Bull. Korean Math. Soc. **48** (2011), No. 2, pp. 353–363
DOI 10.4134/BKMS.2011.48.2.353

DEGREE CONDITIONS AND FRACTIONAL k-FACTORS OF GRAPHS

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Abstract. Let $k \geq 1$ be an integer, and let G be a 2-connected graph of order n with $n \geq \max\{7, 4k+1\}$, and the minimum degree $\delta(G) \geq k+1$. In this paper, it is proved that G has a fractional k-factor excluding any given edge if G satisfies $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices x, y of G. Furthermore, it is showed that the result in this paper is best possible in some sense.

1. Introduction

We investigate the fractional factor problem in graphs, which can be considered as a relaxation of the well-known cardinality matching problem. The fractional factor problem has wide-ranging applications in areas such as network design, scheduling and the combinatorial polyhedron. For example, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph of order n with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We write $N_G[x]$ for $N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, we define $d_G(S) = \sum_{x \in S} d_G(x)$. We denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. If $G[S]$ has no edges, then S is called independent. Let S and T be disjoint subsets of $V(G)$, we denote the number of edges joining S and T by $e_G(S,T)$. We use $\delta(G)$ for the

Received July 30, 2009; Revised February 4, 2010.
2010 *Mathematics Subject Classification.* 05C70.
Key words and phrases. graph, degree, k-factor, fractional k-factor.
This research was supported by Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (10KJB110003) and Jiangsu University of Science and Technology (2010SL101J, 2009SL148J, 2009SL154J), and was sponsored by Qing Lan Project of Jiangsu Province.

minimum degree of G. We write $I(G)$ for the set of isolated vertices in G and $i(G) = |I(G)|$.

Let k be a positive integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. If $k = 1$, then a k-factor is simply called a 1-factor. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. If $k = 1$, then a fractional k-factor is a fractional 1-factor. Some other terminologies and notations can be found in [1].

Zhou [9, 11] obtained some results on factors of graphs. Iida and Nishimura [2] gave a degree sum condition for graphs to have k-factors. Nishimura [6] showed a degree condition for graphs to have k-factors. Wang [7] obtained a degree condition for the existence of k-factors with prescribed properties. Zhou and Liu [12] showed a neighborhood condition for graphs to have fractional k-factors. Zhou and Shen [13] gave a binding number condition for graphs to have fractional k-factors. Zhou [10] obtained some sufficient conditions for graphs to have fractional k-factors. Liu and Zhang [5] showed a toughness condition for graphs to have fractional k-factors. Yu and Liu [8] obtained a sufficient condition for graphs to have fractional k-factors.

The following results on k-factors and fractional k-factors are known.

Theorem 1 ([6]). *Let k be an integer such that $k \geq 3$, and let G be a connected graph of order n with $n \geq 4k - 3$, kn even, and $\delta(G) \geq k$. Suppose that $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices x, y of G. Then G has a k-factor.*

Theorem 2 ([7]). *Let k be an integer such that $k \geq 3$, and let G be a 2-connected graph of order n with $n \geq 4k + 1$, kn even, and $\delta(G) \geq k + 1$. Suppose that $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices x, y of G. Then G has a k-factor excluding any given edge.*

Theorem 3 ([12]). *Let k be an integer such that $k \geq 1$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, and the minimum degree $\delta(G) \geq k$. If $|N_G(x) \cup N_G(y)| \geq \max\{\frac{n}{2}, \frac{1}{2}(n + k - 2)\}$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.*

In this paper, we obtain a degree condition for the existence of fractional k-factors with prescribed properties. The main result is an extension of Theorem 2. The main result will be given in the following section.

2. Results and proofs

The following results are essential to the proofs of our main theorems.

Lemma 2.1 ([4]). *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le k\}$.

Lemma 2.2 ([3]). *A graph G has a fractional k-factor excluding any given edge if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le k\}$*

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \ge \varepsilon(S,T),$$

or

$$k|S| - \sum_{j=0}^{k-1}(k-j)p_j(G-S) \ge \varepsilon(S,T),$$

where $p_j(G-S) = |\{x : d_{G-S}(x) = j\}|$, and $\varepsilon(S,T)$ is defined as follows,

$$\varepsilon(S,T) = \begin{cases} 2, & \text{if } T \text{ is not independent,} \\ 1, & \text{if } T \text{ is independent, and } e_G(T, V(G) \setminus (S \cup T)) \ge 1, \\ 0, & \text{otherwise.} \end{cases}$$

In the following, we give our main theorems and prove them.

Theorem 4. *Let G be a 2-connected graph of order n with $n \ge 7$ and $\delta(G) \ge 2$. If*

$$\max\{d_G(x), d_G(y)\} \ge \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of G, then G has a fractional 1-factor excluding any given edge.

Proof. Suppose that G satisfies the assumption of the theorem, but it has no desired fractional 1-factor. Then by Lemma 2.2, there exists some $S \subseteq V(G)$ such that

$$i(G-S) > |S| - \varepsilon(S,T), \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le 1\}$. Clearly, $S \ne \emptyset$ since $\delta(G) \ge 2$ and G is 2-connected.

Claim 1. $|T| > |S|$.

Proof of Claim 1. If T is not independent, then $\varepsilon(S,T) = 2$ and $|T| \ge i(G-S) + 2$. According to (1), we obtain

$$|T| \ge i(G-S) + 2 > |S| - \varepsilon(S,T) + 2 = |S|.$$

If T is independent and $e_G(T, V(G) \setminus (S \cup T)) \ge 1$, then $\varepsilon(S,T) = 1$ and $|T| \ge i(G-S) + 1$. In view of (1), we get

$$|T| \ge i(G-S) + 1 > |S| - \varepsilon(S,T) + 1 = |S|.$$

Otherwise, $\varepsilon(S,T) = 0$ and $|T| \ge i(G-S)$. From (1), we have

$$|T| \ge i(G-S) > |S| - \varepsilon(S,T) = |S|.$$

This completes the proof of Claim 1. □

According to Claim 1 and $|S| + |T| \leq n$, we have

$$|S| < \frac{n}{2}. \tag{2}$$

Case 1. $|S| = 1$.

In view of Claim 1, we have $|T| \geq 2$.

Subcase 1.1. T is not independent.

In this case, $\varepsilon(S,T) = 2$. We prove firstly the following claim.

Claim 2. $|T| \geq 3$.

Proof of Claim 2. If $T = 2$, then $e = uv \in E(G[T])$ for $u, v \in T$. Since G is 2-connected, $G - S$ is connected. From $n \geq 7$, then there exists $x \in V(G) \setminus (S \cup T)$ such that x is adjacent to u or v. Hence, $d_{G-S}(u) \geq 2$ or $d_{G-S}(v) \geq 2$, which contradicts $d_{G-S}(y) \leq 1$ for any $y \in T$. This completes the proof of Claim 2. □

According to $|T| \geq 3$ and the definition of T, then there exist $u, v \in T$ such that $uv \notin E(G)$. By the hypothesis of Theorem 4, we obtain

$$\frac{n}{2} \leq \max\{d_G(u), d_G(v)\} \leq \max\{d_{G-S}(u), d_{G-S}(v)\} + |S| \leq |S| + 1 = 2,$$

which implies

$$n \leq 4.$$

This contradicts $n \geq 7$.

Subcase 1.2. T is independent.

In this case, $\varepsilon(S,T) \leq 1$. Since G is 2-connected, then $G - S$ is connected. Thus, we have

$$i(G - S) = 0. \tag{3}$$

On the other hand, from (1) we obtain

$$i(G - S) > |S| - \varepsilon(S,T) \geq |S| - 1 = 0,$$

which contradicts (3).

Case 2. $|S| = 2$.

By Claim 1, we have $|T| \geq 3$. According to (1) and $\varepsilon(S,T) \leq 2$, we obtain

$$i(G - S) > |S| - \varepsilon(S,T) \geq |S| - 2 = 0.$$

Therefore, there exists $x \in T$ with $d_{G-S}(x) = 0$. Since $|T| \geq 3$, we have $y \in T \setminus \{x\}$ such that $d_{G-S}(y) \leq 1$ and $xy \notin E(G)$. In view of $n \geq 7$ and the hypothesis of Theorem 4, we have

$$\frac{7}{2} \leq \frac{n}{2} \leq \max\{d_G(x), d_G(y)\} \leq d_{G-S}(y) + |S| \leq 3,$$

which is a contradiction.

Case 3. $|S| \geq 3$.

According to $\varepsilon(S,T) \leq 2$ and (1), we get that

$$i(G - S) \geq |S| - \varepsilon(S,T) + 1 \geq |S| - 1 \geq 2.$$

Let $I(G-S)=\{x_1,x_2,\ldots,x_{i(G-S)}\}$. Clearly, $x_ix_j \notin E(G)$ ($i\neq j$ and $i,j\in\{1,2,\ldots,i(G-S)\}$). Hence, for any two vertices x_i and x_j, we obtain $\max\{d_G(x_i),d_G(x_j)\}\geq \frac{n}{2}$ by the hypothesis of Theorem 4. But by (2), $d_G(x_i)\leq |S|<\frac{n}{2}$ for all $x_i\in I(G-S)$. This is a contradiction. This completes the proof of Theorem 4. □

Theorem 5. *Let G be a 2-connected graph of order n with $n\geq 9$ and $\delta(G)\geq 3$. If*

$$\max\{d_G(x),d_G(y)\}\geq \frac{n}{2}$$

for each pair of nonadjacent vertices x,y of G, then G has a fractional 2-factor excluding any given edge.

Proof. Suppose that G satisfies the assumption of Theorem 5, but it has no desired fractional 2-factor. Then by Lemma 2.2, there exists some $S\subseteq V(G)$ such that

$$2|S|-2p_0(G-S)-p_1(G-S)\leq \varepsilon(S,T)-1, \tag{4}$$

where $T=\{x : x\in V(G)\backslash S, d_{G-S}(x)\leq 2\}$. If $S=\emptyset$, then we have $p_0(G-S)=p_1(G-S)=0$ by $\delta(G)\geq 3$. In this case, it is easy to see that $\varepsilon(S,T)=0$. From (4), we obtain

$$0=2|S|-2p_0(G-S)-p_1(G-S)\leq \varepsilon(S,T)-1=-1,$$

this is a contradiction. If $|S|=1$, then we have $p_0(G-S)=p_1(G-S)=0$ by $\delta(G)\geq 3$. According to (4) and $\varepsilon(S,T)\leq 2$, we get

$$2=2|S|=2|S|-2p_0(G-S)-p_1(G-S)\leq \varepsilon(S,T)-1\leq 1,$$

a contradiction. In the following we may assume that $|S|\geq 2$. We prove firstly the following claim.

Claim 3. $|T|>|S|$.

Proof of Claim 3. If T is not independent, then $\varepsilon(S,T)=2$. Let $p_1(G-S)\geq 2$. Then $|T|\geq p_0(G-S)+p_1(G-S)$. Complying this with (4), we have

$$|T|\geq p_0(G-S)+p_1(G-S)\geq |S|+\frac{p_1(G-S)-\varepsilon(S,T)+1}{2}\geq |S|+\frac{1}{2}.$$

According to the integrity of $|T|$ and $|S|$, we have $|T|>|S|$. Let $p_1(G-S)\leq 1$. Then $|T|\geq p_0(G-S)+2$. In view of (4), we obtain

$$\begin{aligned}|T|&\geq p_0(G-S)+2\\&\geq p_0(G-S)+p_1(G-S)+1\geq |S|+\frac{p_1(G-S)-\varepsilon(S,T)+1}{2}+1\\&\geq |S|+\frac{1}{2}.\end{aligned}$$

By the integrity of $|T|$ and $|S|$, we have $|T|>|S|$.

If T is independent and $e_G(T, V(G) \setminus (S \cup T)) \geq 1$, then $\varepsilon(S,T) = 1$. Let $p_1(G-S) \geq 1$. Then we have $|T| \geq p_0(G-S) + p_1(G-S)$. By (4),

$$|T| \geq p_0(G-S) + p_1(G-S) \geq |S| + \frac{p_1(G-S) - \varepsilon(S,T) + 1}{2} \geq |S| + \frac{1}{2}.$$

In view of the integrity of $|T|$ and $|S|$, we obtain $|T| > |S|$. Let $p_1(G-S) = 0$. From T is independent and $e_G(T, V(G) \setminus (S \cup T)) \geq 1$, we get $|T| > p_0(G-S)$. By $p_1(G-S) = 0$, $\varepsilon(S,T) = 1$ and (4),

$$p_0(G-S) \geq |S|. \tag{5}$$

From $|T| > p_0(G-S)$ and (5), we have

$$|T| > p_0(G-S) \geq |S|.$$

Otherwise, $\varepsilon(S,T) = 0$ and $|T| \geq p_0(G-S) + p_1(G-S)$. Complying this with (4), we obtain

$$|T| \geq p_0(G-S) + p_1(G-S) \geq |S| + \frac{1}{2}.$$

According to the integrity of $|T|$ and $|S|$, we obtain $|T| > |S|$. This completes the proof of Claim 3. □

In view of Claim 3 and $|S| + |T| \leq n$, we have

$$|S| < \frac{n}{2}. \tag{6}$$

Case 1. $p_0(G-S) \geq 2$.

Obviously, $|I(G-S)| = p_0(G-S) \geq 2$. For any $x, y \in I(G-S)$, we have $xy \notin E(G)$. According to the hypothesis of Theorem 5, there must exist at least one vertex, say y, such that $d_G(y) \geq \frac{n}{2}$. Complying this with $d_{G-S}(y) = 0$ and (6), we obtain

$$\frac{n}{2} \leq d_G(y) \leq d_{G-S}(y) + |S| = |S| < \frac{n}{2}.$$

This is a contradiction.

Case 2. $p_0(G-S) = 1$.

Subcase 2.1. $p_1(G-S) = 0$.

According to (4) and $\varepsilon(S,T) \leq 2$, we have

$$2|S| \leq 2p_0(G-S) + p_1(G-S) + 1 = 3,$$

which contradicts $|S| \geq 2$.

Subcase 2.2. $p_1(G-S) \geq 1$.

Since $p_0(G-S) = 1$ and $p_1(G-S) \geq 1$, there exist $x, y \in T$ such that $d_{G-S}(x) = 0$ and $d_{G-S}(y) = 1$. Obviously, $xy \notin E(G)$. According to the hypothesis of Theorem 5, we obtain

$$\frac{n}{2} \leq \max\{d_G(x), d_G(y)\} \leq d_{G-S}(y) + |S| = |S| + 1,$$

that is,

$$|S| \geq \frac{n}{2} - 1. \tag{7}$$

If n is odd and since $|S|$ is an integer, we have $|S| \geq \frac{n-1}{2}$. Moreover, $p_1(G-S) \leq n - p_0(G-S) - |S| \leq n-1-\frac{n-1}{2} = \frac{n-1}{2}$. Hence, by (4) we obtain

$$1 \geq \varepsilon(S,T)-1 \geq 2|S|-2p_0(G-S)-p_1(G-S) \geq 2\cdot\frac{n-1}{2}-2-\frac{n-1}{2} = \frac{n-5}{2},$$

which implies $n \leq 7$. This contradicts $n \geq 9$.

If n is even, then by (6) and (7) we have $|S| = \frac{n}{2} - 1$. Furthermore, $p_1(G-S) \leq n - p_0(G-S) - |S| = n-1-(\frac{n}{2}-1) = \frac{n}{2}$. Since $n \geq 9$ and n is even, we have $n \geq 10$.

For $n = 10$, we have $p_1(G-S) \neq \frac{n}{2} = 5$. Otherwise, $n = |S| + p_0(G-S) + p_1(G-S)$. We write $P_1(G-S) = \{x : x \in V(G) \setminus S, d_{G-S}(x) = 1\}$ and $p_1(G-S) = |P_1(G-S)|$. Then for each $x \in P_1(G-S)$, x is only adjacent to the vertices of $P_1(G-S)$ in $G-S$. It is impossible by $p_1(G-S) = 5$ and the definition of $P_1(G-S)$. Thus, we obtain $p_1(G-S) \leq 4$. Complying this with (4) and $\varepsilon(S,T) \leq 2$,

$$1 \geq \varepsilon(S,T) - 1 \geq 2|S| - 2p_0(G-S) - p_1(G-S) \geq 8-2-4 = 2,$$

it is a contradiction.

For $n \geq 12$, we obtain by $|S| = \frac{n}{2} - 1$ and $p_1(G-S) \leq \frac{n}{2}$

$$2|S| - 2p_0(G-S) - p_1(G-S) \geq 2(\frac{n}{2}-1) - 2 - \frac{n}{2} = \frac{n}{2} - 4 \geq 2 \geq \varepsilon(S,T).$$

This contradicts (4).

Case 3. $p_0(G-S) = 0$.

Subcase 3.1. $0 \leq p_1(G-S) \leq 2$.

From (4) and $|S| \geq 2$, we have

$$\varepsilon(S,T) - 1 \geq 2|S| - 2p_0(G-S) - p_1(G-S) \geq 2 \geq \varepsilon(S,T),$$

which is a contradiction.

Subcase 3.2. $p_1(G-S) \geq 3$.

In this case, there must exist $x, y \in P_1(G-S)$ such that $xy \notin E(G)$. According to the hypothesis of Theorem 5, we obtain $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$. Suppose that $d_G(y) \geq \frac{n}{2}$, we have $\frac{n}{2} \leq d_G(y) \leq d_{G-S}(y) + |S| = |S| + 1$, which implies $|S| \geq \frac{n}{2} - 1$. Furthermore, $p_1(G-S) \leq n - |S| \leq \frac{n}{2} + 1$. By (4) and $n \geq 9$, we obtain

$$1 \geq \varepsilon(S,T) - 1 \geq 2|S| - p_1(G-S) \geq 2(\frac{n}{2}-1) - (\frac{n}{2}+1) = \frac{n}{2} - 3 \geq \frac{3}{2}.$$

It is a contradiction. This completes the proof of Theorem 5. □

Theorem 6. *Let $k \geq 1$ be an integer, and let G be a 2-connected graph of order n with $n \geq \max\{7, 4k+1\}$, and $\delta(G) \geq k+1$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of G, then G has a fractional k-factor excluding any given edge.

Proof. By Theorem 4 and Theorem 5, the result obviously holds for $k = 1$ and $k = 2$. If $k \geq 3$ and kn is even, then G has a k-factor excluding any given edge by Theorem 2. We have known that a k-factor is a special fractional k-factor. Hence, G has a fractional k-factor excluding any given edge. Now we consider the case that k and n are both odd.

Suppose that G satisfies the assumption of the theorem, but has not desired fractional k-factor. From Lemma 2.2, there exists a subset S of $V(G)$ such that

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S,T) - 1, \tag{8}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$. Firstly, we prove the following claims.

Claim 4. $S \neq \emptyset$.

Proof of Claim 4. If $S = \emptyset$, then by $\delta(G) \geq k+1$ and (8) we have

$$\begin{aligned} \varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &= d_G(T) - k|T| \geq (\delta(G) - k)|T| \geq |T| \geq \varepsilon(S,T), \end{aligned}$$

which is a contradiction. □

Claim 5. $|T| \geq k+1$.

Proof of Claim 5. Suppose that $|T| \leq k$. Then by $\delta(G) \geq k+1$, we obtain

$$\begin{aligned} \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \geq |T||S| + d_{G-S}(T) - k|T| \\ &= \sum_{x \in T}(|S| + d_{G-S}(x) - k) \geq \sum_{x \in T}(\delta(G) - k) \geq |T| \geq \varepsilon(S,T). \end{aligned}$$

This contradicts (8). □

Claim 6. $|T| \geq |S| + 1$.

Proof of Claim 6. If $|T| \leq |S|$, then by (8) we have

$$\varepsilon(S,T) - 1 \geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq d_{G-S}(T) \geq \varepsilon(S,T),$$

which is a contradiction. □

Claim 7. $|S| \leq \frac{n-1}{2}$.

Proof of Claim 7. According to Claim 6 and $|S| + |T| \leq n$, we obtain

$$n \geq |S| + |T| \geq 2|S| + 1,$$

which implies

$$|S| \leq \frac{n-1}{2}.$$

□

According to the hypothesis of Theorem 6 and since n is odd, we have

$$\max\{d_G(x), d_G(y)\} \geq \frac{n+1}{2} \tag{9}$$

for each pair of nonadjacent vertices x, y of G.

By Claim 5, $T \neq \emptyset$. Now we define

$$h_1 = \min\{d_{G-S}(x) : x \in T\},$$

and let x_1 be a vertex in T satisfying $d_{G-S}(x_1) = h_1$. Furthermore, if $T \setminus N_T[x_1] \neq \emptyset$, we define

$$h_2 = \min\{d_{G-S}(x) : x \in T \setminus N_T[x_1]\},$$

and let x_2 be a vertex in $T \setminus N_T[x_1]$ satisfying $d_{G-S}(x_2) = h_2$. Then we have $0 \leq h_1 \leq h_2 \leq k$ by the definition of T and $d_G(x_i) \leq |S| + h_i$ for $i = 1, 2$.

If $T \setminus N_T[x_1] \neq \emptyset$, then we have $|S| + h_2 \geq \frac{n+1}{2}$. Otherwise we get $|S| + h_1 \leq |S| + h_2 < \frac{n+1}{2}$, and this implies $d_G(x_1) < \frac{n+1}{2}$ and $d_G(x_2) < \frac{n+1}{2}$. Since $x_1x_2 \notin E(G)$, that would contradict (9).

Now in order to prove the correctness of Theorem 6, we will deduce some contradictions according to the following cases.

Case 1. $T = N_T[x_1]$.

According to Claim 5, we know that T is not independent. In this case, $\varepsilon(S, T) = 2$. In view of Claim 5 and $T = N_T[x_1]$, we have $k \geq h_1 = d_{G-S}(x_1) \geq |T| - 1 \geq k$. Hence, $h_1 = k$. By (8), Claim 4 and the definition of h_1, we obtain

$$\begin{aligned} 1 &= \varepsilon(S,T) - 1 \geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|T| - k|T| = k|S| \geq k \geq 3, \end{aligned}$$

a contradiction.

Case 2. $T \setminus N_T[x_1] \neq \emptyset$.

By Claim 7 and $|S| + h_2 \geq \frac{n+1}{2}$, we have

$$\frac{n-1}{2} \geq |S| \geq \frac{n+1}{2} - h_2,$$

that is,

$$h_2 \geq 1. \tag{10}$$

According to (10), $n \geq 4k+1$, $|S| + |T| \leq n$, $|S| + h_2 \geq \frac{n+1}{2}$, $h_1 \leq h_2 \leq k$ and $|N_T[x_1]| \leq d_{G-S}(x_1) + 1 = h_1 + 1$, we obtain

$$\begin{aligned} \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - k|T| \\ &= k|S| + (h_1 - h_2)|N_T[x_1]| + (h_2 - k)|T| \\ &\geq k|S| + (h_1 - h_2)(h_1 + 1) + (h_2 - k)(n - |S|) \\ &= (2k - h_2)|S| + (h_1 - h_2)(h_1 + 1) + (h_2 - k)n \\ &\geq (2k - h_2)(\frac{n+1}{2} - h_2) + (h_1 - h_2)(h_1 + 1) + (h_2 - k)n \end{aligned}$$

$$\begin{aligned} &= (\frac{h_2-1}{2}-h_1)^2+\frac{3}{4}h_2^2+(\frac{n}{2}-2k-1)h_2+k-\frac{1}{4} \\ &\geq \frac{3}{4}h_2^2+(\frac{n}{2}-2k-1)h_2+k-\frac{1}{4} \\ &\geq \frac{3}{4}h_2^2-\frac{1}{2}h_2+k-\frac{1}{4} \\ &\geq \frac{3}{4}-\frac{1}{2}+k-\frac{1}{4} \\ &= k>2\geq \varepsilon(S,T), \end{aligned}$$

this contradicts (8).

From all the cases above, we deduce the contradictions. Hence, G has a fractional k-factor excluding any given edge. This completes the proof of Theorem 6. □

Remark 1. Let us show that the condition $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ in Theorem 6 cannot be replaced by $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}-1$. Let $t \geq 2$ and $k \geq 1$ be two integers. We construct a graph $G = ((kt-2)K_1 \cup K_2) \vee (kt+1)K_1$. Obviously, G is 2-connected, $\delta(G) = kt \geq 2k \geq k+1$, $n = |V(G)| = 2kt+1 \geq 4k+1$ and

$$\frac{n}{2} > \max\{d_G(x), d_G(y)\} > \frac{n}{2} - 1$$

for each pair of nonadjacent vertices x, y of $(kt+1)K_1 \subset G$. Let $G' = G - E(K_2)$, $S = V((kt-2)K_1 \cup K_2)) \subseteq V(G)$ and $T = V((kt+1)K_1) \subseteq V(G)$. Then $|S| = kt$, $|T| = kt+1$ and $d_{G'-S}(T) = 0$. Thus, we get

$$\begin{aligned} \delta_{G'}(S,T) &= k|S| + d_{G'-S}(T) - k|T| \\ &= k^2t - k(kt+1) \\ &= -k < 0. \end{aligned}$$

By Lemma 2.1, G' has no fractional k-factor, that is, Theorem 6 does not hold. In the above sense, the result in Theorem 6 is best possible.

Remark 2. In the following, we show that the bound on n in Theorem 6 is also best possible. Let $k \geq 2$ be an integer. We construct a graph $G = K_{2k-1} \vee (K_1 \cup kK_2)$. Then G satisfies all conditions of Theorem 6 except $n = 4k$. Let $S = V(K_{2k-1})$ and $T = V(K_1 \cup kK_2)$. Clearly, $d_{G-S}(T) = 2k$ and T is not independent. In this case, $\varepsilon(S,T) = 2$. Thus, we obtain

$$\begin{aligned} \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &= k(2k-1) + 2k - k(2k+1) \\ &= 0 < 2 = \varepsilon(S,T). \end{aligned}$$

According to Lemma 2.2, Theorem 6 does not hold.

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