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A STRONG LIMIT THEOREM FOR SEQUENCES OF BLOCKWISE AND PAIRWISE m-DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, we establish a Marcinkiewicz-Zygmund type strong law for sequences of blockwise and pairwise m-dependent random variables. The sharpness of the results is illustrated by an example.

1. Introduction and preliminaries

The concept of blockwise m-dependence and blockwise quasiorthogonality for a sequence of random variables was introduced by Móricz [20]. Móricz [20] and Gaposhkin [9] extended a classical strong law of large numbers (SLLN) of Kolmogorov to the blockwise m-dependent case, and they also extended the Rademacher-Menshov strong law of large numbers to the blockwise quasiorthogonal case. Thanh [26] studied the Brunk-Chung strong law of large numbers for sequences of blockwise m-dependent random variables. The SLLN problems for blockwise independent and blockwise p-orthogonal random variables in Rademacher type p Banach spaces were studied by Rosalsky and Thanh [23]. In all those papers, the authors only considered the non-identically distributed case. Li and Wang [17] proved some results on SLLN and the law of the iterated logarithm for sequences of blockwise m-dependent Banach spaces valued random variables with respect to the blocks $[2^k, 2^{k+1}), k \geq 0$, and gave positive answers to two conjectures raised by Móricz [20]. They also considered the identically distributed case. However, their results and ours do not imply each other.

There is an interesting literature of investigation on the SLLN problem for sequences of pairwise independent random variables; see Etemadi [7], Csörgo, Tandori and Totik [5], Rosalsky [22], Martikainen [18], Hu [13], Fazekas and Tómács [8], Hong and Hwang [12], Czerebak-Mrozowicz, Klesov and Rychlik [6], Kruglov [14, 15]. On the Marcinkiewicz-Zygmund law of large numbers, see Sung, Lisawadi and Volodin [24] for an interesting open problem (Sung,

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Lisawadi and Volodin [24] also suggested two recent papers by Kruglov [14], and Li, Rosalsky and Volodin [16] that contains a new technique that may lead to a solution of the problem they mentioned). For a practical example of pairwise independent but not mutually independent random variables, we can see in Geisser and Mantel [10], for importance of pairwise independent random variables, we can see in O'Brien [21] and Harber [11], and finally for a survey of their applications in computer science, we can see in Wigderson [28].

Choi and Sung [4] established a Marcinkiewicz-Zygmund type strong law of large numbers for sequences of pairwise independent random variables. Chandra and Goswami [3], Bose and Chandra [2] considered this problem under uniformly integrable condition. Hong and Hwang [12], and Czerebak-Mrozowicz, Klesov and Rychlik [6] extended the result of Choi and Sung to the multidimensional case. Thanh [25] also established a strong law of large numbers for blockwise and pairwise m-dependent random variables which extended the result of Choi and Sung [4]. In Thanh [25], the author considered a sequence of random variables which is blockwise and pairwise m-dependent with respect to the blocks $[2^k, 2^{k+1}), k \geq 0$.

In this note, we consider a sequence of blockwise and pairwise m-dependent random variables $\{X_n, n \geq 1\}$ which is stochastically dominated by a random variable X. We establish a Marcinkiewicz-Zygmund type strong law of large numbers which extends the result of Thanh [25] to the arbitrary blocks case. We also provide an example to illustrate the main result. In this example, the result in Thanh [25] is not applicable. The current work also extends a result of Bose and Chandra [2].

Let m be a fixed nonnegative integer. We say that a collection $\{X_j, 1 \leq j \leq n\}$ of n random variables is *pairwise m-dependent* if either $n \leq m+1$ or $n > m+1$ and X_i and X_j are independent whenever $j - i > m$.

Let $\{\beta_k, k \geq 1\}$ be a strictly increasing sequence of positive integers with $\beta_1 = 1$ and set $B_k = [\beta_k, \beta_{k+1})$.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *blockwise and pairwise m-dependent* with respect to the blocks $\{B_k, k \geq 1\}$ if for each $k \geq 1$, the random variables $\{X_j, j \in B_k\}$ are pairwise m-dependent. Thus the random variables with indices in each block are pairwise m-dependent but there are no independence requirements between the random variables with indices in different blocks; even repetitions are permitted (see Example 2.6 in Section 2).

A sequence of random variables $\{X_n, n \geq 1\}$ satisfying $EX_n^2 = \sigma_n^2 < \infty$, $n \geq 1$, is said to be *blockwise quasiorthogonal* with respect to the blocks $\{B_k, k \geq 1\}$ if for each $k \geq 1$, there exists a collection of constants $\{f_k(j), j = 0, 1, \ldots, \beta_{k+1} - \beta_k - 1\}$ such that

$$|E(X_iX_j)| \leq \sigma_i\sigma_j f_k(|i-j|) \ (i, j \in B_k) \text{ and } \sum_{j=0}^{\beta_{k+1}-\beta_k-1} f_k(j) \leq C,$$

where the constant C is independent of k.

Note that if $\{X_n, n \geq 1\}$ is sequence of random variables with $EX_n = 0$ and $EX_n{}^2 < \infty$, $n \geq 1$, and if $\{X_n, n \geq 1\}$ is blockwise and pairwise m-dependent with respect to the blocks $\{B_k,\, k \geq 1\}$, then $\{X_n, n \geq 1\}$ is blockwise quasiorthogonal with respect to the blocks $\{B_k,\, k \geq 1\}$.

For $\{\beta_k,\, k \geq 1\}$ and $\{B_k,\, k \geq 1\}$ as above, we introduce the following notation:

$$\begin{aligned}
B^{(l)} &= \{k : 2^l \leq k < 2^{l+1}\},\ l \geq 0,\\
B_k^{(l)} &= B_k \cap B^{(l)}, k \geq 1, l \geq 0,\\
I_l &= \{k \geq 1 : B_k^{(l)} \neq \emptyset\}, l \geq 0,\\
r_k^{(l)} &= \min\{r : r \in B_k^{(l)}\}, k \in I_l, l \geq 0,\\
c_l &= \mathrm{card} I_l, l \geq 0,\\
d_l &= \max_{k \in I_l} \mathrm{card} B_k^{(l)}, l \geq 0,\\
\varphi(n) &= \sum_{l=0}^{\infty} c_l I_{B^{(l)}}(n), n \geq 1,\\
\phi(n) &= \sum_{l=0}^{\infty} \log^2(2d_l) I_{B^{(l)}}(n), n \geq 1,\\
\psi(n) &= \max_{k \leq n} \varphi(k), n \geq 1,
\end{aligned}$$

where $I_{B^{(l)}}$ denotes the indicator function of the set $B^{(l)}, l \geq 0$.

Random variables $\{X_n,\, n \geq 1\}$ are said to be a *stochastically dominated* by random variable X if for some constant $C < \infty$

$$P(|X_n| > t) \leq CP(|X| > t), t \geq 0, n \geq 1.$$

2. Main result

We can now state our main result. Throughout this section, the logarithms are to the base 2, the symbol C denotes a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

Theorem 2.1. *Let $1 \leq r < 2$ and $\{X_n,\, n \geq 1\}$ be a sequence of random variables which is blockwise and pairwise m-dependent with respect to the blocks $\{B_k,\, k \geq 1\}$. Suppose that $\{X_n,\, n \geq 1\}$ is stochastically dominated by a random variables X. If*

$$E(|X|^r (\log^+ |X|)^2) < \infty, \tag{2.1}$$

then

$$\lim_{n \to \infty} \frac{1}{n^{1/r} \psi^{1/2}(n)} \sum_{j=1}^{n} (X_j - EX_j) = 0 \ \textit{a.s.} \tag{2.2}$$

Before beginning the proof, we state two lemmas. The first lemma follows Lemma 3 of Móricz [19] generalizing the Rademacher-Menshov inequality.

Lemma 2.2. *If* $\{X_n, n \geq 1\}$ *is a sequence of random variables which is blockwise quasiorthogonal with respect to the blocks* $\{B_k, k \geq 1\}$*, then for all* k *and for all* $\beta_k \leq m \leq n < \beta_{k+1}$

$$E\left(\max_{m\leq p\leq n}\left|\sum_{j=m}^{p} X_j\right|\right)^2 \leq C\log^2(2(n-m+1))\sum_{j=m}^{n} EX_j^2.$$

The second lemma is an alternative version of Theorem 2 of Gaposhkin [9] and it extends the result of Móricz [20] (which, in turn, is an extension of the Rademacher-Mensov strong law of large numbers). It is the key result used to establish Theorem 2.1 and may be of independent interest.

Lemma 2.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables and let* $\{b_n, n \geq 1\}$ *be a nondecreasing sequence of positive constants such that*

$$\inf_{n\geq 0}\frac{b_{2^{n+1}}}{b_{2^n}} > 1 \text{ and } \sup_{n\geq 0}\frac{b_{2^{n+1}}}{b_{2^n}} < \infty. \tag{2.3}$$

If $\{X_n, n \geq 1\}$ *is blockwise quasiorthogonal with respect to the blocks* $\{B_k, k \geq 1\}$ *and if*

$$\sum_{n=1}^{\infty}\frac{EX_n^2}{b_n^2}\phi(n) < \infty, \tag{2.4}$$

then the SLLN

$$\lim_{n\to\infty}\frac{1}{b_n\psi^{1/2}(n)}\sum_{i=1}^{n} X_i = 0 \text{ a.s.} \tag{2.5}$$

obtains.

Proof. Set

$$T_k^{(l)} = \max_{j\in B_k^{(l)}}\Big|\sum_{i=r_k^{(m)}}^{j} X_i\Big|, k\in I_l, l\geq 0$$

and

$$T_l = \frac{1}{(b_{2^{l+1}} - b_{2^l})(\psi(2^l))^{1/2}}\sum_{k\in I_l} T_k^{(l)}, l \geq 0.$$

Note that for $l \geq 0$,

$$\begin{aligned} ET_l^2 &\leq \frac{C}{b^2_{2^{l+1}}(\psi(2^l))}c_l\sum_{k\in I_l}E(T_k^{(l)})^2 \text{ (by the first half of (2.3))} \\ &\leq \frac{C}{b^2_{2^{l+1}}}\sum_{k\in I_l}E(T_k^{(l)})^2 \end{aligned}$$

$$\leq \frac{C}{b^2_{2^{l+1}}} \sum_{k \in I_l} \sum_{i \in B_k^{(l)}} \log^2(2\text{card}B_k^{(l)}) EX_i^2 \text{ (by Lemma 2.2)}$$

$$\leq \frac{C}{b^2_{2^{l+1}}} \log^2(2d_l) \sum_{i=2^l}^{2^{l+1}-1} EX_i^2$$

$$\leq C \sum_{i=2^l}^{2^{l+1}-1} \frac{EX_i^2}{b_i^2} \phi(i).$$

It follows from (2.4) that $\sum_{l=0}^{\infty} ET_l^2 < \infty$ and so by the Markov inequality and the Borel-Cantelli lemma

$$\lim_{l \to \infty} T_l = 0 \text{ a.s.} \tag{2.6}$$

Note that for $n \geq 1$, letting $M \geq 0$ be such that $2^M \leq n < 2^{M+1}$,

$$\frac{|\sum_{i=1}^n X_i|}{b_n(\psi(n))^{1/2}} \leq \frac{\sum_{l=0}^M \sum_{k \in I_l} T_k^{(l)}}{b_{2^M}(\psi(2^M))^{1/2}}$$

$$\leq \sum_{l=0}^{M} \frac{b_{2^{l+1}} - b_{2^l}}{b_{2^M}} T_l$$

$$= \frac{b_{2^{M+1}}}{b_{2^M}} \sum_{l=0}^{M} \frac{b_{2^{l+1}} - b_{2^l}}{b_{2^{M+1}}} T_l$$

$$\leq C \sum_{l=0}^{M} \frac{b_{2^{l+1}} - b_{2^l}}{b_{2^{M+1}}} T_l \text{ (by the second half of (2.3)).} \tag{2.7}$$

The conclusion (2.5) follow immediately from (2.6), (2.7) and the Toeplitz lemma. □

Proof of Theorem 2.1. Set

$$Y_n = X_n I(|X_n| \leq n^{1/r}),\ n \geq 1.$$

We get the following conclusion which was given by Thanh [25] that

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/r}} EY_n^2 \leq CE(|X|^r (\log^+ |X|)^2) < \infty \tag{2.8}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/r}} E|X_n - Y_n| \leq CE(|X|^r \log^+ |X|) < \infty. \tag{2.9}$$

Note that $\phi(n) \leq 4 \log^2 n, n \geq 2$ so by (2.8) and Lemma 2.3 we get

$$\lim_{n \to \infty} \frac{1}{n^{1/r} \psi^{1/2}(n)} \sum_{j=1}^{n} (Y_j - EY_j) = 0 \text{ a.s.} \tag{2.10}$$

By using Kronecker lemma, it follows from (2.9) that

$$\lim_{n\to\infty} \frac{1}{n^{1/r}} \sum_{j=1}^{n} E(X_j - Y_j) = 0. \tag{2.11}$$

Note that $\psi(n) \geq 1, n \geq 1$ so (2.11) implies

$$\lim_{n\to\infty} \frac{1}{n^{1/r}\psi^{1/2}(n)} \sum_{j=1}^{n} E(X_j - Y_j) = 0. \tag{2.12}$$

Next,

$$\begin{aligned}\sum_{n=1}^{\infty} P(X_n \neq Y_n) &= \sum_{n=1}^{\infty} P(|X_n| > n^{1/r}) \\ &\leq C \sum_{n=1}^{\infty} P(|X| > n^{1/r}) \\ &\leq CE|X|^r < \infty. \qquad (2.13)\end{aligned}$$

By (2.10), (2.13) and the Borel-Cantelli lemma we get

$$\lim_{n\to\infty} \frac{1}{n^{1/r}\psi^{1/2}(n)} \sum_{j=1}^{n} (X_j - EY_j) = 0 \text{ a.s.} \tag{2.14}$$

The conclusion (2.2) follows from (2.12) and (2.14). □

Remark 2.4. It follows from Lemma 3 of Wei and Taylor [27] that stochastic dominance can be accomplished by the sequence of random variables having a bounded absolute r^{th} moment ($r > 0$). Specifically, if $\sup_{n\geq 1} E|X_n|^r < \infty$ for some $r > 0$, then there exists a random variable X with $E|X|^p < \infty$ for all $0 < p < r$ such that $P(|X_n| > t) \leq P(|X| > t), t \geq 0, n \geq 1$ (The proviso that $r > 1$ in Lemma 3 of Wei and Taylor [27] is not needed as was pointed out by Adler, Rosalsky, and Taylor [1]). So that Theorem 2.1 reduces to Corollary 4 of Chandra and Goswami [3] when the $\{X_n, n \geq 1\}$ are pairwise independent and $\sup_{n\geq 1} E|X_n|^r < \infty$ for some $r > 1$.

Note that if $\beta_k = [q^{k-1}]$ for all large k and $q > 1$, then $c_l = \mathcal{O}(1)$, $\psi(n) = \mathcal{O}(1)$. So we get the following corollary which is the main result of Thanh [25].

Corollary 2.5. *Let* $\{X_n, n \geq 1\}$ *be a sequence of blockwise and pairwise m-dependent random variables with respect to the blocks* $\{[2^{k-1}, 2^k), k \geq 1\}$ *(or, more generally, with respect to the blocks* $\{[\beta_k, \beta_{k+1}), k \geq 1\}$ *where* $\beta_k = [q^{k-1}]$ *for all large* k *and* $q > 1$*) and if* (2.1) *is satisfied, then*

$$\lim_{n\to\infty} \frac{1}{n^{1/r}} \sum_{j=1}^{n} (X_j - EX_j) = 0 \text{ a.s.} \tag{2.15}$$

The following example illustrates Theorem 2.1. We also show that in this example, the conclusion of Corollary 2.5 fails.

Example 2.6. Let $\{Y_n, n \geq 1\}$ be a sequence of 0-dependent identically distributed of $N(0,1)$ random variables and let $4/3 \leq r < 2$. Let

$$X_n = Y_{n-k^2+1}, k^2 \leq n < (k+1)^2), k \geq 1.$$

Then $\{X_n, n \geq 1\}$ is blockwise 0-dependent with respect to the blocks $\{[k^2, (k+1)^2, k \geq 1\}$ and (2.1) is satisfied, but $\{X_n, n \geq 1\}$ is not blockwise m-dependent with respect to the blocks $\{[2^k, 2^{(k+1)}), k \geq 0\}$ for any non-negative integer m. Now, by noting that in this case $\psi(n) = \mathcal{O}(n^{1/2})$, so that by Theorem 2.1 we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{n^{1/r+1/4}} = 0 \text{ a.s.}$$

Now, for $n = (M+1)^2 - 1$, we have

$$\begin{aligned}\frac{\sum_{i=1}^n X_i}{n^{1/r}} &= \frac{MY_1 + M(Y_2+Y_3) + (M-1)(Y_4+Y_5) + \cdots + (Y_{2M} + Y_{2M+1})}{((M+1)^2-1)^{1/r}} \\ &= S_{(M+1)^2-1},\end{aligned}$$

where

$$S_{(M+1)^2-1} \sim N\Big(0, \frac{2M^3+6M^2+M}{3((M+1)^2-1)^{2/r}}\Big),$$

so that (2.15) fails since $r \geq 4/3$.

Remark 2.7. Example 2.6 also shows that Theorem 2.1 is sharp. More precisely, it shows that for all $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{|\sum_{i=1}^n X_i|}{n^{1/r-\epsilon}\psi^{1/2}(n)} = \infty \text{ a.s.} \tag{2.16}$$

To see this, for $\epsilon > 0$ be arbitrary, let $r = \frac{2}{1+\epsilon} < 2$. Then (2.1) is satisfied, but for $n = (M+1)^2 - 1$

$$\begin{aligned}&\frac{|\sum_{i=1}^n X_i|}{n^{1/r-\epsilon}\psi^{1/2}(n)} \\ &\geq C((M+1)^2-1)^{\epsilon/2}\Big|\frac{MY_1 + M(Y_2+Y_3) + (M-1)(Y_4+Y_5) + \cdots + (Y_{2M}+Y_{2M+1})}{((M+1)^2-1)^{3/4}}\Big| \\ &\geq CM^{\epsilon}|T_{(M+1)^2-1}|,\end{aligned}$$

where

$$T_{(M+1)^2-1} \sim N\Big(0, \frac{2M^3+6M^2+M}{3((M+1)^2-1)^{3/2}}\Big),$$

so that we get (2.16).

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