

PRECONDITIONED GAUSS-SEIDEL ITERATIVE METHOD FOR Z-MATRICES LINEAR SYSTEMS

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ABSTRACT. For $Ax = b$, it has recently been reported that the convergence of the preconditioned Gauss-Seidel iterative method which uses a matrix of the type $P = I + S(\alpha)$ to perform certain elementary row operations on is faster than the basic Gauss-Seidel method. In this paper, we discuss the adaptive Gauss-Seidel iterative method which uses $P = I + S(\alpha) + \tilde{K}(\beta)$ as a preconditioner. We present some comparison theorems, which show the rate of convergence of the new method is faster than the basic method and the method in [7] theoretically. Numerical examples show the effectiveness of our algorithm.

1. Introduction

We consider iterative methods for solving a linear system

$$(1) \quad Ax = b,$$

where A is an $n \times n$ matrix with unit diagonal elements, x and b are n -dimensional vectors.

If we write $A = M - N$ with a nonsingular matrix M , then the basic iterative scheme for (1) is defined by

$$(2) \quad Mx_{k+1} = Nx_k + b, \quad k = 0, 1, 2, \dots$$

Let $T = M^{-1}N$ and $c = M^{-1}b$. Then (2) can be also be written as

$$(3) \quad x_{k+1} = Tx_k + c, \quad k = 0, 1, 2, \dots$$

Writing $A = I - L - U$ and taking $M = I - L$ in (2) or (3) yield the classic Gauss-Seidel method, where I is the identity matrix, and L and D are strictly lower and strictly upper triangular matrices, respectively. The matrix $T = (I - L)^{-1}U$ is then called the Gauss-Seidel iteration matrix.

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We now transform the original system (1) into the preconditioned form

$$(4) \quad PAx = Pb.$$

Then the corresponding basic iterative scheme is

$$(5) \quad M_p x_{k+1} = N_p x_k + Pb, \quad k = 0, 1, 2, \dots,$$

where $PA = M_p - N_p$ is a regular splitting of PA , and M_p is a nonsingular matrix, the nonsingular matrix P is called a preconditioner, the iterative method used in linear equations $PAx = Pb$ is called the preconditioned iterative method.

In 1987, Milaszewicz [8] used the following preconditioner

$$(6) \quad P = I + C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{pmatrix}.$$

In 2003, Hadjidimos et al. [3] presented the following preconditioner

$$(7) \quad P(\alpha) = I + C(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\alpha_2 a_{21} & 1 & 0 & \cdots & 0 \\ -\alpha_3 a_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n a_{n1} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The preconditioner $P(\alpha)$ in (7) is the general preconditioner of the matrix P in (6), when $\alpha_i = 1, i = 2, 3, \dots$, the preconditioner became the matrix P .

In 1991, Gunawardena et al. [2] employed the following preconditioner

$$(8) \quad P_S = I + S = \begin{pmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -a_{23} & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In 1997, Kohno et al. [5] employed the general preconditioned form of the matrix

$$(9) \quad P_S(\alpha) = I + S(\alpha) = \begin{pmatrix} 1 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2 a_{23} & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

When $\alpha_i = 1, i = 1, 2, \dots, n - 1$, the preconditioner $P_S(\alpha)$ is equal to P_S .

In [6], $P = I + S(\alpha) + K(\beta)$ is presented, where

$$(10) \quad K(\beta) = \begin{cases} -\beta_i a_{ij}, & i = j + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The comparison theorem and numerical examples show that the method is the improvement for the iterative method in [5].

In this paper, we first consider the new preconditioner $P = I + S(\alpha) + \tilde{K}(\beta)$, where

$$(11) \quad \tilde{K}(\beta) = \begin{cases} -\beta_i a_{i1}, & i = 2, 3, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Next we discuss its convergence. Finally, we show with numerical examples that this method yields a considerable improvement in the rate of convergence for the iterative method.

2. Basic results

First, we need the following definitions and results.

Definition 1 ([4]). An $n \times n$ real matrix $A = (a_{ij})$ is called Z -matrix, if $a_{ij} \leq 0, \forall i \neq j$, and $a_{ii} > 0$; and an A is called diagonal dominant (or strictly diagonal dominant) Z -matrix, if $\sum_{j=1}^n a_{ij} \geq 0$ (or $\sum_{j=1}^n a_{ij} > 0$).

Definition 2 ([4]). A matrix A is irreducible if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix},$$

where A_1 and A_3 are square matrices.

Definition 3 ([1]). Let $A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}$. We say $A > B$ if $a_{ij} > b_{ij}, \forall 1 \leq i, j \leq n$.

Definition 4 ([1]). Let $A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}$. We say $A \geq B$ if $a_{ij} \geq b_{ij}, \forall 1 \leq i, j \leq n$.

Lemma 1 ([4]). Let A be a nonnegative matrix. Then

- (1) A has a positive real eigenvalue equal to its spectral radius.
- (2) To the spectral radius $\rho(A)$ there corresponds an eigenvector $x > 0$.
- (3) If A is an irreducible matrix, then $\rho(A) > 0$ and there exists $x > 0$ such that $Ax = \rho(A)x$.

Lemma 2 ([4]). Let A be a nonnegative matrix. Then

- (1) If $\alpha x \leq Ax$ for some nonnegative vector $x \neq 0$, then $\alpha \leq \rho(A)$.
- (2) A is a irreducible matrix if and only if $(I + A)^{n-1} > 0$.
- (3) If $Ax \leq \beta x$ for some positive vector x , then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x, \alpha x \neq Ax$ and $Ax \neq \beta x$ for some nonnegative vector x , then $\alpha < \rho(A) < \beta$, and x is a positive vector.
- (4) If B is an $n \times n$ matrix, $0 \leq A \leq B, A \neq B, A + B$ are all irreducible matrices, then $\rho(A) < \rho(B)$.

Lemma 3 ([6]). *Let $A = D - E - F$ be a Z-matrix, where $D, -E$ and $-F$ are diagonal, strictly lower triangular and strictly upper triangular parts of A , respectively. Then an upper bound of the spectral radius for the Gauss-Seidel iterative matrix \tilde{T} is given by*

$$\rho(T) \leq \max \frac{\tilde{f}_i}{\tilde{d}_i - \tilde{e}_i}, \quad \forall i,$$

where \tilde{d}_i, \tilde{e}_i and \tilde{f}_i are sums of elements in the i -th row of D, E and F , respectively.

3. New preconditioned iterative method and convergence analysis

We propose a preconditioned iterative method with

$$P = I + S_\alpha^\beta = \begin{pmatrix} 1 & -\alpha_1 a_{12} & & & \\ -\beta_2 a_{21} & 1 & -\alpha_2 a_{23} & & \\ -\beta_2 a_{21} & & 1 & \ddots & \\ \vdots & & & \ddots & -\alpha_{n-1} a_{n-1,n} \\ -\beta_n a_{n1} & & & & 1 \end{pmatrix},$$

where $S_\alpha^\beta = S(\alpha) + \tilde{K}(\beta)$,

$$S(\alpha) = \begin{pmatrix} 0 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{23} & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\tilde{K}(\beta) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -\beta_2 a_{21} & 0 & 0 & \ddots & \vdots \\ -\beta_3 a_{31} & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\beta_n a_{n,1} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$A_\alpha^\beta = (I + S_\alpha^\beta) A = I - L - U + S_\alpha^\beta - S_\alpha^\beta (L + U),$$

$$b_\alpha^\beta = (I + S_\alpha^\beta) b.$$

Let $S_\alpha^\beta (L + U) = D_0 + L_0 + U_0$, where D_0, L_0 and U_0 are diagonal, strictly lower triangular and strictly upper triangular parts of $S_\alpha^\beta (L + U)$, respectively, and

$$D_0 = \text{diag}\{\beta_1 a_{12} a_{21}, \alpha_2 a_{21} a_{12} + \beta_2 a_{23} a_{32}, \dots, \alpha_{n-1} a_{n-1,1} a_{1,n-1}\}$$

$$+ \beta_{n-1}a_{n-1,1}a_{1,n-1}, \alpha_n a_{n,1}a_{1,n}\},$$

then

$$(12) \quad A_\alpha^\beta = (I - D_0 - L + S_\alpha - L_0) - (U - S^\beta + U_0).$$

If

$$\begin{cases} \alpha_1 a_{12} a_{21} \neq 1, & i = 1, \\ \alpha_i a_{i,i+1} a_{i+1,i} + \beta_i a_{i,1} a_{1,i} \neq 1, & i = 2, \dots, n-1, \\ \beta_n a_{n,1} a_{1,n} \neq 1, & i = n, \end{cases}$$

then matrix $(I - D_0 - L + S_\alpha - L_0)^{-1}$ exists and the Gauss-Seidel iteration matrix T_α^β for A_α^β is defined by

$$T_\alpha^\beta = (I - D_0 - L + S_\alpha - L_0)^{-1} (U - S^\beta + U_0).$$

Let

$$\alpha = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n), \quad \beta = \text{diag}(\beta_1, \beta_2, \dots, \beta_n)$$

and

$$(13) \quad S = \begin{pmatrix} 0 & -a_{12} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -a_{n,1} & 0 & \cdots & 0 \end{pmatrix}.$$

Then $S_\alpha^\beta = S(\alpha) + \tilde{K}(\beta) = \alpha S + \beta \tilde{K}$. If α and β are identity matrices, then

$$(14) \quad \tilde{A} = (I + S + \tilde{K})A = \tilde{D} - \tilde{L} - \tilde{U}, \quad \tilde{T} = (\tilde{D} - \tilde{L})^{-1} \tilde{U}.$$

Obviously, when $\alpha_i = 0, i = 1, 2, \dots, n$, the new method in this paper becomes the method in [3]; when $\beta_i = 0, i = 1, 2, \dots, n$, the new method in this paper becomes the method in [5].

Theorem 1. *Let $A \in R^{n \times n} (n \geq 3)$ be a nonsingular diagonal dominant Z-matrix with unit diagonal elements. If for $0 < \alpha_i \leq 1, 0 < \beta_i \leq 1 (i = 1, 2, \dots, n)$, A satisfies the following conditions, then A_α^β is a strictly diagonal dominant Z-matrix and $\rho(T_\alpha^\beta) < 1$.*

(1) $a_{ii} - |a_{i1}| - |a_{i,i+1}| > 0, a_{11} - |a_{12}| > 0, a_{nn} - |a_{n1}| > 0$ for any $i = 2, 3, \dots, n-1$.

(2) If $\sum_{j=1}^n a_{ij} = 0$ for $i = 1, 2, \dots, n-1$, then $a_{i,1} a_{i,i+1} \neq 0$.

(3) There exists at least a row for i -th row and $(i+1)$ -th row such that it is strictly diagonal dominant row for any $i = 1, 2, \dots, n-1$.

(4) There exists at least a strictly diagonal dominant row for the first row and n -th row; and if $\sum_{j=1}^n a_{nj} = 0$, then $a_{n1} \neq 0$.

Proof. In fact,

$$(A_\alpha^\beta)_{ii} = a_{ii} - \alpha_i a_{i,i+1} a_{i+1,i} - \beta_i a_{i1} a_{1i} \geq a_{ii} - |a_{i,i+1}| - |a_{i,1}| > 0.$$

For $i > j, i = 2, \dots, n$, when $j = 1$,

$$(15) \quad (A_\alpha^\beta)_{i1} = a_{i1} - \alpha_i a_{i,i+1} a_{i+1,i} - \beta_i a_{i1} a_{11} = (1 - \beta_i) a_{i1} - \alpha_i a_{i,i+1} a_{i+1,i} \leq 0,$$

when $j = 2, \dots, n - 1$,

$$(16) \quad (A_\alpha^\beta)_{ij} = a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j} - \beta_i a_{i1} a_{1j} \leq 0.$$

For $i < j, i = 1, 2, \dots, n - 1, j = 2, \dots, n$, when $i + 1 = j$,

$$(17) \quad (A_\alpha^\beta)_{i,i+1} = (1 - \alpha_i) a_{i,i+1} - \beta_i a_{i1} a_{1,i+1} \leq 0.$$

Otherwise

$$(18) \quad (A_\alpha^\beta)_{ij} = a_{i,j} - \alpha_i a_{i,i+1} a_{i+1,j} - \beta_i a_{i1} a_{1j} \leq 0.$$

Hence, A_α^β is a Z -matrix.

From (2), (3) and (4), there exists at least an inequality such that it holds for the following inequalities:

$$\sum_{j=1}^n a_{ij} > 0, \quad -\alpha_i a_{i,i+1} \sum_{j=1}^n a_{i+1,j} > 0, \quad -\beta_i a_{i1} \sum_{j=1}^n a_{1j} > 0.$$

Then

$$(19) \quad \sum_{j=1}^n (A_\alpha^\beta)_{ij} = \sum_{j=1}^n a_{ij} - \alpha_i a_{i,i+1} \sum_{j=1}^n a_{i+1,j} - \beta_i a_{i1} \sum_{j=1}^n a_{1j} > 0.$$

Hence, we know that A_α^β is a strictly diagonal dominant Z -matrix.

Now we decompose A_α^β , let $A_\alpha^\beta = D_\alpha^\beta - L_\alpha^\beta - U_\alpha^\beta$, where $D_\alpha^\beta, -L_\alpha^\beta$ and $-U_\alpha^\beta$ are diagonal, strictly lower triangular and strictly upper triangular parts of A_α^β , respectively, and let $d_{\alpha,i}^\beta, l_{\alpha,i}^\beta$ and $u_{\alpha,i}^\beta$ are the sum of i -th row elements of $D_\alpha^\beta, L_\alpha^\beta$ and U_α^β , respectively. From (18), we have $d_{\alpha,i}^\beta - l_{\alpha,i}^\beta - u_{\alpha,i}^\beta > 0$; and from (17) and (18), we have $u_{\alpha,i}^\beta \geq 0$. So

$$\frac{u_{\alpha,i}^\beta}{d_{\alpha,i}^\beta - l_{\alpha,i}^\beta} < 1.$$

From Lemma 3, $\rho(T_\alpha^\beta) < 1$. The proof is complete. □

Theorem 2. Let $A = I - L - U$ be a Z -matrix, $0 < a_{i1} a_{1i} + a_{i,i+1} a_{i+1,i} < 1$ for $i = 2, \dots, n - 1$, and $0 < a_{12} a_{21} < 1, 0 < a_{n1} a_{1n} < 1$. If $\rho(T) = 1$, then $\rho(\tilde{T}) = \rho(T) = 1$.

Proof. It is obviously that \tilde{A} in (14) is a Z -matrix. Since

$$\begin{aligned} \tilde{A} &= (I + \tilde{K} + S) A = (I + \tilde{K} + S) (I - L - U) \\ &= I + \tilde{K} - L - SL - \tilde{K}L - \tilde{K}U - U - SU + S, \end{aligned}$$

we have $-\tilde{K}L = 0, I + \tilde{K} - L - SL$ is a lower triangular matrix; $-U - SU + S$ is an upper triangular matrix. But

$$-\tilde{K}U = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -a_{21}a_{12} & -a_{21}a_{13} & \cdots & -a_{21}a_{1n} \\ 0 & -a_{31}a_{12} & -a_{31}a_{13} & \cdots & -a_{31}a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{n1}a_{12} & -a_{n1}a_{13} & \cdots & -a_{n1}a_{1n} \end{pmatrix}.$$

If $-\tilde{K}U = M_1 + M_2$, where M_2 is a strictly upper triangular part of $-\tilde{K}U$, then

$$\tilde{T} = (\tilde{D} - \tilde{L})^{-1} \tilde{U} = (I + \tilde{K} - L - SL + M_1)^{-1} (U + SU - S - M_2).$$

Hence T and \tilde{T} are nonnegative matrices, and

$$(20) \quad T = \begin{pmatrix} 0 & T_{12} \\ 0 & T_{22} \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} 0 & \tilde{T}_{12} \\ 0 & \tilde{T}_{22} \end{pmatrix},$$

where T_{22} and \tilde{T}_{22} are all $(n - 1) \times (n - 1)$ irreducible matrices. From $\rho(T) = 1$, we have $\rho(T_{22}) = 1$.

From Lemma 4, $T_{22}x' = x'$ for some vector $x' > 0$. Now we construct the positive vector $x = \begin{pmatrix} T_{12}x' \\ x' \end{pmatrix}$, it satisfies $Tx = x$.

Since $(I - L)^{-1}Ux = x, Ux = (I - L)x$ and

$$\tilde{T}x = (I + \tilde{K} - L - SL + M_1)^{-1} (U + SU - S - M_2)x,$$

so we have

$$\begin{aligned} (U + SU - S - M_2)x &= (U + SU - S + \tilde{K}U + M_1)x \\ &= Ux + SUx - Sx + \tilde{K}Ux + M_1x \\ &= (I - L)x + S(I - L)x - Sx + \tilde{K}(I - L)x + M_1x \\ &= (I + \tilde{K} - L - SL + M_1)x. \end{aligned}$$

Hence $\tilde{T}x = x$. From Lemma 2, we have $\rho(\tilde{T}) = 1$. The proof is complete. \square

Theorem 3. Let $A = I - L - U$ be an $n \times n$ ($n \geq 3$) Z -matrix. $0 < a_{i1}a_{1i} + a_{i,i+1}a_{i+1,i} < 1$ for $i = 2, \dots, n - 1$, and $0 < a_{12}a_{21} < 1, 0 < a_{n1}a_{1n} < 1$. Then

- (1) $\rho(\tilde{T}) < \rho(T)$ for $\rho(T) < 1$,
- (2) $\rho(\tilde{T}) > \rho(T)$ for $\rho(T) > 1$.

Proof. From Theorem 2, there exists a positive vector $x = \begin{pmatrix} T_{12}x' \\ \rho(T)x' \end{pmatrix}$ such that $Tx = \rho(T)x$. We have

$$\tilde{T}x - Tx$$

$$\begin{aligned}
 &= (\tilde{D} - \tilde{L})^{-1} (U + SU - S - M_2)x - (I - L)^{-1} Ux \\
 &= (\tilde{D} - \tilde{L})^{-1} \left[(U + SU - S - M_2)x - (\tilde{D} - \tilde{L}) \rho(T)x \right] \\
 &= (\tilde{D} - \tilde{L})^{-1} \left[(U + SU - S - M_2)x - (I + \tilde{K} - L - SL + M_1) \rho(T)x \right] \\
 &= (\tilde{D} - \tilde{L})^{-1} [\rho(T)Sx - Sx + M_1x - \rho(T)M_1x] \\
 &= (\rho(T) - 1) (\tilde{D} - \tilde{L})^{-1} (S - M_1)x.
 \end{aligned}$$

We know easily that $(\tilde{D} - \tilde{L})^{-1} (S - M_1)$ is a nonnegative matrix, so $\tilde{T}x \leq Tx = \rho(T)x$ when $\rho(T) < 1$; $\tilde{T}x \geq Tx = \rho(T)x$ when $\rho(T) > 1$. From (20), Lemma 4 and Lemma 5, $\rho(\tilde{T}) < \rho(T)$ for $\rho(T) < 1$, and $\rho(\tilde{T}) > \rho(T)$ for $\rho(T) > 1$. The proof is complete. \square

Theorem 4. Let $A = I - L - U$ be a Z -matrix. $0 < a_{i1}a_{1i} + a_{i,i+1}a_{i+1,i} < 1$ for $i = 2, \dots, n - 1$, and $0 < a_{12}a_{21} < 1, 0 < a_{n1}a_{1n} < 1$. Then for any $0 \leq \alpha_i \leq 1, 0 \leq \beta_i \leq 1$, we have

- (1) $\rho(T_\alpha^\beta) < \rho(T)$ for $\rho(T) < 1$,
- (2) $\rho(T_\alpha^\beta) > \rho(T)$ for $\rho(T) > 1$.

Proof. Since

$$\begin{aligned}
 T_\alpha^\beta x - Tx &= (D_\alpha^\beta - L_\alpha^\beta)^{-1} (U + \alpha SU - \alpha S - \beta M_2)x - (I - L)^{-1} Ux \\
 &= (\rho(T) - 1) (D_\alpha^\beta - L_\alpha^\beta)^{-1} (S - \beta M_1)x,
 \end{aligned}$$

and $0 \leq \alpha_i \leq 1, 0 \leq \beta_i \leq 1$, $(D_\alpha^\beta - L_\alpha^\beta)^{-1} (S - \beta M_1)$ is also a nonnegative matrix. Then we have $T_\alpha^\beta x \leq Tx = \rho(T)x$ for $\rho(T) < 1$; $T_\alpha^\beta x \geq Tx = \rho(T)x$ for $\rho(T) > 1$. Hence we have our conclusion. \square

4. Numerical examples

Example 1. Let

$$A = \begin{pmatrix} 1 & -0.4 & -0.3 \\ -0.5 & 1 & -0.2 \\ -0.6 & -0.5 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0.3 \\ 0.3 \\ -0.1 \end{pmatrix}.$$

It is obviously that A is a nonsingular Z -matrix, but it is not diagonal dominant matrix, so it does not satisfy the conditions in [5]. At the same time,

$$(I + S)A = \begin{pmatrix} 0.8 & 0 & -0.38 \\ -0.62 & 0.9 & 0 \\ -0.6 & -0.5 & 1 \end{pmatrix}$$

is not strictly diagonal dominant Z -matrix. But

$$\tilde{A} = (I + S + \tilde{K}) A = \begin{pmatrix} 0.8 & 0 & -0.38 \\ -0.12 & 0.7 & -0.15 \\ 0 & -0.74 & 0.82 \end{pmatrix}$$

is a strictly diagonal dominant Z -matrix.

Now the classical Gauss-Seidel iterative method and the preconditioned Gauss-Seidel iterative method are considered for solving the linear system of equation (1). The stopping criterion $\|r_k\|_\infty < 10^{-3}$ was used in the computations, where $r_k = b - Ax^{(k)}$ and $x^{(k)}$ is the k -th iteration for each of the methods. The number of iterations (IT) for convergence needed for the classical Gauss-Seidel iterative method and the preconditioned Gauss-Seidel iterative method are listed in Table 1 and Table 2, respectively. Thus it can be seen that the preconditioned Gauss-Seidel method proposed in this paper has a faster convergence rate than that of the classical Gauss-Seidel iterative method. Let $x^{(0)} = (0, 0, 0)^T$, the classical Gauss-Seidel iterative method took 16 iterations for the convergence, while it took 6 iterations for the convergence of the preconditioned Gauss-Seidel method. But the two methods took the same CPU times.

Table 1. Numerical results of classical Gauss-Seidel method

Ite.	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\ x^{(k)} - x^*\ _\infty$
0	0	0	0	1
1	0.3000	0.4500	0.3050	0.7000
2	0.5215	0.6467	0.5663	0.4337
3	0.7286	0.7775	0.7259	0.2741
4	0.8288	0.8596	0.8271	0.1729
5	0.8920	0.9114	0.8909	0.1091
6	0.9318	0.9441	0.9311	0.0689
7	0.9570	0.9647	0.9565	0.0435
8	0.9728	0.9777	0.9726	0.0274
9	0.9829	0.9859	0.9827	0.0173
10	0.9892	0.9911	0.9891	0.0109
11	0.9932	0.9944	0.9931	0.0069
12	0.9957	0.9965	0.9957	0.0044
13	0.9973	0.9978	0.9973	0.0027
14	0.9983	0.9986	0.9983	0.0017
15	0.9989	0.9991	0.9989	0.0011
16	0.9993	0.9994	0.9993	0.0007

Table 2. Numerical results of preconditioned Gauss-Seidel method with $P = I + S + \tilde{K}$.

Ite.	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\ x^{(k)} - x^*\ _\infty$
0	0	0	0	1
1	0.5250	0.7043	0.7331	0.4750
2	0.8732	0.9211	0.9288	0.1268
3	0.9662	0.9789	0.9810	0.0338
4	0.9910	0.9944	0.9949	0.0090
5	0.9976	0.9985	0.9986	0.0024
6	0.9994	0.9996	0.9996	0.0006
7	0.9998	0.9999	0.9999	0.0002
8	1.0000	1.0000	1.0000	0.0000

Example 2. Let

$$A = \begin{pmatrix} 1 & -0.1 & -0.2 & -0.01 & 0 \\ -0.4 & 1 & -0.4 & 0 & -0.1 \\ -0.2 & -0.5 & 1 & -0.1 & -0.1 \\ -0.45 & -0.1 & -0.2 & 1 & -0.1 \\ -0.1 & -0.12 & -0.2 & -0.6 & 1 \end{pmatrix}.$$

Then we report the spectral radius of three iterative methods in Table 3.

Table 3. Comparisons of the three iterative methods in [5], [7] and this paper

(α_i, β_i)	(0.7, 0.7)	(0.9, 0.9)	(0.8, 1)	(1, 1)	(0.8, 0.9)	(0.9, 1)
GS1: $\rho(I + \alpha S)$	0.4059	0.3629	0.3848	0.3403	0.3848	0.3629
GS2: $\rho(I + \alpha S + \beta K)$	0.3785	0.3442	0.3540	0.3309	0.3572	0.3422
GS3: $\rho(I + \alpha S + \beta \tilde{K})$	0.3590	0.2947	0.2385	0.2651	0.3196	0.2851

The above matrix S is the form in [5], matrices K and \tilde{K} are as follows:

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 \\ 0.45 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\rho(\cdot)$ is the spectral radius of the corresponding iterative method. Numerical results for Example 1 are shown in Fig. 1.

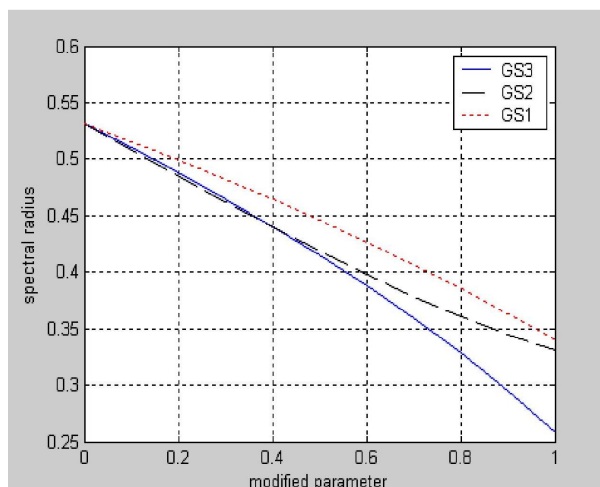


Fig. 1.

From the above numerical examples, we see that the preconditioning effectiveness of $(I + \alpha S + \beta \tilde{K})$ -type preconditioners constructed in this paper is obvious.

5. Conclusion

In this paper, we have presented the new preconditioned Gauss-Seidel iterative method for the Z -matrices linear systems. It remains to construct the comparison theorems for the iterative methods. We conclude that the rate of convergence of the new method in this paper is faster than the rate of convergence of the methods in [5] and [7] by theoretical analysis and numerical examples.

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