Bull. Korean Math. Soc. **48** (2011), No. 2, pp. 315–324
DOI 10.4134/BKMS.2011.48.2.315

DEFORMATIONS OF d/BCK-ALGEBRAS

Paul J. Allen, Hee Sik Kim, and Joseph Neggers

Abstract. In this paper, we study the effects of a deformation mapping on the resulting deformation d/BCK-algebra obtained via such a deformation mapping. Besides providing a method of constructing d-algebras from BCK-algebras, it also highlights the special properties of the standard BCK-algebras of posets as opposed to the properties of the class of divisible d/BCK-algebras which appear to be of interest and which form a new class of d/BCK-algebras insofar as its not having been identified before.

1. Introduction

J. Neggers and H. S. Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs ([9]). After that some further aspects were studied ([4, 6, 8]). As a generalization of BCK-algebras, d-algebras are obtained by deleting two identities. Given one of these deleted identities related identities are constructed by replacing one of the terms involving the original operation by an identical term involving a second (companion) operation, thus producing the notion of companion d-algebra which (precisely) generalizes the notion of BCK-algebra and is such that not every d-algebra is one of these. Recently, the present authors ([1]) developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type.

In this paper we introduce the notion of deformation in d/BCK-algebras. Using such deformations we are able to construct d-algebras from BCK-algebras in such a manner as to maintain control over properties of the deformed BCK-algebras via the nature of the deformation employed. We also observe that certain BCK-algebras cannot be deformed at all, leading to the notion of a rigid d-algebra and consequently of a rigid BCK-algebra as well. Although we have not done so here, it is clear that similar methods as we have used

Received July 2, 2009.
2010 *Mathematics Subject Classification.* 06F35.
Key words and phrases. d/BCK-algebra, deformation (function, point), rigid, invariant.

can be applied to other types of algebras related to these classes, such as $BCC/BCH/BCI$-algebras etc. ([2, 3, 5, 11]).

2. Preliminaries

A *d-algebra* ([9]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y \in X$.

A BCK-algebra is a d-algebra X satisfying the following additional axioms:

(IV) $(x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$

for all $x, y, z \in X$.

If X is a BCK-algebra, then it is well-known that

(VI) $x * 0 = x$,
(VII) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

3. Deformations of d/BCK-algebras

Let $(X, *, 0)$ be an algebra. A map $\varphi : X \to X$ is said to be a *deformation function* of X if

(i) $x \neq 0$ implies $x * \varphi(x) \neq 0$,
(ii) there exists $a \in X$ such that $a * \varphi(a) \neq a$.

We call a a deformation point of X, and $(X, *, 0)$ is said to be a *deformation algebra*.

Note that if φ is a deformation function of d/BCK-algebra $(X, *, 0)$ and a is a deformation point of X, then $a \neq 0$ and $a * \varphi(a) \notin \{0, a\}$.

Example 3.1. Let $X := [0, 1]$ be the unit interval. Define a binary operation "$*$" on X by

$$x * y := \max\{0, x - y\}.$$

Then $(X, *, 0)$ is a BCK-algebra. Define a map $\varphi : X \to X$ by $\varphi(0) = 0$ and $\varphi(x) = \frac{1}{3}x$ for any non-zero x in X. Then φ is a deformation function of X, and hence $(X, *, 0)$ is a deformation algebra. Note that $a * \varphi(a) = \max\{0, a - \varphi(a)\} = \frac{2}{3}a \neq a$ if $a > 0$. Hence the set of all deformation points is (0, 1].

Proposition 3.2. *Let $(X, *, 0)$ be a BCK-algebra and let φ be a deformation function of X. If we define a binary operation on X by*

$$x \bigtriangledown y := (x * y) * \varphi(x * y)$$

for any $x, y \in X$, then $(X, \bigtriangledown, 0)$ is a d-algebra which is not a BCK-algebra.

Proof. Since $(X, *, 0)$ is a BCK-algebra, $x \bigtriangledown x = 0 \bigtriangledown x = 0 * \varphi(0) =$ for any $x \in X$. Assume that $x \bigtriangledown y = 0 = y \bigtriangledown x$. Then $(x*y)*\varphi(x*y) = 0 = (y*x)*\varphi(y*x)$. Since φ is a deformation function of X, we obtain $x * y = 0 = y * x$, which implies $x = y$, since X is a BCK-algebra. This proves $(X, \bigtriangledown, 0)$ is a d-algebra. Let a be a deformation point of X. Then $a \bigtriangledown 0 = (a*0)*\varphi(a*0) = a*\varphi(a) \neq a$, proving $(X, \bigtriangledown, 0)$ is not a BCK-algebra. □

We say $(X, \bigtriangledown, 0)$ in Proposition 3.2 a *deformed BCK-algebra* of a BCK-algebra $(X, *, 0)$.

Example 3.3. Consider a BCK-algebra $(X, *, 0)$ ([7, p. 244]).

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

Define two maps $\varphi(0) = \varphi(1) = 0, \varphi(2) = 1, \varphi(3) = 0$ and $\psi(0) = \psi(1) = 0, \psi(2) = \psi(3) = 1$. If we define $x \bigtriangledown y := (x * y) * \varphi(x * y), x \bullet y := (x * y) * \psi(x * y), \forall x, y \in X$, then we have two deformed BCK-algebras $(X, \bigtriangledown, 0)$ and $(X, \bullet, 0)$ respectively as follows:

$\bigtriangledown$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	1	1	0	0
3	1	1	1	0

$\bullet$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	1	1	0	0
3	2	1	1	0

Proposition 3.4. *Let $(X, *, 0)$ be a BCK-algebra and let φ be a deformation function of X. If we define a binary operation $\bigtriangledown$ on X by $x \bigtriangledown y := (x*y)*\varphi(x*y)$ for any $x, y \in X$, then*

(i) $x * y = 0$ *implies* $x \bigtriangledown y = 0$,
(ii) $x \bigtriangledown 0 = x * \varphi(x)$,
(iii) $(x \bigtriangledown y) * x = 0$,
(iv) $(x \bigtriangledown y) \bigtriangledown x = 0$.

Proof. For (iv). By (iii), $(x \bigtriangledown y) * x = 0$. By applying (i), we obtain $(x \bigtriangledown y) \bigtriangledown x = 0$. □

Example 3.5. Consider a BCK-algebra $(X, *, 0)$ ([7, p. 245]).

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

If we define a map $\varphi(0) = 0, \varphi(1) = 3, \varphi(2) = 1, \varphi(3) = 2$, and define $x \bigtriangledown y := (x * y) * \varphi(x * y)$, then it is easy to see that φ is a deformation function, but $(X, \bigtriangledown, 0)$ is not a BCK-algebra, since $2 \bigtriangledown 0 = 1 \neq 2$, i.e., $(X, \bigtriangledown, 0)$ is a deformed BCK-algebra.

The following question naturally arises: Are all d-algebras deformed BCK-algebras? The answer is negative.

Example 3.6. Consider a BCK-algebra $(X, *, 0)$ ([7, p. 246]).

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

If we define a map $\varphi(0) = 0, \varphi(1) = 2, \varphi(2) = 1, \varphi(3) = 2$, and define $x \bigtriangledown y := (x * y) * \varphi(x * y)$, then it is easy to see that φ is a deformation function and $(X, *, 0) = (X, \bigtriangledown, 0)$.

We call such a function in Example 3.6 an *invariant* deformation function of X. In Example 3.6, by routine calculations, there is no invariant deformation function of X.

In a BCK-algebra X, we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$. We then observe that a BCK-algebra determines a poset structure on it. Let $(X, \leq)$ be a poset with the least element 0. If we define a binary operation $*$ on X as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise,} \end{cases}$$

then the algebraic structure $(X, *, 0)$ turns out to be a BCK-algebra (see [10]), we call such an algebra a *standard BCK-algebra* inherited from the poset $(X, \leq)$.

Theorem 3.7. *The standard BCK-algebra does not have a deformed BCK-algebra.*

Proof. Let $(X, *, 0)$ be a standard BCK-algebra and φ be any deformation function. Define a binary operation $\bigtriangledown$ on X by $x \bigtriangledown y := (x * y) * \varphi(x * y)$ for any $x, y \in X$. If $x \leq y$, then $x * y = 0$ and hence $x \bigtriangledown y = (x * y) * \varphi(x * y) = 0 * \varphi(0) = 0$.

In the other case, we have $x \not\leq y$ and $x \bigtriangledown y = (x * y) * \varphi(x * y) = x * \varphi(x)$. Since $(X, *, 0)$ is a standard BCK-algebra, it follows that

$$x * \varphi(x) = \begin{cases} 0 & \text{if } x \leq \varphi(x) \\ x & \text{otherwise} \end{cases}$$

If we assume $x \leq \varphi(x)$, then $x * \varphi(x) = 0$. Since φ is a deformation function, we have $x = 0$, which means $x = 0 \leq y$ for any $y \in X$, a contradiction. Hence the case "$x \leq \varphi(x)$" cannot happen. Thus $x \bigtriangledown y = x * \varphi(x) = x$, i.e., $(X, \bigtriangledown, 0) = (X, *, 0)$, proving the theorem. □

We say such a BCK-algebra $(X, *, 0)$ an *rigid* BCK-algebra under any deformation function φ.

The following question arises: Are there any rigid BCK-algebras other than the standard BCK-algebra? By routine calculations we see that the BCK-algebra in Example 3.6 is a rigid BCK-algebra.

Since there are BCK-algebras which are not rigid, we may consider deformation functions φ such that $x \bigtriangledown y = x * y$ for all $x, y \in X$ as invariant deformation functions. Thus, if $(X, *, 0)$ is a rigid BCK-algebra, then all deformation functions are invariant deformation functions.

Now, we apply the notion of deformation functions to d-algebras.

Example 3.8. Let $\mathbb{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbb{R}$ by

$$x * y := x(x - y), \forall x, y \in \mathbb{R}.$$

Then $(\mathbb{R}, *, 0)$ is a d-algebra. If we define a map $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = 2x$, then $x * \varphi(x) = -x^2 \neq 0$ if $x \neq 0$. Moreover, $x * \varphi(x) \neq x$ for any $x \in \mathbb{R} \backslash \{0, -1\}$. Thus φ is a deformation function and $x \bigtriangledown y = -x^2(x - y)^2$. It is easy to see that $(\mathbb{R}, \bigtriangledown, 0)$ is also a d-algebra.

Consider the condition that φ be an invariant deformation function of X such that $x \bigtriangledown y = x * y$. Then $x \bigtriangledown y = x(x - y)(x(x - y) - \varphi(x(x - y))$. Since $z > 0$ means $z = \sqrt{z} * 0$, $z = 0$ means $z = 0 * z$, $z < 0$ means $z = \sqrt{|z|} * 2\sqrt{|z|}$, it follows that $\mathbb{R} * \mathbb{R} = \mathbb{R}$. Hence $x \bigtriangledown y = x * y = x(x - y)$ means $z - \varphi(z) = 1$, and $\varphi(z) = z - 1$. But then $x * \varphi(x) = x(x - (x - 1)) = x$ and φ is not a deformation function. Thus $(\mathbb{R}, *, 0)$ does not have any invariant deformation function defined on it whatsoever.

4. Divisible d/BCK-algebras

An algebra $(X, *, 0)$ is said to be *divisible* if for any non-zero x in X, there exists an element $\widehat{x}$ in X such that $x * \widehat{x} \notin \{0, x\}$. We call such a $\widehat{x}$ an *associator* of x. Note that such an associator is not unique in general. In Example 4.3 below, c, d are associators of b.

Example 4.1. Let K be a field with $|K| \geq 3$. If we define a binary operation "$*$" on K by $x * y := x(x-y)$, then it is easy to see that $(K, *, 0)$ is a d-algebra. Let $u \notin \{0, 1\}$. Given $x \neq 0$ in X, if we let $\widehat{x}(x) := x - u$, then $x * \widehat{x} = x(x - \widehat{x}) = xu \notin \{0, x\}$. Hence $(K, *, 0)$ is a divisible d-algebra.

Example 4.2. Let $X := [0, 1]$. If we define a binary operation "$*$" on X by $x * y := \max\{0, x - y\}$, then it is easy to see that $(X, *, 0)$ is a BCK-algebra. Given $x \neq 0$ in X, if we take $\widehat{x} := x/3$, then $x * \widehat{x} = 2x/3 \notin \{0, x\}$, and $(X, *, 0)$ is a divisible BCK-algebra.

Example 4.3. Let $X := \{0, a, b, c, d\}$ be a set with the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	d	c	b
b	d	b	0	c	a
c	d	b	c	0	b
d	a	b	c	d	0

Then it is easy to see that $(X, *, 0)$ is a divisible d-algebra.

Proposition 4.4. *Every standard BCK-algebra is not divisible.*

Proof. Since X is a standard BCK-algebra, for any $x, y \in X$, we have $x * y \in \{0, x\}$. This means that there is no $\widehat{x}$ in X such that $x * \widehat{x} \notin \{0, x\}$. Hence X is not divisible. □

Theorem 4.5. *Every divisible algebra is a deformation algebra.*

Proof. Let $(X, *, 0)$ be a divisible algebra. If we define a map φ by $\varphi(x) := \widehat{x}$ for any non-zero $x \in X$ and $\varphi(0)$ is an arbitrary, where $\widehat{x}$ is an associator of x, then $x * \varphi(x) = x * \widehat{x} \notin \{0, x\}$ if $x \neq 0$, proving φ is a deformation function. Hence $(X, *, 0)$ is a deformation algebra. □

Corollary 4.6. *Every divisible d/BCK-algebra is a deformation d/BCK-algebra.*

The converse of Theorem 4.5 need not be true in general. In Example 3.6, $(X, \bigtriangledown, 0)$ is a deformation algebra, but not divisible, since 3 does not have $\widehat{3}$ such that $3 * \widehat{3} \notin \{0, 3\}$.

The following is a construction of a deformation algebra on any divisible d-algebra.

Theorem 4.7. *Let $(X, *, 0)$ be a divisible d-algebra and let $a \in X - \{0\}$. Define a map $\varphi_a : X \to X$ by $\varphi_a(x) = 0$ $(x \neq a)$ and $\varphi_a(a) = \widehat{a}$, where the choice of sectors has been fixed. Then φ_a is a deformation function of X.*

Proof. Let $(X,*,0)$ be a divisible d-algebra and let $a \in X-\{0\}$. Define a map $\varphi_a : X \to X$ by $\varphi_a(x) = 0\,(x \neq a)$ and $\varphi_a(a) = \widehat{a}$, the choice of associators has been fixed. Then we have

$$x * \varphi_a(x) = \begin{cases} x*0 & \text{if } x \neq a, \\ a * \widehat{a} & \text{otherwise.} \end{cases}$$

Thus $a * \varphi_a(a) = a * \widehat{a} \notin \{0, a\}$, since X is a divisible algebra. We claim that φ_a is a deformation function of X. If $x \neq 0$, then by (III) we obtain $x * 0 \neq 0$, which means that $x * \varphi_a(x) = x * 0 \neq 0$ when $x \neq a$. □

Note that if the d-algebra has the property that $x * 0 = x$ (e.g., if it is an edge d-algebra or a BCK-algebra), then $x * \varphi_a(x) = x$ if $x \neq a$, and the only non-fixed point of the action $x \to x * \varphi_a(x)$ is a itself. In any case, divisible d-algebras are quite rich in deformation functions.

If S is any subset of $X\backslash\{0\}$, we let φ_S be defined by $\varphi_S(x) = 0$ if $x \notin S$ and $\varphi_S(x) = \widehat{x}$ for $x \in S$. If the choice of the (non-unique) element $\widehat{x}$ has been fixed, then we obtain a correspondence $S \longleftrightarrow \varphi_S$, so that we may think of φ_S as a "characteristic deformation function" for $S \in P(X\backslash\{0\})$.

Proposition 4.8. *Let $(X,*,0)$ be a divisible d-algebra and let $\emptyset \neq S \subset X\backslash\{0\}$. Let φ_S be a characteristic deformation function. If we define $x \bigtriangledown y := (x*y) * \varphi_S(x*y)$ for any $x, y \in X$, then $(X, \bigtriangledown, 0)$ is a d-algebra.*

Proof. If $x*y \notin S$, then $\varphi_S(x*y) = 0$ and hence $x \bigtriangledown y = (x*y)*0$. If $x*y \in S$, then $x \bigtriangledown y = (x*y) * (\widehat{x*y}) \notin \{0, x*y\}$. Hence $x \bigtriangledown x = 0 \bigtriangledown x = 0$ for any $x \in X$. Suppose $x \bigtriangledown y = y \bigtriangledown x = 0$. Then $(x*y)*0 = 0 = (y*x)*0$, since the case $x*y \in S$ does not happen. Since $(X,*,0)$ is a d-algebra, we obtain $x*y = 0 = y*x$ and $x = y$, proving the proposition. □

Example 4.9. In Example 4.3, if we let $S := \{1,2,4\} \subset X\backslash\{0\}$ and if we define $\varphi_S(0) = \varphi_S(3) = 0$ and $\varphi_S(1) = 2, \varphi_S(2) = 3, \varphi_S(4) = 1$, then we have the following table:

$\bigtriangledown$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	b	d	c
b	b	c	0	d	d
c	b	c	d	0	c
d	d	c	d	b	0

If we let $\widehat{a} = c, \widehat{b} = d, \widehat{d} = a$, then it is easy to show that $(X, \bigtriangledown, 0)$ is a divisible d-algebra.

The following question arises: In Proposition 4.8, is the d-algebra $(X, \bigtriangledown, 0)$ always divisible? The answer is negative. Consider the following example.

Example 4.10. In Example 4.3, if we take $S := \{0, 2, 3\}$ and if we define $\varphi_S(0) = \varphi_S(2) = \varphi_S(3) = 0$ and $\varphi_S(1) = 2, \varphi_S(4) = 3$, then we obtain the following table:

$\bigtriangledown$	0	a	b	c	d
0	0	0	0	0	0
a	d	0	d	d	d
b	d	d	0	d	d
c	d	d	d	0	d
d	d	d	d	d	0

which shows that the d-algebra $(X, \bigtriangledown, 0)$ is not divisible.

The following question naturally arises: Under what condition(s) does the d-algebra $(X, \bigtriangledown, 0)$ become a divisible d-algebra?

Theorem 4.11. *Let $(X, *, 0)$ be a divisible d-algebra with conditions*:

(VIII) $(x * y) * 0 = x$ *implies* $x * y = x$,
(IX) *if* $x * \widehat{x} = a$, *then* $a * \widehat{a} \notin \{0, a, x\}$.

Define a binary operation "$\bigtriangledown$" on X by

$$x \bigtriangledown y := (x * y) * \varphi_a(x * y), \forall x, y \in X.$$

Then $(X, \bigtriangledown, 0)$ is a divisible d-algebra.

Proof. Since φ_a is a deformation function, it is easy to show that $(X, \bigtriangledown, 0)$ is a d-algebra. Given $x \neq 0$ in X, suppose that $x*\widehat{x} \neq a$. Then $x \bigtriangledown \widehat{x} = (x*\widehat{x})*\varphi_a(x*\widehat{x}) = (x*\widehat{x})*0$. We claim that $x \bigtriangledown \widehat{x} \notin \{0, x\}$. In fact, assume that $x \bigtriangledown \widehat{x} = 0$. Then $(x * \widehat{x}) * 0 = 0$. Since $(X, *, 0)$ is a d-algebra, we have $x * \widehat{x} = 0$, which means that $(X, *, 0)$ is not divisible, a contradiction. Assume that $x \bigtriangledown \widehat{x} = x$. Then $x = (x*\widehat{x})*0$. By applying (VIII) we obtain $x = x*\widehat{x}$, which means that $(X, *, 0)$ is not divisible, a contradiction. Next, we consider the case $x * \widehat{x} = a$. $x \bigtriangledown \widehat{x} = (x * \widehat{x}) * \varphi_a(x * \widehat{x}) = a * \varphi_a(a) = a * \widehat{a} \notin \{0, x\}$ by Theorem 4.7. By (IX) $x * \widehat{x} = a$ implies $a * \widehat{a} \notin \{0, a, x\}$. Hence $x \bigtriangledown \widehat{x} \notin \{0, a, x\} \supset \{0, x\}$. Hence $(X, \bigtriangledown, 0)$ is divisible. □

Example 4.12. In Example 4.2, if we let $\widehat{x} := x/3$ for any $x \neq 0$ in X, then $(X, *, 0)$ is a divisible BCK-algebra. Consider the condition (VIII). Assume that $x*\widehat{x} = a$. Then $a*\widehat{a} = 2a/3 = 4x/9 < x$ for $x > 0$. Hence $a*\widehat{a} \notin \{0, a, x\}$. The condition (IX) holds trivially. By Theorem 4.11, $(X, \bigtriangledown, 0)$ is a divisible d-algebra. But it is not a BCK-algebra, since $2x/3 \bigtriangledown 0 = 4x/9 \neq 2x/3$.

Consider the following condition for a BCK-algebra $(X, *, 0)$

(X) if $(x * y) * z = x$ and $x \neq 0$, then $y = 0$.

Then also $z = 0$, since $(x * y) * z = (x * z) * y$.

Note that not all BCK-algebras satisfy condition (X).

Example 4.13. Let $(X,*,0)$ be the standard BCK-algebra of a poset $(X,\leq,0)$ with minimal element 0. If $x\not\leq y, y\not\leq x, z\not\leq x, x\not\leq z, y\not\leq z, z\not\leq y$, i.e., $\{x,y\ z\}$ is an antichain, then $(x*y)*z=x*z, x\neq 0$ and $z\neq 0$, whence condition (X) fails for such BCK-algebras.

Example 4.14. The BCK-algebra in Example 4.2 satisfies condition (X). In fact, if we let $x\neq 0, (x*y)*z=x$, then $x=\max\{0,x*y-z\}=x*y-z$. It follows that $\max\{0,x-y\}=x*y=z+x\geq x>0$. Hence $x-y=z+x$ and $y+z=0$, proving $y=z=0$. Thus $(X,*,0)$ satisfies condition (X).

Proposition 4.15. *Let $(X,*,0)$ be a divisible BCK-algebra with condition* (X)*. Then condition* (IX) *holds.*

Proof. Let $(X,*,0)$ be a divisible BCK-algebra with condition (X). Suppose that $x*\widehat{x}=a$ and consider $a*\widehat{a}=b$. We note that from the divisibility we have $a\notin\{0,x\}$ and $b\notin\{0,a\}$. If $a*\widehat{a}\in\{0,a,x\}$, then $a*\widehat{a}=x$ and $(x*\widehat{x})*\widehat{a}=x$. By condition (X) it follows that $\widehat{x}=0$ and $a=x*\widehat{x}=x*0=x\in\{0,x\}$, a contradiction. Hence $a*\widehat{a}\notin\{0,a,x\}$ and condition (IX) holds. □

An element m of a d-algebra $(X,*,0)$ is said to be *maximum* if $x*m=0$ for all $x\in X$.

Proposition 4.16. *Let $(X,*,0)$ be a divisible d-algebra. Then no element can have m as a sector of it.*

Proof. Straightforward. □

Note that $m*\widehat{m}\notin\{0,m\}$ implies $0*\widehat{m}=0, \widehat{m}*m=0$, i.e., $0\neq\widehat{m}\neq m$. Since X is divisible and m is maximal element, $\widehat{\widehat{m}}\notin\{0,m,\widehat{m}\}$. We have a question: Can a divisible d-algebra $(X,*,0)$ be constructed with $|X|=4$? The answer is yes. Consider the following table:

$*$	0	$\widehat{m}$	$\widehat{\widehat{m}}$	m
0	0	0	0	0
$\widehat{m}$	$\widehat{m}$	0	$\widehat{\widehat{m}}$	0
$\widehat{\widehat{m}}$	$\widehat{\widehat{m}}$	$\widehat{m}$	0	0
m	m	$\widehat{m}$	$\widehat{\widehat{m}}$	0

5. Comments

In the foregoing we have observed that from d/BCK-algebras, new examples may be constructed via the deformation process. This brought to the foreground interesting classes of d/BCK-algebras, viz., the rigid d/BCK-algebras and the divisible BCK-algebras. Whereas the rigid BCK-algebras seem to be modeled on edge d-algebras and the standard BCK-algebras associated with posets with minimum elements, the class of divisible d/BCK-algebras seems to be much larger in some sense as well as perhaps more mysterious and in need of closer analysis.

References

[1] P. J. Allen, H. S. Kim, and J. Neggers, *Companion d-algebras*, Math. Slovaca **57** (2007), no. 2, 93–106.
[2] W. A. Dudek and X. H. Zhang, *On ideals and congruences in BCC-algebras*, Czechoslovak Math. J. **48(123)** (1998), no. 1, 21–29.
[3] Q. Hu and X. Li, *On proper BCH-algebras*, Math. Japon. **30** (1985), no. 4, 659–661.
[4] Y. B. Jun, J. Neggers, and H. S. Kim, *Fuzzy d-ideals of d-algebras*, J. Fuzzy Math. **8** (2000), no. 1, 123–130.
[5] Y. Komori, *The class of BCC-algebras is not a variety*, Math. Japon. **29** (1984), no. 3, 391–394.
[6] Y. C. Lee and H. S. Kim, *On d^*-subalgebras of d-transitive d^*-algebras*, Math. Slovaca **49** (1999), no. 1, 27–33.
[7] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa, Seoul, 1994.
[8] J. Neggers, Y. B. Jun, and H. S. Kim, *On d-ideals in d-algebras*, Math. Slovaca **49** (1999), no. 3, 243–251.
[9] J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca **49** (1999), no. 1, 19–26.
[10] S. Tanaka, *Examples of BCK-algebras*, Math. Sem. Notes Kobe Univ. **3** (1975), no. 1, 75–82.
[11] H. Yisheng, *BCI-Algebras*, Science Press, Beijing, 2006.

Paul J. Allen
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350, USA
E-mail address: pallen@as.ua.edu

Hee Sik Kim
Department of Mathematics
Research Institute for Natural Science
Hanyang University
Seoul 133-791, Korea
E-mail address: heekim@hanyang.ac.kr

Joseph Neggers
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350, USA
E-mail address: jneggers@as.ua.edu