to study the rational curves on \mathcal{M} using the modular properties.

In general, for a Fano manifold N of Picard number 1, the *index* of N is defined by the number i such that $-K_N \cong \mathcal{O}_N(i)$, where $\mathcal{O}_N(1)$ is the ample generator of Pic(N). In analogy with the case of projective hypersurfaces, a rational curve $l \subset N$ is called a *line on* N if the index of N equals $-K_N \cdot l$. Also we say that l has degree k if $-K_N \cdot l$ equals k times the index of N. It is an open question if every Fano manifold of Picard number 1 has a line ([3], p. 248).

As a Fano manifold, \mathcal{M} has index 2 ([1]). Ramanan [7] found a family of lines on \mathcal{M} , but they sweep out a sublocus of \mathcal{M} of large codimension. It has also been observed that \mathcal{M} is covered by rational curves of degree r, which are called *Hecke curves* ([6], Corollary 5.16). Hwang ([2], Question 1) asked if the Hecke

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curves are the rational curves of minimal degree passing through a general point of \mathcal{M} . This was recently answered affirmatively by Sun ([9], Theorem 1). As a corollary, he gave another proof of nonabelian Torelli theorem. He also proved that there are no lines on \mathcal{M} different from those found by Ramanan ([9], Theorem 2).

The aim of this paper is to get more detailed information on the rational curves on \mathcal{M} of degree between 1 and r. In particular, we show that the locus inside \mathcal{M} swept out by the rational curves of degree < r, consists of r - 1 irreducible components, each of which comes from certain extensions of fixed type. We also study the intersections of these components. As a by-product, we get another way to recover the curve C from the moduli space $\mathcal{M} = SU_C(r, d)$.

This paper is organized as follows. In $\S2$, we study certain extension spaces parameterizing the bundles which are not (1, 1)-stable. It will turn out that these spaces provide the locus of rational curves of small degree.

In §3, we study the intersections of the images of the extension spaces inside \mathcal{M} . In most cases, their intersection is empty, but there are special pairs whose intersection property can be described in terms of certain scroll over C.

In §4, we first briefly review Sun's results on rational curves on \mathcal{M} . Combining with the informations from §2 and §3, we prove in Proposition 4.2 that for every ε with $1 \le \varepsilon \le r-1$, the locus in \mathcal{M} swept out by the rational curves of degree $\le \varepsilon$ consists of ε irreducible components. Finally in Proposition 4.5, we observe that these results on the loci of the rational curves of small degree provide us another proof of Torelli theorem.

2. Extension spaces

From now on, we fix a curve C of genus $g \ge 4$ and simplify our notation so that $U(\rho, \delta)$ denotes the moduli space of semistable bundles over C of rank ρ and degree δ . Let $U^s(\rho, \delta)$ be the open subset of $U(\rho, \delta)$ consisting of stable bundles. Also for a line bundle Λ on C, let $SU(\rho, \Lambda)$ be the moduli space of semistable bundles over C of rank ρ with determinant Λ .

Definition 2.1. (1) For each $0 < \rho' < \rho$ and $W \in U^s(\rho, \delta)$, define

$$s_{\rho'}(W) := \min\{\rho'\delta - \rho \deg(W')\},\$$

where the minimum is taken over the subbundles W' of W of rank ρ' .

(2) For any fixed integers $s \ge 0$ and ρ' , we define

$$U_{\rho',s}(\rho,\delta) := \{ W \in U^s(\rho,\delta) : s_{\rho'}(W) > s \}.$$

(3) We say that a bundle $W \in U(\rho, \delta)$ is Lange-stable if $W \in U_{\rho',s}(\rho, \delta)$ where $s = \rho'(\rho - \rho')(g - 1) - 1$ for every ρ' .

(4) A bundle $W \in U(\rho, \delta)$ is called (1, 1)-stable if $W \in U_{\rho',\rho}(\rho, \delta)$ for every ρ' (cf. [6], Definition 5.1).

Lemma 2.2. (1) For each ρ' , if $s < \rho'(\rho - \rho')(g - 1)$, then $U_{\rho',s}(\rho, \delta)$ and $U_{\rho',s}(\rho, \delta) \cap SU(\rho, \Lambda)$ are nonempty open subsets of $U(\rho, \delta)$ and $SU(\rho, \Lambda)$ respectively.

(2) A general bundle $W \in SU(\rho, \Lambda)$ is Lange-stable and (1, 1)-stable.

Remark 2.3. The following inequalities on the slope $\mu(W) := \deg(W)/\operatorname{rk}(W)$ will be used in the proof below.

(1) For $s = \rho'(\rho - \rho')(g - 1) - 1$, the inequality defining $U_{\rho',s}(\rho, \delta)$ implies

$$\mu(W) - \mu(W') \ge 3(\rho - \rho')/\rho,$$

since $g \ge 4$. Also, we note that $2(\rho - \rho')/\rho \ge 1/\rho'$ for any $1 \le \rho' < \rho$. (2) The inequality defining (1,1)-stability is equivalent to $\mu(W) - \mu(W') > 1/\rho'$.

Proof. (1) By dimension count, one can show that a general bundle $W \in U(\rho, \delta)$ satisfies

$$s_{\rho'}(E) \ge \rho'(\rho - \rho')(g - 1),$$

see [4], p. 448. The statement for $SU(\rho, \Lambda)$ follows from the observation that for every line bundle L of degree 0, we have $s_{\rho'}(W) = s_{\rho'}(W \otimes L)$.

(2) The statement on Lange-stability is an immediate consequence of (1). Also by Remark 2.3, Lange-stability implies (1, 1)-stability.

Now we consider the bundles $V \in \mathcal{M} = SU(r, d)$ which are not (1, 1)-stable, or equivalently, the bundles which admit a subbundle V_1 such that

$$\operatorname{rk}(V_1)d - r \operatorname{deg}(V_1) \le r.$$

Here the equality cannot hold since (r, d) = 1. For each integer $0 < \varepsilon < r$, let r_1 and d_1 be the integers satisfying $0 < r_1 < r$ and $r_1d - rd_1 = \varepsilon$. Let $r_2 = r - r_1$ and $d_2 = d - d_1$. For $V_1 \in U(r_1, d_1)$ and $V_2 \in U(r_2, d_2)$, consider those bundles V associated with nontrivial extensions

$$(2.1) 0 \to V_1 \to V \to V_2 \to 0.$$

Note the following identities: for $0 < r_1, r_2, \varepsilon < r$,

(2.2)
$$r_1d - rd_1 = rd_2 - r_2d = r_1d_2 - r_2d_1 = \varepsilon.$$

We have the following which generalizes and refines [7], Lemma 2.1 and [9], Lemma 3.1.

Lemma 2.4. Suppose that V_1 and V_2 in the nontrivial extension (2.1) are Lange-stable. Then for a proper subbundle V' of V of rank r', the inequality

$$\mu(V) - \mu(V') < 1/r'$$

implies that $\operatorname{rk}(V') = r_2, \operatorname{deg}(V') = d_2 - 1$, and the composition $V' \to V \to V_2$ is a sheaf injection, unless V' coincides with the subbundle V_1 . In particular, $V \in U_{r',r}(r,d)$ for each $r' \neq r_1, r_2$. *Proof.* We need to show that

$$\mu(V) - \mu(V') > 1/r'$$

for every subbundle V' of rank r' and degree d' if V' is neither V_1 nor a bundle of rank r_2 with the specified properties. This can be checked case by case. Let V'_2 be the image of the composition $V' \to V \to V_2$ and let V'_1 be the kernel of $V' \to V'_2$. Let r'_1 and d'_1 (resp. r'_2 and d'_2) be the rank an degree of V'_1 (resp. V'_2).

(i) First assume $V'_2 = 0$ and $V' = V'_1 \neq V_1$. Since V' is a subbundle of V, $r' = r'_1 < r_1$. Hence by the identity (2.2) and the (1, 1)-stability of V_1 , we get

$$\mu(V) - \mu(V') = \mu(V) - \mu(V_1) + \mu(V_1) - \mu(V')$$

> $\frac{\varepsilon}{rr_1} + \frac{1}{r'_1} > \frac{1}{r'_1} = \frac{1}{r'}.$

(ii) Next assume $V'_1 = 0$ and $V' = V'_2 \neq V_2$. If $r'_2 = r_2$, then $d'_2 < d_2$ and

$$\mu(V) - \mu(V') = \frac{r_2 d - rd'_2}{rr_2} = \frac{r(d_2 - d'_2) - \varepsilon}{rr_2}.$$

Thus $\mu(V) - \mu(V') > 1/r_2 = 1/r'$ if and only if $d_2 - d'_2 \ge 2$. If $r'_2 < r_2$, by the similar argument as (i), we get $\mu(V) - \mu(V') > 1/r'_2 = 1/r'$. (iii) Now assume $0 \neq V'_1 \neq V_1$ and $V'_2 = V_2$. By Remark 2.3 (1) and (2.2),

(iii) Now assume $0 \neq V'_1 \neq V_1$ and $V'_2 = V_2$. By Remark 2.3 (1) and (2.2), we get

$$\mu(V'_1) = (\mu(V'_1) - \mu(V_1)) + \mu(V_1)$$

$$\leq -\frac{3(r_1 - r'_1)}{r_1} + \mu(V) - \frac{\varepsilon}{rr_1}.$$

Thus,

$$\begin{aligned} r'\mu(V') &= & \mu(V_1')r_1' + \mu(V_2)r_2 \\ &\leq & \left(\mu(V) - \frac{3(r_1 - r_1')}{r_1} - \frac{\varepsilon}{rr_1}\right)r_1' + \left(\mu(V) + \frac{\varepsilon}{rr_2}\right)r_2 \\ &= & r'\mu(V) - \frac{3(r_1 - r_1')}{r_1} \cdot r_1' - \frac{r_1'\varepsilon}{rr_1} + \frac{\varepsilon}{r} \\ &< & r'\mu(V) - \frac{2(r_1 - r_1')}{r_1} \cdot r_1' - \frac{(r_1 - r_1')}{r_1} \cdot r_1' + \frac{(r_1 - r_1')}{r_1} \\ &\leq & r'\mu(V) - \frac{2(r_1 - r_1')}{r_1} \cdot r_1' \\ &\leq & r'\mu(V) - 1. \end{aligned}$$

(iv) The remaining cases $(V'_1 = V_1, \ 0 \neq V'_2 \neq V_2)$ and $(0 \neq V'_1 \neq V_1, \ 0 \neq V'_2 \neq V_2)$ can be checked similarly.

For a fixed line bundle $\Lambda \in Pic^d(C)$, consider the subvariety of $U(r_1, d_1) \times U(r_2, d_2)$ defined by

$$\mathcal{D}_{\varepsilon} := \{ (V_1, V_2) : \det V_1 \otimes \det V_2 \cong \Lambda \},\$$

where r_1, d_1 and r_2, d_2 are determined by ε and the identities (2.2). Let $P(V_1, V_2) := \mathbb{P}H^1(C, V_2^* \otimes V_1)$ be the projective space parameterizing the non-trivial extensions up to nonzero scalars.

Lemma 2.5. If $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$ is general, then the classifying map

$$\Phi: P(V_1, V_2) \to SU(r, \Lambda)$$

is an injective morphism.

Proof. We assume that V_1 and V_2 are Lange-stable. By Lemma 2.4, every bundle V associated to a point of $P(V_1, V_2)$ is stable. We claim that there exists a unique injection $V_1 \to V$ up to homothety, which implies that Φ is an injective morphism. When $r_1 \neq r_2$, this was already shown in Lemma 2.4.

Now assume $r_1 = r_2$ and suppose that there are two different subbundle maps, say $\alpha, \beta : V_1 \to V$, where α induces the original exact sequence

$$0 \to V_1 \stackrel{\alpha}{\to} V \to V_2 \to 0.$$

Applying Lemma 2.4 for $V' = (\beta : V_1 \to V)$, we deduce that $d_1 = d_2 - 1$ and the composition $V_1 \xrightarrow{\beta} V \to V_2$ is a sheaf injection. This implies that the induced map $\alpha \oplus \beta : V_1 \oplus V_1 \to V$ is also a sheaf injection, whose quotient should be a skyscraper sheaf \mathbb{C}_x for some $x \in C$. Hence for some $\lambda \in \mathbb{C}$, the map given by the composition

$$V_1 \stackrel{(id,\lambda \cdot id)}{\longrightarrow} V_1 \oplus V_1 \stackrel{\alpha \oplus \beta}{\longrightarrow} V$$

has rank $r_1 - 1$ at x. Taking saturation, we get a nonzero map $\widetilde{V_1} \to V$, where deg $\widetilde{V_1} = d_1 + 1$. This contradicts to the stability of V (The final step of this argument appeared in the proof of [6] Lemma 5.6, which was concerned with the case of rank 2).

Definition 2.6. For each integer ε with $0 < \varepsilon < r$, let R_{ε} be the subset of $\mathcal{M} = SU(r, d)$ consisting of the bundles which admit a subbundle of rank r_1 and degree d_1 with $r_1d - rd_1 = \varepsilon$.

Note that $V \in \mathcal{M}$ is not (1,1)-stable if and only if $V \in R_{\varepsilon}$ for some ε .

Lemma 2.7. For each $0 < \varepsilon < r$, R_{ε} is nonempty and irreducible. A general $V \in R_{\varepsilon}$ is contained in the image of $P(V_1, V_2)$ for some general $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$.

Proof. The nonemptyness of R_{ε} follows from Lemma 2.4. The remaining statements follow from a more general statement proven by Teixidor i Bigas (see [8], Proposition 1.6). In fact, the case of U(r, d) was worked out there, but the same argument applies to SU(r, d).

3. Scrolls

Now we study the intersection of R_i and R_j inside \mathcal{M} for $0 < i \neq j < r$.

Lemma 3.1. Consider $(V_1, V_2) \in \mathcal{D}_i$, where V_1 and V_2 are Lange-stable. If $i + j \neq r$, then the image of $P(V_1, V_2)$ inside \mathcal{M} does not intersect R_j .

Proof. Consider $V \in P(V_1, V_2)$. By Lemma 2.4, there are at most two subbundles, V_1 and W_1 , which violate the inequality for the (1, 1)-stability of V. Also, $\operatorname{rk}(W_1) = r - \operatorname{rk}(V_1)$ and $\operatorname{deg}(W_1) = d - \operatorname{deg}(V_1) - 1$. If $V \in R_j$, this means that V admits a subbundle W_1 such that $j = \operatorname{rk}(W_1)d - r \operatorname{deg}(W_1)$. Hence from the identity (2.2) defining i, we see that i + j = r.

To study the case i + j = r in more detail, we need first to consider a scroll in $P(V_1, V_2)$. For $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$, $0 < \varepsilon < r$, consider the rational map

$$\Upsilon: \mathbb{P}\mathrm{Hom}(V_2, V_1) \dashrightarrow P(V_1, V_2)$$

defined by the complete linear system

$$P(V_1, V_2) \cong \mathbb{P}H^0(C, K_C \otimes V_1^* \otimes V_2)^{\vee}$$

$$\cong \mathbb{P}H^0(\mathbb{P}\mathrm{Hom}(V_2, V_1), p^*K_C \otimes \mathcal{O}(1))^{\vee},$$

where $p : \mathbb{P}\text{Hom}(V_2, V_1) \to C$ is the projection.

Lemma 3.2. Assume $g \ge 4$. If $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$ is general. Then the map Υ is an embedding.

Proof. For any $x \in C$, the line bundle $p^*K_C \otimes \mathcal{O}(1)$ restricted to the fiber $p^{-1}(x)$ is very ample. Thus it is very ample on $\mathbb{P}\text{Hom}(V_2, V_1)$ if for every $x, y \in C$, the case x = y included, the restriction map

$$H^{0}(\mathbb{P}\mathrm{Hom}(V_{2}, V_{1}), p^{*}K_{C} \otimes \mathcal{O}(1)) \to H^{0}(\pi^{-1}(x+y), p^{*}K_{C} \otimes \mathcal{O}(1)|_{\pi^{-1}(x+y)})$$

is surjective. By the projection formula, this is guaranteed if for any $x, y \in C$,

 $h^{0}(C, K_{C} \otimes V_{1}^{*} \otimes V_{2}) - h^{0}(C, K_{C}(-x-y) \otimes V_{1}^{*} \otimes V_{2}) = 2r_{1}r_{2}.$

Since $V_1^* \otimes V_2$ is semistable, this is equivalent to the vanishing of $h^0(C, V_2^* \otimes V_1(x+y))$. In the below, we will show this vanishing.

If r = 2, then V_1 and V_2 are line bundles and $\varepsilon = 1$. In this case, $h^0(C, V_2^* \otimes V_1(x+y)) > 0$ for some x, y if and only if $V_1^* \otimes V_2 \in C + C - C$ in $Pic^1(C)$. Since $g \ge 4$, we can take $V_1^* \otimes V_2$ outside the locus C + C - C.

If $r \ge 3$, we may use the "twisted Brill-Noether for one section" proven by Russo and Teixidor i Bigas([8], Theorem 0.3):

For a general $G \in U(r_G, d_G)$, the locus of the bundles H inside $U(r_H, d_H)$ such that $h^0(H^* \otimes G) > 0$ has dimension

$$\gamma := r_H (r_H - r_G)(g - 1) + r_H d_G - r_G d_H,$$

if it is nonempty.

First assume that $r_2 > 1$ and apply this to $G = V_1(x + y)$ and $H = V_2$: for a fixed $G \in U(r_1, d_1 + 2r_1)$, consider the locus of the bundles $H \in SU(r_2, d_2)$ admitting a nonzero map $H \to G$. Since we may choose det H general by varying det G, we can reduce the above dimension γ to $\gamma - g$. We get

$$\gamma - g = r_2(r_2 - r_1)(g - 1) + r_2(d_1 + 2r_1) - r_1d_2 - g$$

$$= r_2^2(g-1) - r_1r_2(g-3) - \varepsilon - g$$

and it can be checked that

$$\gamma - g < (r_2^2 - 1)(g - 1) - 2 = \dim SU(r_2, d_2) - 2$$

This implies that for a general choice of $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$, we get the vanishing of $h^0(C, V_2^* \otimes V_1(x+y))$ for any $x, y \in C$.

If $r_2 = 1$, then $r_1 > 1$ and the same argument works through for $G = V_2^*(x+y)$ and $H = V_1^*$.

Hence for a general $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$, we have a scroll $\Upsilon(\mathbb{P}\text{Hom}(V_2, V_1))$ inside $P(V_1, V_2)$. This locus provides a criteria on the existence of a subbundle V' of rank r_2 in Lemma 2.4.

Lemma 3.3. Assume $g \ge 4$. Suppose that V_1 and V_2 in the nontrivial extension (2.1) are Lange-stable. Then V has a subbundle $V' \ne V_1$ of rank r_2 and degree $d_2 - 1$ if and only if the associated point $v \in P(V_1, V_2)$ lies on the scroll $\Upsilon(\mathbb{P}Hom(V_2, V_1))$.

Proof. By Lemma 2.4, the subbundle $V' \neq V_1$ of rank r_2 and degree $d_2 - 1$ induces a sheaf injection $V'(\to V) \to V_2$. In other words, there is an elementary transformation

$$0 \to V' \to V_2 \to \mathbb{C}_x \to 0$$

for some $x \in C$ which lifts to $V \in H^1(C, V_2^* \otimes V_1)$. This is equivalent to that $v \in \mathbb{P} \ker [H^1(C, V_2^* \otimes V_1) \to H^1(C, (V')^* \otimes V_1)]$

for the induced map on the extension spaces. By definition of Υ , this is equivalent to that v lies on the image of the fiber $\mathbb{P}\text{Hom}(V_2, V_1)|_x$.

Summarizing the results in this section, we get the following.

Proposition 3.4. For a general $(V_1, V_2) \in \mathcal{D}_i$ where 0 < i < r, let P be the image of $P(V_1, V_2)$ inside \mathcal{M} . Let $0 < j \neq i < r$.

(1) If $i + j \neq r$, then $P \cap R_j$ is empty.

(2) If i + j = r, then $P \cap R_j$ is the image of the scroll

 $\Upsilon(\mathbb{P}\mathrm{Hom}(V_2, V_1)) \subset P(V_1, V_2).$

(3) In particular, R_i is not contained in R_j .

4. Rational curves

First we recall Sun's results in [9] on rational curves on \mathcal{M} .

Let $\phi : \mathbb{P}^1 \to \mathcal{M}$ be a rational curve. Let $X = C \times \mathbb{P}^1$ together with the projections $f : X \to C$ and $\pi : X \to \mathbb{P}^1$. Since \mathcal{M} admits a universal bundle, there is a bundle E on X such that $\phi(t) = E|_{C \times \{t\}}$ for each $t \in \mathbb{P}^1$. Its restriction to a general fiber $f^{-1}(\xi) = X_{\xi} \cong \mathbb{P}^1$ is of the form

$$E|_{X_{\xi}} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{X_{\xi}}(\alpha_1)^{\oplus r_i}, \quad \alpha_1 > \dots > \alpha_n.$$

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Tensoring E by $\pi^* \mathcal{O}(-\alpha_n)$, we may assume that $\alpha_n = 0$ in this generic splitting type of E. Then E admits a Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the $F_i = E_i/E_{i-1}$ are torsion free with generic splitting type $(\mathcal{O}(\alpha_i)^{\oplus r_i})$ for each *i*. Let $F'_i = F_i \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-\alpha_i)$ so that they have generic splitting type $(\mathcal{O}^{\oplus r_i})$. Let deg (E_i) denote the degree of E_i on the generic fiber of π .

Lemma 4.1 ([9], (2.1), (2.2), Lemma 2.2). (1) For the ample generator $\mathcal{L}_{\mathcal{M}}$ of $Pic(\mathcal{M})$, we have (4.1)

$$\deg \phi^*(\mathcal{L}_{\mathcal{M}}) = r \sum_{i=1}^n c_2(F'_i) + \sum_{i=1}^{n-1} [rk(E_i) \deg(E) - rk(E) \deg(E_i)] (\alpha_i - \alpha_{i+1}).$$

(2) Any torsion free sheaf \mathcal{E} on X with generic splitting type $(\mathcal{O}^{\oplus r_i})$ must have $c_2(\mathcal{E}) \geq 0$. Also, $c_2(\mathcal{E}) = 0$ if and only if $\mathcal{E} = f^*W$ where W is a locally free sheaf on C.

Combining the lemma with our discussion in previous sections, we get the following.

Proposition 4.2. Assume $g \ge 4$, let $0 < \varepsilon < r$.

(1) For a general $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$, consider the morphism $P(V_1, V_2) \to \mathcal{M}$. For every line on $P(V_1, V_2)$, its image inside \mathcal{M} has degree ε .

(2) Any smooth rational curve ϕ on \mathcal{M} of degree ε which passes through a general point of R_{ε} is an image of a line on $P(V_1, V_2)$ for some $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$.

(3) For each ε , the locus swept out by the rational curves on \mathcal{M} of degree $\leq \varepsilon$ consists of ε irreducible components whose closures are given by $R_1, \ldots, R_{\varepsilon}$.

Proof. (1) This is immediate from (4.1).

(2) By Lemma 2.7, a general point V of R_{ε} is contained in the image of $P(V_1, V_2)$ for some general $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$. By Lemma 2.4 and Lemma 3.3, V_1 is the unique subbundle of V violating the inequality for the (1, 1)-stability of V.

Now assume that ϕ passes through $V \in R_{\varepsilon}$. By (4.1), deg $\phi = \varepsilon < r$ implies that n = 2, $c_2(F'_i) = 0$, and $\alpha_1 = 1$. Also by Lemma 4.1 (2),

$$F_1 = E_1 = f^*(V_1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$$
 and $F_2 = E_2/E_1 = f^*(V_2)$.

Hence we get the exact sequence

$$0 \to f^*(V_1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*(V_2) \to 0.$$

This shows that ϕ is an image of a line on $P(V_1, V_2)$.

(3) By Proposition 3.4(3), R_i is not contained in R_j if $j \neq i$. The images of $P(V_1, V_2)$ for $(V_1, V_2) \in \mathcal{D}_{\varepsilon}$ are dense in R_{ε} by Lemma 2.7. By Proposition 4.2(1), the images of the lines on $P(V_1, V_2)$ are rational curves of degree ε . Hence it suffices to show that any rational curve ϕ of degree ε is contained in $R_1 \cup \cdots \cup R_{\varepsilon}$. Indeed, by the formula (4.1), every point on a curve ϕ of

degree $\varepsilon < r$ corresponds to a bundle admitting a subbundle V_1 such that $\operatorname{rk}(V_1)d - r \operatorname{deg}(V_1) \leq \varepsilon$.

For $\varepsilon = 1$, it is easy to see that there are a projective bundle $\mathcal{P} \to \mathcal{D}_1$ whose fiber at (V_1, V_2) is $P(V_1, V_2)$, and a universal bundle $U \to C \times \mathcal{P}$ which induces a rational map $\mathcal{P} \to \mathcal{M}$. In this case, a stronger version of Proposition 4.2(2) has been proven.

Proposition 4.3. (1) ([9], Theorem 2) The classifying map $\mathcal{P} \to \mathcal{M}$ is a morphism whose image coincides with R_1 . And every line on \mathcal{M} is an image of a line on $\mathcal{P}(V_1, V_2)$ for some $(V_1, V_2) \in \mathcal{D}_1$.

(2) ([5], §4) Moreover, $\Phi : P(V_1, V_2) \to \mathcal{M}$ is an embedding for a general $(V_1, V_2) \in \mathcal{D}_1$.

Consider a general $(V_1, V_2) \in \mathcal{D}_1$ so that $P(V_1, V_2)$ is embedded in \mathcal{M} . We denote the image by P. By formula (4.1), $\mathcal{L}_{\mathcal{M}}|_P \cong \mathcal{O}_P(1)$. The converse also holds.

Lemma 4.4. Assume $g \geq 4$. Let P be a subvariety of \mathcal{M} passing through a general point of R_1 such that $P \cong \mathbb{P}^k$ for some k and $\mathcal{L}_{\mathcal{M}}|_P \cong \mathcal{O}_P(1)$. Then P is the image of a k-dimensional linear subspace of $P(V_1, V_2)$ for some $(V_1, V_2) \in \mathcal{D}_1$.

Proof. As before, we may assume that P passes through an image of the point $V \in P(V_1, V_2)$ for a general $(V_1, V_2) \in \mathcal{D}_1$, such that V_1 is the unique subbundle violating the (1, 1)-stability of V. The proof of Proposition 4.2(2) shows that every line on P passing through V is an image of a line on $P(V_1, V_2)$. Hence P is contained in the image of $P(V_1, V_2)$.

Finally we give another proof of nonabelian Torelli theorem based on the above discussions. As was mentioned in §1, Sun [9] found a way to recover the curve C from the geometry of rational curves on $\mathcal{M} = SU(r, d)$ of degree r passing through a general point of \mathcal{M} . In the following, we give a way to recover C from the geometry of rational curves on \mathcal{M} of degree 1 and r-1, which sweep out a small sublocus of \mathcal{M} .

Proposition 4.5. Let C_1 and C_2 be smooth algebraic curves of genus $g \ge 4$. If $SU_{C_1}(r, d) \cong SU_{C_2}(r, d)$, then $C_1 \cong C_2$.

Proof. By Proposition 4.2(3), each irreducible variety R_{ε} , $0 < \varepsilon < r$, is characterized in terms of the rational curves of degree $\leq \varepsilon$. More precisely, R_{ε} is the closure of $R_{\varepsilon}^{\circ} \setminus R_{\varepsilon-1}^{\circ}$, where R_{k}° is the sublocus swept out by the rational curves of degree $\leq k$.

By Lemma 4.4, we can pick out a subvariety P of R_1 which is the image of $P(V_1, V_2)$ for a general $(V_1, V_2) \in \mathcal{D}_1$. Note that the composition $\mathbb{P}\text{Hom}(V_2, V_1) \to P(V_1, V_2) \to \mathcal{M}$ is an embedding by Lemma 3.2 and Proposition 4.3(2). If r > 2, then the locus $P \cap R_{r-1}$ is isomorphic to a scroll over Cby Proposition 3.4(2). Now assume r = 2. We claim that the locus of points $v \in P$ such that $v \in l$ for a line l on \mathcal{M} which is not contained in P, is isomorphic to C. Indeed, if there is a line l on \mathcal{M} which is not contained in P such that $v \in l \cap P$, then the associated bundle V admits two different (line) subbundles V_1 and V'_1 of degree $d_1(=d_2-1)$ by Proposition 4.3(1), and vice versa. By Lemma 3.3, the locus of such bundles inside P is the image of $\mathbb{P}\text{Hom}(V_2, V_1) \cong C$.

In this way, we can recover C from $\mathcal{M} = SU_C(r, d)$ using the properties of \mathcal{M} as a projective variety with $Pic(\mathcal{M}) \cong \mathbb{Z}$.

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