Bull. Korean Math. Soc. **48** (2011), No. 2, pp. 277–290
DOI 10.4134/BKMS.2011.48.2.277

CHARACTERIZATIONS OF ELEMENTS IN PRIME RADICALS OF SKEW POLYNOMIAL RINGS AND SKEW LAURENT POLYNOMIAL RINGS

Jeoung Soo Cheon, Eun Jeong Kim, Chang Ik Lee, and Yun Ho Shin

Abstract. We show that the θ-prime radical of a ring R is the set of all strongly θ-nilpotent elements in R, where θ is an automorphism of R. We observe some conditions under which the θ-prime radical of R coincides with the prime radical of R. Moreover we characterize elements in prime radicals of skew Laurent polynomial rings, studying (θ, θ^{-1})-(semi)primeness of ideals of R.

1. Introduction

Throughout R denotes a ring with identity and $\theta : R \to R$ is an automorphism of R. We use $\mathbb{Z}$ to denote the ring of integers. An ideal I of R is called a θ-ideal if $\theta(I) \subseteq I$, and is called *θ-invariant* if $\theta(I) = I$. There are some examples of θ-ideals which are not θ-invariant.

Example 1.1. Let K be any ring and $T = K[x_i \mid i \in \mathbb{Z}]$ be the free algebra over K in the commuting indeterminates x_i, $i \in \mathbb{Z}$. Define a K-homomorphism $\theta : T \to T$ by $\theta(x_i) = x_{i+1}$, $i \in \mathbb{Z}$.

(1) Put $I_1 = \sum_{i\leq -1} Tx_i^2 \oplus \sum_{i\geq 0} Tx_i$. Then it is a θ-ideal of T. However, it is not θ-invariant, since $x_0 \in I_1 \setminus \theta(I_1)$.

(2) Consider the ideal N of T generated by the monomials $x_{i_1} \cdots x_{i_n}$, where $n \geq 2$, then it is a θ-invariant ideal of T. Thus, θ induces an automorphism of $R = T/N \cong K \oplus \sum_{i\in\mathbb{Z}} K\bar{x}_i$, where $\bar{x}_i = x_i + N$. Put $I_2 = \sum_{i\geq 1} K\bar{x}_i$. Then it is a θ-ideal of R. However, it is not θ-invariant since $\theta(I) = \sum_{i\geq 2} K\bar{x}_i$.

According to Pearson and Stephenson [4], a proper θ-ideal I of R is called *θ-prime* provided that if $AB \subseteq I$ for an ideal A and a θ-ideal B in R, then $A \subseteq I$ or $B \subseteq I$; a proper θ-ideal I of R is called *θ-semiprime* provided that whenever A is an ideal of R and m is an integer such that $A\theta^k(A) \subseteq I$ for all

Received June 2, 2009.
2010 *Mathematics Subject Classification.* 16N40, 16N60, 16S36.
Key words and phrases. θ-ideal, θ-prime ideal, θ-semiprime ideal, strongly θ-nilpotent element, θ-prime radical, prime radical, skew polynomial ring, skew Laurent polynomial ring.

$k \geq m$ we have $A \subseteq I$. It is not difficult to check that a θ-invariant ideal I of R is θ-prime if and only if $A \subseteq I$ or $B \subseteq I$ whenever A, B are ideals in R and m is an integer such that $A\theta^k(B) \subseteq I$ for all integers $k \geq m$. Thus θ-prime ideals are θ-semiprime.

A ring is called θ-prime (θ-semiprime) if the zero ideal is θ-prime (θ-semiprime). The (left) skew polynomial ring by θ over R is denoted by $R[x;\theta]$. Note that θ extends to an automorphism $\theta^* : R[x;\theta] \to R[x;\theta]$ defined by $\theta^*(\sum_{i\geq 0} a_i x^i) = \sum_{i\geq 0} \theta(a_i)x^i$ for all $\sum_{i\geq 0} a_i x^i \in R[x;\theta]$.

We first recall the following result proved by Pearson and Stephenson.

Lemma 1.2 ([4, Proposition 1.1]). (1) *$R[x;\theta]$ is a prime ring if and only if R is θ-prime.*

(2) *$R[x;\theta]$ is a semiprime ring if and only if R is θ-semiprime.*

$P(R)$ denotes the prime radical of R (i.e., the intersection of all prime ideals in R). Analogously we define the *θ-prime radical* of R by

$$\bigcap\{P \mid P \text{ is a } \theta\text{-invariant prime ideal of } R\},$$

which is written by $P_\theta(R)$.

The prime radical of $R[x;\theta]$ had been completely described as follows.

Lemma 1.3 ([4, Theorem 1.3]). *The prime radical of $R[x;\theta]$ is*

$$\begin{aligned} P(R[x;\theta]) &= \Big\{\sum_{i\geq 0} a_i x^i \mid a_0 \in P(R)\cap P_\theta(R) \text{ and } a_i \in P_\theta(R) \text{ for } i \geq 1\Big\} \\ &= (P(R)\cap P_\theta(R)) + \sum_{i\geq 1} P_\theta(R)x^i. \end{aligned}$$

Corollary 1.4. *$R[x;\theta]$ is semiprime if and only if R is θ-semiprime if and only if $P_\theta(R) = 0$.*

Remark 1.5. For a proper θ-invariant ideal I of R, the map $\bar\theta : R/I \to R/I$, defined by $\bar\theta(a+I) = \theta(a) + I$ for $a \in R$, is an automorphism. Moreover for an ideal P of R with $I \subseteq P$, P is θ-prime (θ-semiprime) if and only if P/I is $\bar\theta$-prime ($\bar\theta$-semiprime). Thus we have $P_{\bar\theta}(R/I) = Q/I$ where $Q = \bigcap\{P \mid P \text{ is a } \theta\text{-invariant prime ideal of } R \text{ and } I \subseteq P\}$.

The following lemma is an immediate consequence of Remark 1.4.

Lemma 1.6. *A proper θ-invariant ideal I of R is θ-prime (θ-semiprime) if and only if R/I is $\bar\theta$-prime ($\bar\theta$-semiprime).*

Proposition 1.7. *Let I be a proper θ-invariant ideal of R and $\bar\theta$ be the automorphism of R/I defined as above. Then the following conditions are equivalent*:

(1) *I is a θ-semiprime ideal of R*;
(2) *R/I is a $\bar\theta$-semiprime ring*;
(3) *$P_{\bar\theta}(R/I) = 0$*;

(4) *I is the intersection of some θ-invariant prime ideals of R;*
(5) *$(R/I)[x;\bar{\theta}]$ is a semiprime ring;*
(6) *$I[x;\theta] = \{\sum_{i\geq 0} a_i x^i \mid a_i \in I\}$ is a semiprime ideal of $R[x;\theta]$.*

Proof. (1)⇔(2) is obtained by Lemma 1.5. Corollary 1.3 gives (2)⇔(3)⇔(5). (3)⇔(4) is proved by the argument in Remark 1.4. (5)⇔(6) follows from the fact that $\frac{R}{I}[x;\bar{\theta}] \cong \frac{R[x;\theta]}{I[x;\theta]}$. □

Recall that $P(R)$ is the smallest semiprime ideal of R. The following is a similar result for $P_\theta(R)$, obtained from Proposition 1.6.

Corollary 1.8. *$P_\theta(R)$ is the smallest θ-invariant semiprime ideal, i.e., $P_\theta(R)$ is contained in every θ-invariant semiprime ideal of R.*

Note that $P(R)$ is the set of all strongly nilpotent elements in R [2, Proposition 3.2.1]. Similarly we can characterize elements in $P_\theta(R)$ as follows.

An element a in R shall be called *strongly θ-nilpotent* provided that for any sequence $(t_n)_{n=0}^{\infty}$ of positive integers such that $t_{n+1} \geq 1 + \sum_{i=0}^{n} t_i$, and for any sequence $(a_n)_{n=0}^{\infty}$ in R such that $a_0 = a$ and $a_{n+1} \in a_n R\theta^{t_n}(a_n)$ for all $n \geq 0$, there is an integer m such that $a_m = 0$. We will prove that $P_\theta(R)$ is the set of all strongly θ-nilpotent elements in R.

Lemma 1.9. *Let P be a θ-prime ideal of R. If $a \in R\backslash P$, then for any integer n there exists an integer $t_n \geq n$ such that $aR\theta^{t_n}(a) \nsubseteq P$.*

Proof. Since P is θ-invariant and $a \notin P$, we have $\theta^m(a) \notin P$ for each integer m. For a fixed integer n, let $A_n = \sum_{k=n}^{\infty} R\theta^k(a)R$, then A_n is a θ-ideal of R and $(RaR)A_n \nsubseteq P$ since P is θ-prime. Now we get $(RaR)A_n = \sum_{k=n}^{\infty} RaR\theta^k(a)R$, hence $aR\theta^{t_n}(a) \nsubseteq P$ for some $t_n \geq n$. □

Theorem 1.10. *$P_\theta(R)$ is the set of all strongly θ-nilpotent elements in R.*

Proof. Suppose $a \in P_\theta(R)$, then $ax \in P(R[x;\theta])$ by Lemma 1.2. So ax is strongly nilpotent in $R[x;\theta]$ by [2, Proposition 3.2.1]. Let $(a_n)_{n=0}^{\infty}$ be a sequence in R such that $a_0 = a$, $a_{n+1} = a_n r_n \theta^{t_n}(a_n)$, where $r_n \in R$ and t_n is a positive integer satisfying $t_{n+1} \geq 1 + \sum_{i=0}^{n} t_i$ for all $n \geq 0$. For convenience, let $s_0 = 1$, $s_n = 1 + \sum_{i=0}^{n-1} t_i$, $y_0 = ax$ and $y_n = a_n x^{s_n}$ for all $n \geq 1$. Then $s_{n+1} = s_n + t_n$, $s_n \leq t_n$ and hence we have

$$\begin{aligned} y_{n+1} &= a_{n+1}x^{s_{n+1}} = a_n r_n \theta^{t_n}(a_n) x^{t_n} x^{s_n} = a_n x^{s_n} \theta^{-s_n}(r_n) x^{t_n - s_n} a_n x^{s_n} \\ &= y_n z_n y_n \in y_n R[x;\theta] y_n, \end{aligned}$$

where $z_n = \theta^{-s_n}(r_n) x^{t_n - s_n}$ for all $n \geq 0$. Since $y_0 = ax$ is strongly nilpotent in $R[x;\theta]$, $y_n = 0$ eventually and so does $a_n = 0$, proving that a is strongly θ-nilpotent.

Conversely let $a \notin P_\theta(R)$, then $a \notin P$ for some θ-prime ideal P of R. Thus by Lemma 1.8 there is an integer $t_0 \geq 1$ and $r_0 \in R$ with $a_0 r_0 \theta^{t_0}(a_0) \notin P$. Let $a_1 = a_0 r_0 \theta^{t_0}(a_0)$, then we get $a_2 = a_1 r_1 \theta^{t_1}(a_1) \notin P$ for $r_1 \in R$ and

$t_1 \geq 1 + t_0$ by applying Lemma 1.8 to a_1. Repeating this process, we obtain sequences $(r_n)_{n=0}^{\infty}$, $(a_n)_{n=0}^{\infty}$ in R and $(t_n)_{n=0}^{\infty}$ of positive integers such that $t_{n+1} \geq 1 + \sum_{i=0}^{n} t_i$ and $a_0 = a$, $a_{n+1} = a_n r_n \theta^{t_n}(a_n)$ with $a_n \notin P$ for all $n \geq 0$. This shows that a is not strongly θ-nilpotent. □

Since θ^{-1} is also an automorphism of R, we can define θ^{-1}-primeness and θ^{-1}-semiprimeness analogously. In general $P_{\theta^{-1}}(R)$ need not coincide with $P_\theta(R)$ by Example 3.13 below. But Pearson, Stephenson, and Watters [5] gave an affirmative answer for positive powers of θ as in the following.

Lemma 1.11 ([5, Proposition 4.9])**.** *$P_{\theta^n}(R) = P_\theta(R)$ for any positive integer n; that is, $a \in R$ is strongly θ-nilpotent if and only if a is strongly θ^n-nilpotent.*

Remark 1.12. In [3] Lam, Leroy, and Matczuk defined the notion of strongly θ-nilpotency and θ-prime radical $\mathrm{rad}(R;\theta)$ to discuss the prime radicals of $R[x;\theta]$ and $R[x,x^{-1};\theta]$. But the notions in [3] are different from ours. In Section 3 we will prove that

$$\mathrm{rad}(R;\theta) = P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R).$$

Lam, Leroy and Matczuk [3, Definition 3.1(b)] introduce the notion of θ-nilpotency as follows. An element a in R is θ-nilpotent if for each integer $k \geq 1$ there exists an integer $n = n(k) \geq 1$ such that $a\theta^k(a)\theta^{2k}(a)\cdots\theta^{nk}(a) = 0$. A θ-invariant ideal I of R is said to be θ-nil if every element in I is θ-nilpotent.

Proposition 1.13. *$P_\theta(R)$ is θ-nil.*

Proof. It is obvious that $P_\theta(R)$ is θ-invariant. Thus it suffices to prove that every strongly θ-nilpotent element is θ-nilpotent. Let a be strongly θ-nilpotent and $k \geq 1$. Put $t_n = 2^n k$. Then $t_0 = k \geq 1$ and $t_{n+1} = 2^{n+1}k \geq 1+(1+2+\cdots+2^n)k = 1 + \sum_{i=0}^{n} t_i$ for all $n \geq 0$. Also let $a_0 = a$ and $a_{n+1} = a_n\theta^{t_n}(a_n)$; then $a_{n+1} \in a_n R \theta^{t_n}(a_n)$. Thus $a_n = 0$ for some $n \geq 1$ because $a_0 = a$ is strongly θ-nilpotent. Consequently $0 = a_n = a\theta^k(a)\theta^{2k}(a)\cdots\theta^{(2^n-1)k}(a)$, entailing that a is θ-nilpotent. □

Lemma 1.14 ([3, Theorem 3.5])**.** *Every ring R contains the largest θ-nil ideal, written by $N_\theta(R)$, such that $N_{\bar\theta}(R/N_\theta(R)) = 0$, where $\bar\theta$ is the induced automorphism of $R/N_\theta(R)$ defined as in Remark 1.4.*

The ideal $N_\theta(R)$ in Lemma 1.13 is called the *θ-nil radical* of R.

2. Relations between $P(R)$ and $P_\theta(R)$

In this section we first give some examples of θ and R and next consider some conditions under which $P(R)$ and $P_\theta(R)$ are equal.

Example 2.1. Let F be a field and $A = F\{x_i \mid i \in \mathbb{Z}\}$ be the free algebra with noncommuting indeterminates $\{x_i \mid i \in \mathbb{Z}\}$ over F. Let I be the ideal of A generated by the subset $\{u^2 \mid u \in \sum_{i\in\mathbb{Z}} Fx_i\}$ and set $R = A/I$. Then R is the exterior algebra on the set $\{\bar{x}_i \mid i \in \mathbb{Z}\}$, where $\bar{x}_i = x_i + I$. Let $\theta : R \to R$ be the F-automorphism of R induced by the assignment $\bar{x}_i \to \bar{x}_{i+1}$ for all $i \in \mathbb{Z}$. Then clearly $P(R) = \sum_{i \in \mathbb{Z}} R\bar{x}_i$. However $P_\theta(R) = 0$ since R has no nonzero strongly θ-nilpotent elements. In this case $P_\theta(R) \subsetneqq P(R)$.

Example 2.2. Let F be a field and $B = \prod_{i \in \mathbb{Z}} F_i$ with $F_i = F$ for all i. Let R be the F-subalgebra of B generated by $\oplus_{i \in \mathbb{Z}} F_i$ and 1_B. For each i set e_i to be the idempotent of B such that $e_i(j) = \delta_{ij} 1_F$, where δ_{ij} is the Kronecker delta. Let $\theta : R \to R$ be the automorphism of R induced by the assignment $e_i \mapsto e_{i+1}$ for each i. Since R is a reduced ring, we have $P(R) = 0$. But each e_i is strongly θ-nilpotent (also θ^{-1}-nilpotent); hence $P_\theta(R) = P_{\theta^{-1}}(R) = \oplus_{i \in \mathbb{Z}} F_i$. In this case $P(R) \subsetneqq P_\theta(R)$.

Example 2.3. Let R_1, θ_1 be the ring and automorphism respectively as in Example 2.1; and R_2 and θ_2 be the ring and automorphism respectively as in Example 2.2. Set $R = R_1 \oplus R_2$ and define $\theta = \theta_1 \oplus \theta_2$ by $\theta(a_1, a_2) = (\theta_1(a_1), \theta_2(a_2))$. Then clearly θ is an automorphism of R, and we have $P(R) = P(R_1) \oplus P(R_2) = P(R_1)$ and $P_\theta(R) = P_{\theta_1}(R_1) \oplus P_{\theta_2}(R_2) = P_{\theta_2}(R_2)$ by Examples 2.1 and 2.2. Thus $P(R)$ and $P_\theta(R)$ are not comparable.

In Examples 2.1, 2.2, and 2.3, we have $P(R) \neq P_\theta(R)$. But $P(R)$ and $P_\theta(R)$ are equal under some ascending chain condition as follows.

Note. If P is a θ-semiprime ideal and A is a θ^{-1}-ideal of R such that $A^2 \subseteq P$, then $A \subseteq P$. In fact, note that $A\theta^k(A) = \theta^k(\theta^{-k}(A)A) \subseteq \theta^k(A^2) \subseteq \theta^k(P) = P$ for any integer $k \geq 0$. Since P is θ-semiprime, we have $A \subseteq P$.

Proposition 2.4. *If R satisfies the ascending chain condition on θ-ideals, then $P(R) = P_\theta(R)$ and especially $P(R[x;\theta]) = P(R)[x;\theta]$.*

Proof. First note that if R satisfies the ascending chain condition on θ-ideals, then every θ-ideal is θ-invariant. Suppose that A is an ideal of R and m is an integer such that $A\theta^k(A) \subseteq P(R)$ for all $k \geq m$. Then $AB \subseteq P(R)$ with $B = \sum_{k=m}^{\infty} \theta^k(A)$. Since B is a θ-ideal, B is θ-invariant. If P is any prime ideal of R, then $AB \subseteq P(R) \subseteq P$ and so $A \subseteq P$ or $B \subseteq P$. If $B \subseteq P$ then $A \subseteq \theta^{-m}(B) = B \subseteq P$. In any case $A \subseteq P$, entailing $A \subseteq P(R)$. This proves $P_\theta(R) \subseteq P(R)$ by Corollary 1.7.

For the converse inclusion, it is enough to show that $P_\theta(R)$ is a semiprime ideal of R. Suppose that I is an ideal of R such that $I^2 \subseteq P_\theta(R)$. Then clearly $(\theta^k(I))^2 = \theta^k(I^2) \subseteq \theta^k(P_\theta(R)) = P_\theta(R)$ for any integer k. Choose any element $a \in I$ and let $C = \sum_{k=0}^{\infty} R\theta^k(a)R$. Then C is a θ-ideal of R and hence is θ-invariant, entailing $a \in C = \theta(C) = \sum_{k=1}^{\infty} R\theta^k(a)R$. This implies that $a \in R\theta(a)R + \cdots + R\theta^m(a)R$ for some $m \geq 1$. Let $D = \sum_{k=1}^{m} R\theta^k(a)R$; then D is a θ^{-1}-ideal such that

$$D^{m+1} = (\sum_{k=1}^{m} R\theta^k(a)R)^{m+1} \subseteq \sum_{k=1}^{m} (R\theta^k(a)R)^2 \subseteq \sum_{k=1}^{m} (\theta^k(I))^2 \subseteq P_\theta(R).$$

By the preceding Note combined with induction on $m \geq 1$, we have $D \subseteq P_\theta(R)$. Since $a \in D$, $a \in P_\theta(R)$ and hence $I \subseteq P_\theta(R)$. □

Corollary 2.5. *If R is left or right Noetherian, then $P_\theta(R) = P_{\theta^{-1}}(R) = P(R)$ for any automorphism θ of R.*

An automorphism θ of R is called *of locally finite order* if for any $a \in R$ there is an integer $n = n(a) \geq 1$ such that $\theta^n(a) = a$. Bedi and Ram proved that if θ is of locally finite order, then the Jacobson radicals of $R[x;\theta]$ and $R[x, x^{-1};\theta]$ have much nicer forms [1, Corollary 3.3 and Theorem 3.7]. We also prove that if θ is of locally finite order, then $P_\theta(R) = P(R)$ in the following.

Proposition 2.6. *If θ is of locally finite order, then $P(R) = P_\theta(R) = P_{\theta^{-1}}(R)$.*

Proof. It suffices to prove that $P(R)$ is θ-semiprime and $P_\theta(R)$ is semiprime. Let I be an ideal of R and m be an integer such that $I\theta^k(I) \subseteq P(R)$ for all $k \geq m$. Set $A = RaR$ for $a \in I$. Since θ is of locally finite order there is an integer $n \geq 1$ such that $\theta^n(a) = a$. Thus $\theta^{nk}(a) = a$ for any integer k. Choose a positive integer k such that $k \geq m$, then $nk \geq m$ and hence we have $A^2 = A\theta^{nk}(A) \subseteq I\theta^{nk}(I) \subseteq P(R)$. Thus we obtain $A \subseteq P(R)$ and $a \in P(R)$, proving that $I \subseteq P(R)$ and $P(R)$ is θ-semiprime.

To show that $P_\theta(R)$ is semiprime, let J be an ideal of R with $J^2 \subseteq P_\theta(R)$. Since $P_\theta(R)$ is θ-invariant, $\theta^k(J^2) \subseteq \theta^k(P_\theta(R)) = P_\theta(R)$ for each integer k. Let $b \in J$ and choose an integer $n \geq 1$ such that $\theta^n(b) = b$. Set $B = \sum_{k=0}^{n-1} R\theta^k(b)R$. Then B is a θ-ideal of R and $B^{n+1} \subseteq \sum_{k=0}^{n-1}(R\theta^k(b)R)^2 \subseteq \sum_{k=0}^{n-1}\theta^k(J^2) \subseteq P_\theta(R)$. Since $P_\theta(R)$ is θ-semiprime and B is a θ-ideal in R we have $b \in B \subseteq P_\theta(R)$, entailing $J \subseteq P_\theta(R)$. Thus $P_\theta(R)$ is a semiprime ideal of R. □

An important class of automorphisms is the class of power-quasi-inner ones. According to Pearson et al. [5], an automorphism θ of R is called *quasi-inner* (*QI* for short) if there exists a regular element (i.e., neither left nor right zero-divisor) $u \in R$ such that $ur = \theta(r)u$ for all $r \in R$, and θ is called *power-quasi-inner* (*PQI* for short) if θ^n is QI for some positive integer n.

Remark 2.7. Let θ be a PQI automorphism of R. Then there are an integer $n \geq 1$ and a regular element $u \in R$ such that $ur = \theta^n(r)u$ for all $r \in R$. We call such a regular element u an *axis* for θ. If u is an axis for θ, then $Ru = uR$ (hence this is a two-sided ideal of R). Moreover if u is an axis for θ, then u^n is also an axis for θ for all $n \geq 1$. Pearson, Stephenson and Watters proved that if θ is PQI, then there is an axis u for θ such that $\theta(u) = u$ [5, Proposition 4.7].

We will find relations among $P(R)$, $P_\theta(R)$ and $P_{\theta^{-1}}(R)$ for a PQI automorphism θ of R. First notice the following lemma, obtained from [5, Lemma 4.11].

Lemma 2.8. *Suppose that θ is a PQI automorphism of R and $u \in R$ is an axis for θ. Then $P(R)u \subseteq P_\theta(R)$ and $P_\theta(R)u \subseteq P(R)$.*

Lemma 2.9. *Let θ be a QI automorphism of R with an axis u satisfying $ur = \theta(r)u$ for all $r \in R$. Then*

(1) *If $a \in R$ with $au \in P(R)$, then $a \in P_\theta(R)$.*

(2) *If $a \in R$ with $au \in P_{\theta^{-1}}(R)$, then $a \in P(R)$.*

Proof. The proofs are similar to that of Theorem 1.9. Since θ is QI, $\theta(u) = u$ by definition.

(1) If $au \in P(R)$, then au is strongly nilpotent in R. It suffices to show that a is strongly θ-nilpotent by Theorem 1.9. Let $(a_n)_{n=0}^{\infty}$ be a sequence in R such that $a_0 = a$ and $a_{n+1} = a_n r_n \theta^{t_n}(a_n)$, where $r_n \in R$ and $t_0 \geq 1$, $t_{n+1} \geq 1+\sum_{i=0}^{n} t_i$ for all $n \geq 0$. For convenience let $s_0 = 1$, $s_{n+1} = 1+\sum_{i=0}^{n} t_i$. Then $s_{n+1} = s_n + t_n$ and $s_n \leq t_n < s_{n+1}$. Letting $b_n = a_n u^{s_n}$ for all $n \geq 0$, then we have $b_0 = au$ and

$$\begin{aligned} b_{n+1} &= a_{n+1}u^{s_{n+1}} = a_n r_n \theta^{t_n}(a_n) u^{t_n} u^{s_n} \\ &= a_n u^{s_n} \theta^{-s_n}(r_n) u^{t_n - s_n} a_n u^{s_n} = b_n(\theta^{-s_n}(r_n)u^{t_n-s_n})b_n \in b_n R b_n \end{aligned}$$

for all $n \geq 0$. Since $b_0 = au$ is strongly nilpotent, $b_m = 0$ for some $m \geq 0$, causing $a_m = 0$ because u is regular. Thus a is strongly θ-nilpotent.

(2) If $au \in P_{\theta^{-1}}(R)$, then au is strongly θ^{-1}-nilpotent. We will show that a is strongly nilpotent. Let $(c_n)_{n=0}^{\infty}$ be a sequence in R such that $c_0 = a$ and $c_{n+1} = c_n r_n c_n$, where $r_n \in R$ for all $n \geq 0$. Let $t_n = 2^n$, then clearly $t_0 \geq 1$, $t_{n+1} \geq 1 + \sum_{i=0}^{n} t_i$ for all $n \geq 0$. Put $d_n = c_n u^{t_n} = c_n u^{2^n}$. Then we have $d_0 = c_0 u = au$ and

$$\begin{aligned} d_{n+1} &= c_{n+1}u^{t_{n+1}} = c_n r_n c_n u^{2^{n+1}} = c_n r_n c_n u^{2^n} u^{2^n} \\ &= c_n u^{2^n} \theta^{-2^n}(r_n) \theta^{-2^n}(c_n u^{2^n}) = d_n \theta^{-2^n}(r_n) \theta^{-t_n}(d_n) \in d_n R \theta^{-t_n}(d_n), \end{aligned}$$

for all $n \geq 0$. Since $d_0 = au$ is strongly θ^{-1}-nilpotent, we get $d_k = 0$ for some $k \geq 1$, causing $c_k = 0$ because u is regular. Thus a is strongly nilpotent. □

Note. In Lemma 2.9 we also obtain that $ua \in P(R)$ (resp. $ua \in P_{\theta^{-1}}(R)$) implies $a \in P_\theta(R)$ (resp. $a \in P(R)$), by similar proofs.

Lemma 2.10. *If θ is a PQI automorphism of R, then we have the following assertions*:

(1) $P_{\theta^{-1}}(R) \subseteq P(R) \subseteq P_\theta(R)$;

(2) *Every axis u for θ is regular modulo $P_\theta(R)$.*

Proof. (1) Suppose that θ^n is QI for some $n \geq 1$, then $P_{\theta^{-n}}(R) \subseteq P(R) \subseteq P_{\theta^n}(R)$ by Lemma 2.9. But $P_{\theta^n}(R) = P_\theta(R)$ and $P_{\theta^{-n}}(R) = P_{\theta^{-1}}(R)$ by Lemma 1.10; hence we have $P_{\theta^{-1}}(R) \subseteq P(R) \subseteq P_\theta(R)$.

(2) Suppose that u is an axis for θ and $a \in R$ with $au \in P_\theta(R)$. Then $au^2 \in P(R)$ by Lemma 2.8. Since u^2 is also an axis for θ, we have $a \in P_\theta(R)$ by Lemma 2.9(1) and Lemma 1.10. We also obtain that $ua \in P_\theta(R)$ implies $a \in P_\theta(R)$ by Note of Lemma 2.9. Thus u is regular modulo $P_\theta(R)$. □

The class of QI automorphisms is large as can be seen by the following construction.

Example 2.11. Let θ be any automorphism of R and $S = R[x;\theta]$. Define $\theta^* : S \to S$ by $\theta^*(\sum_i a_i x^i) = \sum_i \theta(a_i)x^i$. Then θ^* is a QI automorphism of S with an axis x, i.e., $xf(x) = \theta^*(f(x))x$ for all $f(x) \in S$.

Corollary 2.12. *Let $S = R[x;\theta]$ and $\sigma = \theta^*$ be as in Example 2.11. Then $P_{\sigma^{-1}}(S) \subseteq P(S) \subseteq P_\sigma(S)$.*

Proof. By Lemma 2.10(1) and Example 2.11. □

Note. For any automorphism θ of R we have $P(S) = (P(R) \cap P_\theta(R)) + \sum_{i\geq 1} P_\theta(R)x^i$ by Lemma 1.2, where $S = R[x;\theta]$. So we also obtain $P_\sigma(S) = \sum_{i\geq 0} P_\theta(R)x^i = P_\theta(R)[x;\theta]$ with the help of Lemma 2.10(2), where $\sigma = \theta^*$ as in Corollary 2.12. Therefore $P(S) = P_\sigma(S)$ if and only if $P_\theta(R) \subseteq P(R)$.

In Section 3 we will show $P_{\sigma^{-1}}(S) = (P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R))[x;\theta]$ and also give an example of R with a QI automorphism θ such that $P_{\theta^{-1}}(R) \subsetneqq P(R) \subsetneqq P_\theta(R)$. But we have the following equality.

Proposition 2.13. *Let θ be a PQI automorphism of R. Then the following conditions are equivalent*:

(1) $P(R) = P_\theta(R)$;

(2) *Every axis u for θ is regular modulo $P(R)$*;

(3) *Some axis u for θ is regular modulo $P(R)$.*

Proof. (1)⇒(2) follows directly from Lemma 2.10(2) and (2)⇒(3) is obvious. To prove (3)⇒(1), let u be an axis for θ which is regular modulo $P(R)$. If $a \in P_\theta(R)$, then $au \in P(R)$ by Lemma 2.8, forcing $a \in P(R)$ since u is regular modulo $P(R)$. Thus we have $P(R) = P_\theta(R)$ with the help of Lemma 2.10(1). □

Corollary 2.14. *Let θ be a PQI automorphism of R. If there is an axis u for θ which is regular modulo $P(R)$, then $P(R[x;\theta]) = P(R)[x;\theta]$.*

Proof. By Proposition 2.13 and Note after Corollary 2.12. □

3. The prime radical of $R[x, x^{-1};\theta]$

Let θ be an automorphism of R. We use S and T to denote $R[x;\theta]$ and $R[x, x^{-1};\theta]$ respectively, where $R[x, x^{-1};\theta]$ is the skew Laurent polynomial ring with an indeterminate x over R. Let $\sigma = \theta^*$ be the automorphism of S defined by $\sigma(\sum_{i\geq 0} a_i x^i) = \sum_{i\geq 0} \theta(a_i)x^i$. Then σ is QI with an axis x. Note that $T = SX^{-1}$, the (right) quotient ring of S by the set $X = \{x^n \mid n \in \mathbb{Z} \text{ with } n \geq 0\}$.

In this section we will prove $P(T) = P_{\sigma^{-1}}(S)X^{-1} = (P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R))[x, x^{-1};\theta]$ and characterize elements in $P(T)$. The following lemma is obvious.

Lemma 3.1. *Let K be a proper ideal of T. Then we have the following assertions*:

(1) $K \cap S = \{f(x) \in S \mid f(x)x^{-m} \in K$ *for some integer* $m \geq 0\}$.

(2) *$K \cap S$ is a σ-invariant ideal of S and $(K \cap S)X^{-1} = K$.*
(3) *$x \notin K \cap S$ and x is regular modulo $K \cap S$.*

Now we need some technical definitions. A proper θ-invariant ideal P of R is said to be (θ, θ^{-1})-*prime* provided that $A \subseteq P$ or $B \subseteq P$ whenever $AB \subseteq P$ for θ-invariant ideals A, B in R. A proper θ-invariant ideal Q of R is said to be (θ, θ^{-1})-*semiprime* provided that if A is a θ-invariant ideal of R with $A^2 \subseteq Q$ then $A \subseteq Q$. Clearly every θ-prime and every θ^{-1}-prime (resp. every θ-semiprime and every θ^{-1}-semiprime) ideal is (θ, θ^{-1})-prime (resp. (θ, θ^{-1})-semiprime). Observe that $P(R)$, $P_\theta(R)$ and $P_{\theta^{-1}}(R)$ are all (θ, θ^{-1})-semiprime. Also note that an intersection of any set of (θ, θ^{-1})-semiprime ideals is (θ, θ^{-1})-semiprime. In particular the intersection of all the (θ, θ^{-1})-semiprime ideals of R is (θ, θ^{-1})-semiprime, and hence R contains the smallest (θ, θ^{-1})-semiprime ideal. As in the classical case, we define the (θ, θ^{-1})-prime radical $P_{(\theta,\theta^{-1})}(R)$ by

$$P_{(\theta,\theta^{-1})}(R) = \bigcap \{P \mid P \text{ is a } (\theta, \theta^{-1})\text{-prime ideal of } R\}.$$

Notice that an ideal I of R is θ-invariant if and only if $I[x; \theta] = \{\sum_{i \geq 0} a_i x^i \mid a_i \in I \text{ for all } i\}$ is a σ-invariant ideal of S if and only if the set $I[x, x^{-1}; \theta] = \{\sum_{i=-m}^{n} a_i x^i \mid m, n \geq 0, a_i \in I \text{ for all } i\}$ is an ideal of T.

Proposition 3.2. *Let I be a proper θ-invariant ideal of R. Then we have the following assertions*:
(1) *I is (θ, θ^{-1})-prime if and only if $I[x, x^{-1}; \theta]$ is a prime ideal of T.*
(2) *I is (θ, θ^{-1})-semiprime if and only if $I[x, x^{-1}; \theta]$ is a semiprime ideal of T.*

Proof. (1) Suppose that I is a (θ, θ^{-1})-prime ideal of R and let H, K be ideals of T such that $HK \subseteq I[x, x^{-1}; \theta]$. Assume on the contrary that $H \not\subseteq I[x, x^{-1}; \theta]$ and $K \not\subseteq I[x, x^{-1}; \theta]$. Then by Lemma 3.1, there are polynomials $f(x) = \sum_{i=0}^{m} a_i x^i \in H \cap S$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in K \cap S$ such that $f(x) \notin I[x, x^{-1}; \theta]$ and $g(x) \notin I[x, x^{-1}; \theta]$. Let p and q be the first integers such that $a_p \notin I$ and $b_q \notin I$, respectively. Then for any integer $k \in \mathbb{Z}$ and any $r \in R$, the coefficient of x^{p+q} in $\sigma^k(f(x))rg(x)$ is

$$\sum_{i=0}^{p+q} \theta^k(a_i)\theta^i(r)\theta^i(b_{p+q-i}) \in I.$$

Since $\theta^k(a_i) \in I$ and $b_j \in I$ for all i, j with $0 \leq i < p, 0 \leq j < q$, we have $\theta^k(a_p)\theta^p(r)\theta^p(b_q) \in I$ for all $k \in \mathbb{Z}, r \in R$. Thus for any integers k and l we get

$$\theta^k(a_p)R\theta^l(b_q) = \theta^{l-p}(\theta^{k-l+p}(a_p)R\theta^p(b_q)) \subseteq I,$$
$$\left(\sum_{k=-\infty}^{\infty} R\theta^k(a_p)R\right)\left(\sum_{k=-\infty}^{\infty} R\theta^l(b_q)R\right) \subseteq I.$$

But $\sum_{k=-\infty}^{\infty} R\theta^k(a_p)R$, $\sum_{k=-\infty}^{\infty} R\theta^l(b_q)R$ are θ-invariant and I is (θ, θ^{-1})-prime, we obtain that $\sum_{k=-\infty}^{\infty} R\theta^k(a_p)R \subseteq I$ or $\sum_{k=-\infty}^{\infty} R\theta^l(b_q)R \subseteq I$. Consequently $a_p \in I$ or $b_q \in I$, a contradiction to the choice of p and q. Therefore $I[x, x^{-1}; \theta]$ is a prime ideal of T.

Conversely suppose that $I[x, x^{-1}; \theta]$ is a prime ideal of T and that A, B are θ-invariant ideals of R such that $AB \subseteq I$. Then $A[x, x^{-1}; \theta]B[x, x^{-1}; \theta] = AB[x, x^{-1}; \theta] \subseteq I[x, x^{-1}; \theta]$ and hence $A[x, x^{-1}; \theta] \subseteq I[x, x^{-1}; \theta]$ or $B[x, x^{-1}; \theta] \subseteq I[x, x^{-1}; \theta]$, forcing $A \subseteq I$ or $B \subseteq I$. Thus I is (θ, θ^{-1})-prime.

(2) The case of semiprimeness can be proved by taking $H = K$ and $A = B$ in the proof of (1). □

Lemma 3.3. *Let H be a σ-invariant ideal of S such that $x \notin H$ and x is regular modulo H. If A and B are ideals of S such that $AB \subseteq H$ and $\sigma^{-1}(B) \subseteq B$, then there exist σ-invariant (hence σ^{-1}-invariant) ideals C and D such that $A \subseteq C$, $B \subseteq D$ and $CD \subseteq H$.*

Proof. Note that $A\sigma^i(B)x^i = Ax^iB \subseteq AB \subseteq H$ for any integer $i \geq 0$, and so $A\sigma^i(B) \subseteq H$ since x is regular modulo H. Let $D = \sum_{i=0}^{\infty} \sigma^i(B)$. Then D is a σ-invariant ideal with $B \subseteq D$ and $AD \subseteq H$. Moreover $\sigma^j(A)D = \sigma^j(AD) \subseteq \sigma^j(H) = H$ for each integer j. Thus if we let $C = \sum_{i=-\infty}^{\infty} \sigma^j(A)$, then C is σ-invariant and $A \subseteq C$, $CD \subseteq H$. □

Proposition 3.4. *Let I be a proper θ-invariant ideal of R.*

(1) *I is (θ, θ^{-1})-prime if and only if $I[x; \theta]$ is a σ^{-1}-prime ideal of S.*

(2) *I is (θ, θ^{-1})-semiprime if and only if $I[x; \theta]$ is a σ^{-1}-semiprime ideal of S.*

Proof. (1) Suppose that I is (θ, θ^{-1})-prime and C, D are ideals of S with $CD \subseteq I[x; \theta]$ and $\sigma^{-1}(D) \subseteq D$. By Lemma 3.3 we can assume that C, D are σ-invariant. If $C \nsubseteq I[x; \theta]$ and $D \nsubseteq I[x; \theta]$, then the same argument as in the proof of Proposition 3.2(1) leads to a contradiction. Thus $C \subseteq I[x; \theta]$ of $D \subseteq I[x; \theta]$.

Conversely suppose that $I[x; \theta]$ is σ^{-1}-prime and A, B are θ-invariant ideals of R such that $AB \subseteq I$. Then $A[x; \theta]$ and $B[x; \theta]$ are σ-invariant ideals of S satisfying $A[x; \theta]B[x; \theta] = AB[x; \theta] \subseteq I[x; \theta]$, whence $A \subseteq I$ or $B \subseteq I$.

(2) Suppose I is (θ, θ^{-1})-semiprime. Let C be an ideal of S and m be an integer such that $C\sigma^{-k}(C) \subseteq I[x; \theta]$ for all $k \geq m$. Let $D = \sum_{k \geq m} \sigma^{-k}(C)$, then D is a σ^{-1}-ideal and $CD \subseteq I[x; \theta]$. Since $x \notin I[x; \theta]$ and x is regular modulo $I[x; \theta]$, we can assume that C and D are σ^{-1}-invariant by Lemma 3.3. Thus we can also assume without loss of generality that C is σ^{-1}-invariant and $C^2 \subseteq I[x; \theta]$. If $C \nsubseteq I[x; \theta]$, then a similar argument as in the proof of Proposition 3.2(1) leads to a contradiction. So $C \subseteq I[x; \theta]$, concluding that $I[x; \theta]$ is σ^{-1}-semiprime.

Conversely suppose that $I[x; \theta]$ is σ^{-1}-semiprime and A is a θ-invariant ideal of R with $A^2 \subseteq I$. Then $A[x; \theta]^2 = A^2[x; \theta] \subseteq I[x; \theta]$. Since $A[x; \theta]$ is σ^{-1}-invariant, $A[x; \theta] \subseteq I[x; \theta]$ and $A \subseteq I$. Thus I is (θ, θ^{-1})-semiprime. □

Corollary 3.5. *The following conditions are equivalent*:

(1) *R is (θ, θ^{-1})-prime* (*resp.* *(θ, θ^{-1})-semiprime*);

(2) *S is σ^{-1}-prime* (*resp.* *σ^{-1}-semiprime*);

(3) *T is prime* (*resp.* *semiprime*).

We may compare Corollary 3.5 with [3, Theorem 4.21].

Lemma 3.6. (1) *If P is a σ^{-1}-prime ideal of S, then $P \cap R$ is a (θ, θ^{-1})-prime ideal of R and $(P \cap R)[x; \theta] \subseteq P$.*

(2) *If Q is a prime ideal of T, then $Q \cap S$ is a σ^{-1}-prime ideal of S. In particular $Q \cap R$ is a (θ, θ^{-1})-prime ideal of R with $(Q \cap R)[x, x^{-1}; \theta] \subseteq Q$.*

Proof. (1) Let P be a σ^{-1}-prime ideal of S. Then $P \cap R$ is θ-invariant and $(P \cap R)[x; \theta] \subseteq P$ because of $\sigma(P) = P$. Let A, B be θ-invariant ideals of R with $AB \subseteq P \cap R$. Then $A[x; \theta]B[x; \theta] = (AB)[x; \theta] \subseteq (P \cap R)[x; \theta] \subseteq P$. Since $A[x; \theta]$ and $B[x; \theta]$ are σ^{-1}-ideals of S, we have $A[x; \theta] \subseteq P$ or $B[x; \theta] \subseteq P$; hence $A \subseteq P \cap R$ or $B \subseteq P \cap R$, showing that $P \cap R$ is (θ, θ^{-1})-prime.

(2) Let Q be a prime ideal of T. Then $Q \cap S$ is a σ-invariant ideal of S with $(Q \cap S)X^{-1} = Q$ by Lemma 3.1. Let C, D be ideals of S with $CD \subseteq Q \cap S$ and $\sigma^{-1}(D) \subseteq D$. Since $x \notin Q \cap S$ and x is regular modulo $Q \cap S$, it follows from Lemma 3.3 that C and D can be assumed σ-invariant. Thus CX^{-1} and DX^{-1} are ideals of T such that $CX^{-1}DX^{-1} = (CD)X^{-1} \subseteq (Q \cap S)X^{-1} = Q$. Since Q is prime we have that $CX^{-1} \subseteq Q$ or $DX^{-1} \subseteq Q$, yielding $C \subseteq Q \cap S$ or $D \subseteq Q \cap S$. Thus $Q \cap S$ is σ^{-1}-prime. Now the (θ, θ^{-1})-primeness of $Q \cap R = (Q \cap S) \cap R$ is an immediate consequence of (1). □

The following proposition is obtained from Propositions 3.2, 3.4 and Lemma 3.6.

Proposition 3.7. (1) $P_{\sigma^{-1}}(S) = P_{(\theta,\theta^{-1})}(R)[x; \theta]$.

(2) $P(T) = P_{(\theta,\theta^{-1})}(R)[x, x^{-1}; \theta] = P_{\sigma^{-1}}(S)X^{-1}$.

Corollary 3.8. *$P_{(\theta,\theta^{-1})}(R)$ is the smallest (θ, θ^{-1})-semiprime ideal of R. Especially $P_{(\theta,\theta^{-1})}(R) \subseteq P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R)$.*

To prove $P_{(\theta,\theta^{-1})}(R) = P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R)$ and characterize elements of $P(T)$, we need one more related definition. An element a in R is said to be *strongly (θ, θ^{-1})-nilpotent* provided that given any sequence $(t_n)_{n=0}^{\infty}$ of integers, every sequence $(a_n)_{n=0}^{\infty}$, such that $a_0 = a$ and $a_{n+1} \in a_n R \theta^{t_n}(a_n)$ for all $n \geq 0$, is eventually zero [3, Definition 1.8].

Lemma 3.9. (1) *For any $a \in R \backslash P_{(\theta,\theta^{-1})}(R)$ there are $r \in R$ and an integer t such that $ar\theta^t(a) \notin P_{(\theta,\theta^{-1})}(R)$.*

(2) *If $a \in R$ is strongly (θ, θ^{-1})-nilpotent, then $a \in P_{(\theta,\theta^{-1})}(R)$.*

(3) *If $a \in P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R)$, then a is strongly (θ, θ^{-1})-nilpotent.*

Proof. (1) Suppose that $a \in R$ and $aR\theta^i(a) \subseteq P_{(\theta,\theta^{-1})}(R)$ for any integer i. Then for all $i, j \in \mathbb{Z}$

$$\theta^i(a)R\theta^j(a) = \theta^i(aR\theta^{j-i}(a)) \subseteq \theta^i(P_{(\theta,\theta^{-1})}(R)) = P_{(\theta,\theta^{-1})}(R).$$

Thus if $A = \sum_{i=-\infty}^{\infty} R\theta^i(a)R$, then $A^2 \subseteq P_{(\theta,\theta^{-1})}(R)$. Since A is θ-invariant, we have $A \subseteq P_{(\theta,\theta^{-1})}(R)$ and $a \in P_{(\theta,\theta^{-1})}(R)$.

(2) Let $a \notin P_{(\theta,\theta^{-1})}(R)$. Then $ar_0\theta^{t_0}(a) \notin P_{(\theta,\theta^{-1})}(R)$ for some $r_0 \in R$ and $t_o \in \mathbb{Z}$ by (1). Let $a_1 = ar_0\theta^{t_0}(a)$ and apply (1) again to a_1. Then we get $a_2 = a_1r_1\theta^{t_1}(a_1) \notin P_{(\theta,\theta^{-1})}(R)$ for some $r_1 \in R$ and $t_1 \in \mathbb{Z}$. Inductively there exists a sequence $(t_n)_{n=0}^{\infty}$ of integers and a sequence $(r_n)_{n=0}^{\infty}$ in R such that $a_n \notin P_{(\theta,\theta^{-1})}(R)$ for all $n \in \{0, 1, 2, \ldots\}$, where $a_0 = a$ and $a_{k+1} = a_kr_k\theta^{t_k}(a_k)$ for all $k \in \{0, 1, 2, \ldots\}$. Thus a is not strongly (θ, θ^{-1})-nilpotent.

(3) Let $a \in P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R)$ and suppose that $(t_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{Z}$ and $(r_n)_{n=0}^{\infty}$ is a sequence in R. Next set $a_0 = a$, $a_{k+1} = a_kr_k\theta^{t_k}(a_k)$ for all $k \in \{0, 1, 2, \ldots\}$ and $s_n = \sum_{i=0}^{n} t_i$ for all $n \geq 0$. Notice that for all $i, j \in \mathbb{Z}$ with $0 \leq i < j$

$$(*) \qquad a_{i+1} \in aR\theta^{s_i}(a) \text{ and } a_{j+1} \in a_{i+1}R\theta^{(s_j - s_i)}(a_{i+1}).$$

In particular if $0 \leq i < j$ and $s_i = s_j$ then $a_{j+1} \in a_{i+1}Ra_{i+1}$. We will show that $a_n = 0$ for some $n \geq 0$. The proof splits into the following two cases.

`Case 1.` When the sequence $(s_n)_{n=0}^{\infty}$ is bounded.
Assume that $(s_n)_{n=0}^{\infty}$ is bounded. Then there is an integer m such that $s_k = m$ for infinitely many k's. Choose a sequence $(n(k))_{k=0}^{\infty}$ of positive integers such that $1 \leq n(0) < n(1) < n(2) < \cdots$ and $s_{n(k)} = m$ for all $k \geq 0$. Let $b_k = a_{n(k)+1}$ for all $k \geq 0$; then by $(*)$ we have

$$b_{k+1} = a_{n(k+1)+1} \in a_{n(k)+1}R\theta^{(s_{n(k+1)} - s_{n(k)})}(a_{n(k)+1}) = b_kR\theta^{m-m}(b_k) = b_kRb_k.$$

Since $b_0 = a_{n(0)+1} \in aR\theta^m(a) \subseteq P(R)$, b_0 is strongly nilpotent; hence $b_k = 0$ and $a_{n(k)+1} = 0$ for some $k \geq 0$.

`Case 2.` When the sequence $(s_n)_{n=0}^{\infty}$ is not bounded.
By symmetry we may assume that $(s_n)_{n=0}^{\infty}$ is not bounded above. So there is a strictly increasing sequence $(n(k))_{k=0}^{\infty}$ of positive integers such that $s_{n(0)} \geq 1$ and $s_{n(k+1)} \geq 1 + 2s_{n(k)}$ for all $k \geq 0$. Let $z_0 = s_{n(0)}$ and $z_{k+1} = s_{n(k+1)} - s_{n(k)}$ for all $k \geq 0$. Then $z_0 \geq 1$ and $z_{k+1} = s_{n(k+1)} - s_{n(k)} \geq 1 + s_{n(k)}$, and so $1 + z_0 + z_1 + \cdots + z_k = 1 + s_{n(k)} \leq z_{k+1}$. Also let $b_0 = a$, $b_{k+1} = a_{n(k)+1}$ for all $k \geq 0$. Then $b_1 = a_{n(0)+1} \in aR\theta^{s_{n(0)}}(a) = b_0R\theta^{z_0}(a)$ and

$$b_{k+1} = a_{n(k)+1} \in a_{n(k-1)+1}R\theta^{(s_{n(k)} - s_{n(k-1)})}(a_{n(k-1)+1}) = b_kR\theta^{z_k}(b_k)$$

for all $k \geq 1$. Since $b_0 = a \in P_\theta(R)$, b_0 is strongly θ-nilpotent; hence $b_k = 0$ and $a_{n(k)+1} = 0$ for some $k \geq 0$.

Therefore a is strongly (θ, θ^{-1})-nilpotent. □

The following, that is obtained from Corollary 3.8 and Lemma 3.9(2), (3), may be compared with [3, Proposition 1.11].

Corollary 3.10. *$P_{(\theta,\theta^{-1})}(R) = P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R)$ and $P_{(\theta,\theta^{-1})}(R)$ consists of all strongly (θ, θ^{-1})-nilpotent elements in R.*

The following theorem is shown by Proposition 3.7 and Corollary 3.10.

Theorem 3.11. (1) $P_{\sigma^{-1}}(S) = (P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R))[x;\theta]$.

(2)
$$\begin{aligned} P(T) &= (P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R))[x, x^{-1};\theta] \\ &= (P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R))[x;\theta]X^{-1}. \end{aligned}$$

Moreover, $P(T)$ is a graded ideal of T and for $f(x) = \sum_{i=m}^{n} a_i x^i \in T$, $f(x) \in P(T)$ if and only if each a_i is strongly (θ, θ^{-1})-nilpotent in R.

Under some available conditions on R and θ, the prime radical of $R[x, x^{-1};\theta]$ is more tractable as in the following.

Corollary 3.12. (1) *If R satisfies the ACC on θ-ideals and θ^{-1}-ideals, then $P(T) = P(R)[x, x^{-1};\theta]$.*

(2) *If θ is of locally finite order, then $P(T) = P(R)[x, x^{-1};\theta]$.*

(3) *If θ is PQI on R, then $P(T) = P_{\theta^{-1}}(R)[x, x^{-1};\theta] \subseteq P(R)[x, x^{-1};\theta]$.*

Now we give an example of a QI automorphism σ of S such that $P_{\sigma^{-1}}(S) \subsetneqq P(S) \subsetneqq P_\sigma(S)$.

Example 3.13. Let R be a ring and θ be an automorphism of R such that $P(R) = 0$ and $P_\theta(R) \neq 0$ as in Example 2.2. Then σ is a QI automorphism of S. In this situation we have $P_{\sigma^{-1}}(S) = (P(R) \cap P_\theta(R) \cap P_{\theta^{-1}}(R))[x;\theta] = 0$, $P(S) = P(R) \cap P_\theta(R) + \sum_{i=1}^{\infty} P_\theta(R)x^i = P_\theta(R)[x;\theta]x$ and $P_\sigma(S) = \sum_{i=0}^{\infty} P_\theta(R)x^i = P_\theta(R)[x;\theta]$. Thus $P_{\sigma^{-1}}(S) \subsetneqq P(S) \subsetneqq P_\sigma(S)$.

References

[1] S. S. Bedi and J. Ram, *Jacobson radical of skew polynomial rings and skew group rings*, Israel J. Math. **35** (1980), no. 4, 327–338.

[2] J. Lambek, *Lectures on Rings and Modules*, Chelsea Publishing Co., New York, 1976.

[3] T. Y. Lam, A. Leroy, and J. Matczuk, *Primeness, semiprimeness and prime radical of Ore extensions*, Comm. Algebra **25** (1997), no. 8, 2459–2506.

[4] K. R. Pearson and W. Stephenson, *A skew polynomial ring over a Jacobson ring need not be a Jacobson ring*, Comm. Algebra **5** (1977), no. 8, 783–794.

[5] K. R. Pearson, W. Stephenson, and J. F. Watters, *Skew polynomials and Jacobson rings*, Proc. London Math. Soc. (3) **42** (1981), no. 3, 559–576.

Jeoung Soo Cheon
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: jeoungsoo@pusan.ac.kr

Eun Jeong Kim
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: ejkim69@pusan.ac.kr

Chang Ik Lee
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: cilee@pusan.ac.kr

Yun Ho Shin
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: yyyi2000@yahoo.co.kr