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SEMI-DIVISORIALITY OF HOM-MODULES AND INJECTIVE COGENERATOR OF A QUOTIENT CATEGORY

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Abstract. In this paper, we study w-nullity and (co-)semi-divisoriality of Hom-modules and the semi-divisorial envelope of $\mathsf{Hom}_R(M,N)$ under suitable conditions on R, M, and N. We also investigate an injective cogenerator of a quotient category.

1. Introduction

Let R be an integral domain. In [17] Wang and McCasland defined semi-divisorial closure, or w-closure for torsion-free R-modules. In [7], H. Kim extended this notion to any R-module and introduced and studied the related notions of co-semi-divisoriality and w-nullity. In [7, 8, 9] these concepts were then used to give new module-theoretic characterizations of t-linkative domains, generalized GCD domains, and strong Mori domains, classes of domains widely considered in multiplicative ideal theory.

Earlier, in [1, 12, 13], Beck, Nishi and Shinagawa investigated injective modules over a Krull domain in terms of co-divisorial modules, pseudo-null modules, and divisorial modules and investigated pseudo-nullity and (co-)divisoriality of Home-modules. In particular, it was shown that in the case of a Krull domain R with quotient field K, the injective envelope $E(K/R)$ of K/R is a cogenerator of the quotient category $\mathsf{Mod}(R)/\mathscr{M}_0$, where $\mathsf{Mod}(R)$ is the category of all unitary R-modules and $\mathscr{M}_0$ is the thick subcategory of the modules with trivial maps into the codivisorial modules. Recently, in [11] Mouçouf characterized the rings of Krull type R with quotient field K such that the (canonical) functorial image of $E(K/R)$ is an injective cogenerator of the quotient category $\mathsf{Mod}(R)/\mathscr{M}_0$. Also in [16], Wang investigated the case when Hom-modules are semi-divisorial in torsion-free.

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In this paper, we study an injective cogenerator of a quotient category and w-nullity and (co-)semi-divisoriality of Hom-modules using methods developed in [1, 11, 12, 13]. As a corollary, for the class of completely integrally closed domains, we characterize Krull domains in terms of an injective cogenerator of a quotient category. We also investigate the semi-divisorial envelope of $\mathsf{Hom}_R(M, N)$ under suitable conditions on R, M, and N.

Throughout this paper, R denotes an integral domain with quotient field K. Let $\mathcal{F}(R)$ denote the set of nonzero fractional ideals of R. Recall that the function on $\mathcal{F}(R)$ defined by $A \mapsto (A^{-1})^{-1} = A_v$ is a star operation called the *v-operation*, where $A^{-1} = R :_K A = \{x \in K \mid xA \subseteq R\}$. An ideal J of R is called a *Glaz-Vasconcelos ideal* if J is a finitely generated ideal of R with $J^{-1} = R$. We abbreviate this as *GV-ideal*, denoted by $J \in \mathsf{GV}(R)$. Following [17], a torsion-free R module M is called a *w-module* if $Jx \subseteq M$ for $J \in \mathsf{GV}(R)$ and $x \in M \otimes K$ implies that $x \in M$, which is said to be semi-divisorial in [4]. For a torsion-free R-module M, Wang and McCasland defined the *w-envelope* of M in [17] as $M_w = \{x \in M \otimes K \mid Jx \subseteq M \text{ for some } J \in \mathsf{GV}(R)\}$. In particular, if I is a nonzero fractional ideal, then $I_w = \{x \in K \mid Jx \subseteq I \text{ for some } J \in \mathsf{GV}(R)\}$. The canonical map $I \mapsto I_w$ on $\mathcal{F}(R)$ is a star-operation, denoted w. It was shown in [17] that a prime ideal P of R is a w-ideal if and only if $P_w \neq R$. Therefore, all prime ideals contained in a proper w-ideal of R are also w-ideals. We denote by w-$\mathrm{Max}(R)$ the set of w-maximal ideals of R. It is also worth noting that w distributes over (finite) intersections [17, Proposition 2.5]. For unexplained terminology and notation, we refer to [2, 3, 14].

2. w-null and (co-)semi-divisorial Hom-modules

In [7], H. Kim introduced the notions of "co-semi-divisoriality" and "w-nullity" of a module as follows. Let M be a module over an integral domain R and let $\tau(M) := \{x \in M \mid (\mathcal{O}(x))_w = R\}$, where $\mathcal{O}(x) := (0 :_R x) = ann_R(x)$ is the order ideal of x. Then $\tau(M)$ is a submodule of M. M is said to be *co-semi-divisorial* (*resp., w-null*) if $\tau(M) = 0$ (resp., $\tau(M) = M$). Note that the notions of co-semi-divisoriality and w-nullity can be interpreted in terms of a suitable torsion theory [2, Proposition IX.6.2 and Proposition IX.6.4] (with $\mathcal{P} = w$-$\mathrm{Max}(R)$).

Let R be an integral domain, let $\mathscr{T}_\tau(R)$ denote the full subcategory of $\mathsf{Mod}(R)$ consisting of all modules M such that $M_P = 0$ for all $P \in w$-$\mathrm{Max}(R)$, and let $\mathscr{F}_\tau(R)$ denote the full subcategory of all R-modules M have no subobject other than zero belonging to $\mathscr{T}_\tau(R)$. Finally let $\mathscr{C}_\tau(R)$ be the full subcategory of $\mathsf{Mod}(R)$ consisting of all co-semi-divisorial and semi-divisorial R-modules.

In an abelian category $\mathcal{A}$, we have the following definitions:

(a) An injective object E is called an *injective cogenerator* if $\mathsf{Hom}_{\mathcal{A}}(M, E) \neq 0$ for every $M \in \mathcal{A}$ that is not a zero object.

(b) A nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ is said to be *thick* if, for each short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, M is an object of $\mathcal{C}$ if and only if L and N are objects of $\mathcal{C}$. It is also called a *Serre subcategory* of $\mathcal{A}$.

It is clear that $\mathscr{T}_\tau(R)$ is a thick subcategory of $\mathsf{Mod}(R)$. Then we can now consider the quotient category $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$ and the canonical functor $T : \mathsf{Mod}(R) \to \mathsf{Mod}(R)/\mathscr{T}_\tau(R)$.

As usual, we denote by $E(M)$ the injective envelope of an R-module M. The following result will be useful later on.

Proposition 2.1. *The following statements are equivalent for an R-module M.*

1. *M is co-semi-divisorial, i.e., $M \in \mathscr{F}_\tau(R)$.*
2. *$\mathcal{O}(x)$ is a w-ideal for every element $x \in M$.*
3. *$(\mathcal{O}(x))_w \neq R$ for every nonzero element $x \in M$.*
4. *$\mathsf{Hom}_R(N, M) = 0$ for every w-null R-module N.*
5. *$\mathsf{Hom}_R(N, E(M)) = 0$ for every w-null R-module N.*

Proof. The equivalences of (1), (2), (3), and (4) are given in [7, Proposition 2.6], while the equivalence of (1) and (5) follows from [6, Proposition 1.2]. □

Note from [17, Proposition 1.4] that the annihilator ideal of any submodule of a co-semi-divisorial module is a w-ideal. Recall from [1] that a module M is said to be *codivisorial* if the annihilator of every nonzero element of M is a divisorial ideal. Thus in a Krull domain, the notions of co-semi-divisoriality and codivisoriality are the same.

Recall from [16, Definition 4.5] that an R-module M is said to be *w-vanishing* if $M_P = 0$ for any maximal w-ideal P of R.

Proposition 2.2. *Let N be an R-module. Then the following statements are equivalent.*

1. *N is w-null, i.e., $M \in \mathscr{T}_\tau(R)$.*
2. *For each $x \in N$, $\mathcal{O}(x)$ is not contained in any maximal w-ideal.*
3. *N is w-vanishing.*
4. *There is a torsion-free R-module F with $N \cong F_w/F$.*
5. *$\mathsf{Hom}_R(N, E(M)) = 0$ for every co-semi-divisorial R-module M.*

Proof. The equivalences of (1), (2), (3), and (4) are given in [7, Proposition 9.3], while the equivalence of (1) and (5) follows from [6, Proposition 1.2]. □

Now we study w-nullity and (co-)semi-divisoriality of Hom-modules. It was shown in [7, Proposition 3.1] that an R-module M is co-semi-divisorial if and only if $\mathsf{Hom}_R(\mathcal{Z}(R), M) = 0$, where $\mathcal{Z}(R) := \bigoplus_{\{I \leq R \ \mid \ I_w = R\}} R/I$.

Proposition 2.3. *Let R be an integral domain and let M and N be R-modules. If M is co-semi-divisorial, then so is $\mathsf{Hom}_R(N, M)$.*

Proof. By [7, Proposition 2.6], it suffices to show that $\mathsf{Hom}_R(L, \mathsf{Hom}_R(N, M)) = 0$ for every w-null R-module L. But this follows from $\mathsf{Hom}_R(L, \mathsf{Hom}_R(N, M)) \cong \mathsf{Hom}_R(N, \mathsf{Hom}_R(L, M)) = 0$ since M is co-semi-divisorial. □

Proposition 2.4. *Let R be an integral domain and let M and N be any R-module. If M is w-null, then so is $\mathsf{Tor}_n^R(N, M)$ for all $n \geq 0$.*

Proof. First we consider the case $n = 0$. For every co-semi-divisorial R-module L we have $\mathsf{Hom}_R(N \otimes_R M, E(L)) \cong \mathsf{Hom}_R(N, \mathsf{Hom}_R(M, E(L))) = 0$ since M is w-null; therefore $N \otimes_R M$ is w-null by Proposition 2.2. For the case when $n \geq 1$, we consider a projective resolution of N:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to P_0 \to N \to 0.$$

Then, since each $P_i \otimes M$ is w-null, we can see that $\mathsf{Tor}_n^R(N, M)$ is w-null for every $n \geq 0$ by noting that the submodules and homomorphic images of w-null modules are also w-null. □

Now we recall some definitions from [7]: Let M be an R-module. Then $W(M) := \pi^{-1}(\tau(E(M)/M))$ is called the *semi-divisorial envelope* of M, where $\pi : E(M) \to E(M)/M$ is the canonical projection, M is said to be *semi-divisorial* if $W(M) = M$, and M is said to be *weakly w-flat* if $\mathsf{Tor}_1^R(\mathcal{Z}(R), M) = 0$. It is clear from the definition that every injective R-module is semi-divisorial. Let N be an R-module. Then we denote $\mathcal{U}_w(N) := \{L \mid L \text{ is a submodule of } N \text{ such that } (L :_R x)_w = R \text{ for every } x \in N\}$.

Proposition 2.5. *The following statements are equivalent for an R-module M.*

(1) *M is weakly w-flat.*
(2) *$M^\flat := \mathsf{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is semi-divisorial.*
(3) *$I \otimes_R M \to M$ is a monomorphism for all $I \in \mathcal{U}_w(R)$.*
(4) *$L \otimes_R M \to N \otimes_R M$ is a monomorphism for all $L \in \mathcal{U}_w(N)$.*

Proof. The equivalence of (1) and (2) is given in [7, Proposition 4.3], while the equivalences of (2), (3), and (4) are given in [14, IX, Exercise 25]. □

Let M be a semi-divisorial R-module and N be an R-module. Then it was shown in [7, Corollary 3.4] that if $\mathsf{Hom}_R(\mathsf{Tor}_1^R(\mathcal{Z}(R), N), M) = 0$, then $\mathsf{Hom}_R(N, M)$ is semi-divisorial.

Theorem 2.6. *Let R be an integral domain, M be a semi-divisorial R-module, and N be an R-module. Then $\mathsf{Hom}_R(N, M)$ is semi-divisorial if one of the following conditions is satisfied*;

(i) *M is co-semi-divisorial*;
(ii) *N is weakly w-flat.*

Proof. It suffices to show that $\mathsf{Hom}_R(\mathsf{Tor}_1^R(\mathcal{Z}(R), N), M) = 0$ by [7, Corollary 3.4].

(i) Note that R/I is w-null for every $I \in \mathcal{U}_w(R)$ ([7, Proposition 2.5]). Thus we have that $\mathsf{Tor}_1^R(R/I, N)$ is w-null for every $I \in \mathcal{U}_w(R)$. Now since Tor commutes with direct sums and w-nullity is closed under direct sums, we have $\mathsf{Tor}_1^R(\mathcal{Z}(R), N)$ is w-null. Therefore $\mathsf{Hom}_R(\mathsf{Tor}_1^R(\mathcal{Z}(R), N), M) = 0$ by the co-semi-divisoriality of M ([7, Proposition 2.6]).

(ii) This follows from the definition of "weakly w-flat". □

It was shown in [5, Proposition 2.2] that for a rank one flat ideal $I \subset K$, the endomorphism $\mathrm{End}_R(I)(= I : I)$ of I is semi-divisorial. We extend this result to any flat module in the following corollary. Note that flat R-modules are torsion-free (and so co-semi-divisorial) for every integral domain R.

Corollary 2.7. *Let R be an integral domain.*

1. *If M is a flat R-module, then $\mathrm{End}_R(M)$ is a semi-divisorial R-module.*
2. *If M is a co-semi-divisorial and semi-divisorial R-module, then so is $\mathrm{End}_R(M)$.*
3. *If M is co-semi-divisorial, then $M^* = \mathsf{Hom}_R(M, R)$ is semi-divisorial.*

3. Semi-divisorial equivalence

In this section, we investigate the semi-divisorial envelope of $\mathsf{Hom}_R(M, N)$ under suitable conditions on R, M, and N. To do so, we need some definitions and results.

Lemma 3.1 ([15, Proposition 1.1])**.** *Let R be an integral domain and let $L \to M \to N$ be an exact sequence of R-modules. If L and N are w-null, then so is M.*

Let M and N be R-modules and let $f : M \to N$ be an R-homomorphism. Then f is said to be *w-injective* (resp., *w-surjective*) if $\ker(f)$ (resp., $\mathrm{coker}(f)$) is w-null. And f is said to be *w-isomorphic* if f is both w-injective and w-surjective.

Lemma 3.2 ([15, Lemma 1.2])**.** *Let R be an integral domain and let $f : L \to M$ and $g : M \to N$ be homomorphisms of R-modules. If f and g are w-injective (resp., w-surjective or w-isomorphic), then so is gf.*

Theorem 3.3 ([7, Theorem 8.1])**.** *The following statements are equivalent for an integral domain R.*

1. *If an R-module M is injective, then so is $\tau(M)$.*
2. *$E(\tau(M)) = \tau(E(M))$ for any R-module M.*
3. *Let N be an essential extension of M. If M is w-null, then so is N.*
4. *Let $I \leq R$ such that $I_w \neq R$. Then $I :_R a$ is a w-ideal for some $a \in R \setminus I_w$.*
5. *If M is not w-null, then M has a nonzero co-semi-divisorial submodule.*
6. *If $I \leq R$, then there exists an ideal J of R such that $J_w = R$ and $I = I_w \cap J$.*

Recall that an integral domain R is said to be *pseudo-t-linkative* if R satisfies one of the equivalent conditions of Theorem 3.3.

Proposition 3.4. *Let R be a pseudo-t-linkative domain with quotient field $K(\neq R)$. Let $f : M \to N$ be a homomorphism of R-modules and $p : M \to M/\tau(M)$, $q : N \to N/\tau(N)$ be the canonical projections.*

(1) *There is a unique homomorphism $f_* : M/\tau(M) \to N/\tau(N)$ such that $f_*p = qf$.*
(2) *If f is w-injective, then f_* is injective, and if f is w-isomorphic, then so is f_*.*
(3) *If f is w-isomorphic and M is semi-divisorial, then f_* is an isomorphism.*

Proof. (1) The existence of f_* follows from [7, Proposition 2.8] and its uniqueness is clear.

(2) Suppose first that f is w-injective. Since $\tau(M) \subseteq f^{-1}(\tau(N))$, we have the following exact sequence

$$0 \to \ker(f) \to f^{-1}(\tau(N)) \to \tau(N).$$

This implies, by Lemma 3.1, that $f^{-1}(\tau(N))$ is w-null; therefore $\tau(M) = f^{-1}(\tau(N))$. Thus f_* must be injective. If, moreover, f is w-surjective, then $\mathrm{coker}(f)$ is w-null. Since the induced homomorphism of $\mathrm{coker}(f)$ to $\mathrm{coker}(f_*)$ is surjective, $\mathrm{coker}(f_*)$ must be w-null.

(3) Suppose that M is semi-divisorial. Then $M \cong \tau(M) \oplus M/\tau(M)$ by [7, Corollary 8.9], and hence $M/\tau(M)$ is also semi-divisorial. Now the assertion follows from [7, Corollary 5.3]. □

It was shown in [16, Proposition 2.1] that $\mathsf{Hom}_R(M, N) = \mathsf{Hom}_R(M_w, N)$ for a torsion-free R-module M and a w-module N. It follows from this result that w, as a functor from the category of all torsion-free R-modules to the category of all w-modules, is a reflector. The following result shows that the functor W is a reflector from the category $\mathscr{F}_\tau(R)$ to the category $\mathscr{C}_\tau(R)$. By the R-dual of an R-module M is meant the R-module $M^* = \mathsf{Hom}_R(M, R)$.

Proposition 3.5. *Let R be an integral domain and let M, N be R-modules. Let i be the canonical injection of M to $W(M)$. If N is co-semi-divisorial, then*

$$\mathsf{Hom}_R(i, W(N)) : \mathsf{Hom}_R(W(M), W(N)) \to \mathsf{Hom}_R(M, W(N))$$

is an isomorphism. In particular, we have $M^* = (W(M))^*$.

Proof. Since N is co-semi-divisorial, so is $W(N)$ by [7, Proposition 2.9]. On the other hand, $W(M)/M$ is w-null by the definition of a semi-divisorial envelope W. Therefore $\mathsf{Hom}(W(M)/M, W(N)) = 0$, which implies that $\mathsf{Hom}_R(i, W(N))$ is an injection. By [7, Proposition 3.2], we can see that $\mathsf{Hom}_R(i, W(N))$ is a surjection. □

Corollary 3.6. *Let R be a pseudo-t-linkative domain with quotient field $K(\neq R)$. Let $f : M \to N$ be a homomorphism of R-modules. Then there exists a unique homomorphism $T(f) : T(M) \to T(N)$ such that $T(f)i = jf$, where i (resp., j) is the canonical homomorphism of M (resp., N) to $T(M)$ (resp., $T(N)$). Moreover, if f is a w-isomorphism, then $T(f)$ is an isomorphism.*

Proof. The homomorphism f induces the homomorphism f_* of $M/\tau(M)$ to $N/\tau(N)$ by Proposition 3.4. Applying Proposition 3.5 to f_*, we can obtain a homomorphism $T(f) : T(M) \to T(N)$ such that $T(f)i = jf$.

It is easy to show that, similarly to the proof of Proposition 3.5, $\mathsf{Hom}(i,T(N))$ is an injection. This shows the uniqueness of $T(f)$.

Suppose now that f is a w-isomorphism. Then by Proposition 3.4, f_* is a w-isomorphism (f_* is necessarily injective). Since the canonical injection of $M/\tau(M)$ to $T(M)$ is an essential extension, $T(f)$ must be an injection. Since both f_* and the canonical injection of $N/\tau(N)$ to $T(N)$ are w-surjective, so is the composition of them by Lemma 3.2. We can conclude from this fact that $T(f)$ is a w-surjection. Since a w-isomorphism of co-semi-divisorial and semi-divisorial modules is an isomorphism by [7, Corollary 5.3], $T(f)$ must be an isomorphism. □

It was shown in [16, Proposition 2.3] that $(\mathsf{Hom}_R(M,N))_w = \mathsf{Hom}_R(M,N_w)$ for a torsion-free finitely generated R-module M and a torsion-free R-module N. As a corollary, Wang obtained that $(\mathrm{End}_R(M))_w = \mathrm{End}_R(M_w)$ for a torsion-free finitely generated R-module M ([16, Corollary 2.4]).

Theorem 3.7. *Let R be a pseudo-t-linkative domain. Let M and N be co-semi-divisorial R-modules. If M is a submodule of a finitely generated R-module L, then we have*

$$W(\mathsf{Hom}_R(M,N)) \cong \mathsf{Hom}_R(W(M),W(N)).$$

Proof. By Proposition 3.5, we have only to prove

$$W(\mathsf{Hom}_R(M,N)) \cong \mathsf{Hom}_R(M,W(N)).$$

Consider the following exact sequence

$$0 \to \mathsf{Hom}_R(M,N) \to \mathsf{Hom}_R(M,W(N)) \to \mathsf{Hom}_R(M,W(N)/N).$$

Since N is co-semi-divisorial, so is $W(N)$; thus, by Proposition 2.3, $\mathsf{Hom}_R(M,N)$ and $\mathsf{Hom}_R(M,W(N))$ are co-semi-divisorial. Also we have that $\mathsf{Hom}_R(M,W(N))$ is semi-divisorial by Theorem 2.6. Since a w-isomorphism of co-semi-divisorial modules is an essential extension, it suffices to show that $\mathsf{Hom}_R(M,W(N)/N)$ is w-null.

In general, for a submodule M_1 of a finitely generated R-module M_2 and a w-null R-module N_1, we will show that $\mathsf{Hom}_R(M_1,N_1)$ is w-null. Set $N_2 := E(N_1)$. Then N_2 is w-null by [7, Theorem 8.1], since R is pseudo-t-linkative. Let $\{x_1,\dots,x_n\}$ be a system of generators of M_2 and let $f \in \mathsf{Hom}_R(M_2,N_2)$. Then $\mathcal{O}(f) = \mathcal{O}(f(x_1)) \cap \cdots \cap \mathcal{O}(f(x_n))$. Since each $(\mathcal{O}(f(x_i)))_w = R$, we have $(\mathcal{O}(f))_w = R$ by the distributivity of the star-operation w over finite intersection. Hence $\mathsf{Hom}_R(M_2, N_2)$ is w-null. Therefore, $\mathsf{Hom}_R(M_1, N_2)$ is w-null, since it is a homomorphic image of $\mathsf{Hom}_R(M_2, N_2)$. Thus $\mathsf{Hom}_R(M_1, N_1)$ is w-null since it is isomorphic to a submodule of $\mathsf{Hom}_R(M_1, N_2)$. $\square$

Corollary 3.8. *Let R be a pseudo-t-linkative domain with quotient field $K(\neq R)$ and let M and N be co-semi-divisorial and semi-divisorial R-modules. If M is a submodule of a finitely generated R-module, then* $\mathsf{Hom}_R(M, N)$ *is semi-divisorial.*

Let M and N be an R-modules. We say that M is *semi-divisorially equivalent to* N if there exists a w-isomorphism of $W(M)$ to $W(N)$.

Proposition 3.9. *Let R be a pseudo-t-linkative domain with quotient field $K(\neq R)$. Let M and N be R-modules.*

(1) *M is semi-divisorially equivalent to N if and only if $W(M/\tau(M))$ is isomorphic to $W(N/\tau(N))$. In particular, the "semi-divisorial equivalence" is an equivalence relation.*
(2) *If M is w-isomorphic to N, then M is semi-divisorially equivalent to N.*

Proof. (1) The necessity follows from the facts that $W(M) \cong W(\tau(M)) \oplus W(M/\tau(M))$ and $W(N) \cong W(\tau(N)) \oplus W(N/\tau(N))$ by [7, Corollary 8.9] and $W(\tau(M))$ and $W(\tau(N))$ are w-null by [7, Theorem 8.1] since R is pseudo-t-linkative. The sufficiency follows from Proposition 3.4.

(2) The assertion follows immediately from Corollary 3.6. $\square$

4. Injective cogenerator of a quotient category

In this section, we generalize some results of [1, 11] related to an injective cogenerator in a quotient category. We recall from [4] that a domain R is said to be an *H-domain* if every ideal I of R with $I^{-1} = R$ is quasi-finite (i.e. $I^{-1} = J^{-1}$ for some finitely generated subideal J of I).

Theorem 4.1. *Let R be an H-domain with quotient field $K(\neq R)$, and let M be any R-module. Then M is w-null if and only if* $\mathsf{Hom}_R(M, E(K/R)) = 0$.

Proof. ($\Rightarrow$): This follows from Proposition 2.1 since $E(K/R)$ is co-semi-divisorial by [7, Corollary 2.11].

($\Leftarrow$): Suppose that M is not w-null and let $N = M/\tau(M)$. By Proposition 2.1 and [7, Proposition 2.8], there is a non-zero element of $x \in N$ such that $\mathcal{O}(x)$ is a proper w-ideal and hence $R : \mathcal{O}(x) \supsetneqq R$ (since R is an H-domain). Let $a \in R : \mathcal{O}(x) \setminus R$. Then $R :_R a \supset \mathcal{O}(x)$. Let $f : R \to K/R$ be the homomorphism defined by $f(b) = \overline{ab}$, where $\overline{ab}$ is the class of ab in K/R. Since $\ker(f) = R :_R a \supset \mathcal{O}(x)$, there is a non-zero homomorphism $g : R/\mathcal{O}(x) \to K/R$ such that $f = gp$, where p is the canonical projection of R to $R/\mathcal{O}(x)$. Let i be the canonical injection of $R/\mathcal{O}(x)(\cong Rx)$ to N. Then there is a non-zero homomorphism h of N to $E(K/R)$ such that $ig = hj$, and hence hq is a non-zero homomorphism of M to $E(K/R)$, where q is the canonical projection of M to N. □

Since $K/R \in \mathscr{F}_\tau(R)$, i.e., K/R has no subobject other than zero belonging to $\mathscr{T}_\tau(R)$, then $T(E(K/R))$ is the injective envelope of the object $T(K/R)$ of $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$.

Corollary 4.2. *If R is an H-domain, then $T(E(K/R))$ is an injective cogenerator in the quotient category $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$. Hence every co-semi-divisorial and semi-divisorial module over an H-domain can be embedded in an injective module.*

Proof. Let $T(N) \in \mathsf{Mod}(R)/\mathscr{T}_\tau(R)$ with $\mathsf{Hom}_{\mathsf{Mod}(R)/\mathscr{T}_\tau(R)}(T(N), T(E(K/R))) = 0$. Then by [11, Lemma 2.6] we have $\mathsf{Hom}_{\mathsf{Mod}(R)}(N, E(K/R)) = 0$, and by Theorem 4.1 we have $N \in \mathscr{T}_\tau(R)$, and then $T(N) = 0$. It is clearly seen that $T(E(K/R))$ is a cogenerator object of $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$. The last assertion follows from [14, Proposition I.6.6]. □

Lemma 4.3. *Let R be an integral domain, let $P \in w\text{-Max}(R)$, let M be a co-semi-divisorial R-module and let $f : R/P \to M$ a homomorphism. Then either $f \equiv 0$ or f is injective.*

Proof. Suppose that $f \not\equiv 0$ and let $f(\bar{1}) = x$. Then we have $x \in M$. Since M is co-semi-divisorial, then $\mathcal{O}(x)$ is a w-ideal, and since $x \neq 0$, there exists $Q \in w\text{-Max}(R)$ such that $\mathcal{O}(x) \subseteq Q$, but since $P \subseteq \mathcal{O}(x)$, we have $P \subseteq Q$ and hence $P = Q$, so $\mathcal{O}(x) = P$ and f is injective. □

We recall from [10, III.1.4] two facts related to $\mathscr{C}_\tau(R)$, $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$, and T.

(a) The subcategory $\mathscr{C}_\tau(R)$ of $\mathsf{Mod}(R)$ may be identified with $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$.

(b) Let M be an R-module. Then $T(M) = W(M/\tau(M))$.

Therefore, we have that $T(E(K/R)) = W(E(K/R)/\tau(E(K/R))) \cong E(K/R)$.

Theorem 4.4. *Let R be an integral domain with quotient field K satisfying $(R :_R x)_v = (R :_R x)$ for every $x \in K$. If $T(E(K/R))$ is an injective cogenerator in the quotient category $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$, then R is an H-domain.*

Proof. Note that if R satisfies that $(R :_R x)_v = (R :_R x)$ for every $x \in K$, then K/R is co-divisorial. Suppose that R is not an H-domain. Then by [17, Proposition 5.7] there exists a prime ideal P which is w-maximal but not a v-ideal. First we show that the module R/P can not be injected in $E(K/R)$. If this were not so, then the kernel of the composition $R \xrightarrow{\Pi} R/P \to E(R/K)$ is P, where Π is the canonical projection. Then by [1, Corollary 1.7] P is a v-ideal, which is a contradiction. Thus by Lemma 4.3, $\mathsf{Hom}_{\mathsf{Mod}(R)}(R/P, E(K/R)) = 0$. So $\mathsf{Hom}_{\mathsf{Mod}(R)/\mathscr{T}_\tau(R)}(T(R/P), T(E(K/R))) \cong \mathsf{Hom}_{\mathsf{Mod}(R)}(W(R/P), E(K/R)) \cong \mathsf{Hom}_{\mathsf{Mod}(R)}(R/P, E(K/R)) = 0$ (note that the last isomorphism follows from Proposition 3.5). Since $T(E(K/R))$ is a cogenerator object in $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$,

$T(R/P) = 0$, and thus $R/P \in \mathscr{T}_\tau(R)$, i.e., R/P is w-null. Hence $P_w = R$, which is a contradiction. Therefore R is an H-domain. □

It is well known that if R is a completely integrally closed domain, then R satisfies the hypothesis of Theorem 4.4. Now the following result follows from Corollary 4.2, Theorem 4.4, and the fact that an integral domain R is a Krull domain if and only if R is a completely integrally closed H-domain ([4, 3.2(d)]).

Corollary 4.5. *Let R be a completely integrally closed domain. Then R is a Krull domain if and only if $E(K/R)$ is an injective cogenerator in the quotient category $\mathsf{Mod}(R)/\mathscr{T}_\tau(R)$.*

Let M be any R-module. We have a canonical mapping:

$$\lambda_M : M \to \mathsf{Hom}_R(\mathsf{Hom}_R(M, E(K/R)), E(K/R)).$$

Let $f \in \mathsf{Hom}_R(M, E(K/R))$. Then define $\lambda_M(m)$ by the equation $\lambda_M(m)(f) = f(m)$ for all $m \in M$.

Theorem 4.6. *Let R be an H-domain with quotient field $K(\neq R)$, and let M be any R-module. Then M is co-semi-divisorial if and only if λ_M is injective.*

Proof. ($\Leftarrow$): This follows from the facts that $E(K/R)$ is co-semi-divisorial and $\mathsf{Hom}_R(L, N)$ is co-semi-divisorial whenever N is co-divisorial.

($\Rightarrow$): Let $x \in M \setminus \{0\}$. Since Rx is not w-null, we can find a homomorphism $f : Rx \to E(K/R)$ such that $f(x) \neq 0$ by Theorem 4.1. Since $E(K/R)$ is injective, we can lift f to a mapping $\bar{f} : M \to E(K/R)$. This shows that λ_M is injective, since $\lambda_M(x)(\bar{f}) = \bar{f}(x) = f(x) \neq 0$ and hence $\lambda_M(x) \neq 0$. □

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