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GENERALIZED ANALYTIC FOURIER-FEYNMAN TRANSFORMS AND CONVOLUTIONS ON A FRESNEL TYPE CLASS

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Abstract. In this paper, we define an L_p analytic generalized Fourier-Feynman transform and a convolution product of functionals in a Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ which is called the Fresnel type class, and in more general class $\mathcal{F}_{A_1,A_2}$ of functionals defined on general function space $C_{a,b}[0,T]$ rather than on classical Wiener space. Also we obtain some relationships between the L_p analytic generalized Fourier-Feynman transform and convolution product for functionals in $\mathcal{F}(C_{a,b}[0,T])$ and in $\mathcal{F}_{A_1,A_2}$.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. The concept of an L_1 analytic Fourier-Feynman transform (FFT) for functionals on Wiener space was introduced by Brue in [2]. Further work involving the L_2–L_2 theory and the L_p–$L_{p'}$ theory, $1/p+1/p'=1$, includes [3, 16]. In [13], Huffman, Park and Skoug defined a convolution product (CP) for functionals on Wiener space, and they obtained various results for the FFT and CP [13, 14, 15]. On the other hand, in [1], Ahn investigated the L_1 FFT theory on the Fresnel class $\mathcal{F}(B)$ of an abstract Wiener space, and in [11] Chang, Song and Yoo studied the FFT and the first variation on an abstract Wiener space and corresponding Fresnel class $\mathcal{F}(B)$. There has been a tremendous amount of papers in the literature on the FFT and CP theory on classical and abstract Wiener spaces. Furthermore, in [18], Kallianpur and Bromley introduced a larger class $\mathcal{F}_{A_1,A_2}$ than the Fresnel class $\mathcal{F}(B)$ for a successful treatment of certain physical problems by means of a Feynman integral.

In recent paper [8], Chang and Skoug established various results involving generalized analytic Feynman integrals and generalized analytic FFTs(GFFT)

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for functionals defined on a very general function space $C_{a,b}[0,T]$ rather than on the Wiener space $C_0[0,T]$. The function space $C_{a,b}[0,T]$ was introduced by Chang and Chung in [6]. In [4], Chang and Choi studied a multiple L_p analytic GFFT on the Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ which was introduced in [8]. On the other hand, in [9, 10], Chang and Lee defined a Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$ of functionals defined on $C_{a,b}[0,T]$ and studied GFFT, conditional GFFT, and multiple L_p analytic GFFT on the Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$.

In this paper, we define an L_p analytic GFFT and a CP of functionals defined on a product function space $C^2_{a,b}[0,T] \equiv C_{a,b}[0,T] \times C_{a,b}[0,T]$ and establish various relationships between the GFFT and CP of functionals in $\mathcal{F}_{A_1,A_2}$ which is a class of functionals defined on the function space $C^2_{a,b}[0,T]$. The Wiener process used in [3, 13–18, 1, 4, and 5] is stationary in time and is free of drift, while the stochastic process used in [6–11], and in this paper, is nonstationary in time and is subject to the drift $a(t)$. Of course, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the $C_{a,b}[0,T]$ reduces to Wiener space $C_0[0,T]$.

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \ldots, Y(t_n,\omega))$ is normally distributed with the density function

$$K(\vec{t},\vec{\eta}) = \Big((2\pi)^n \prod_{j=1}^{n} \big(b(t_j) - b(t_{j-1})\big)\Big)^{-1/2} \cdot \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\Big\}, \tag{2.1}$$

where $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \ldots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [21, pp. 18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [21, p. 187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

Given two $\mathbb{C}$-valued measurable functions F and G on $C_{a,b}[0,T]$, F is said to be equal to G scale almost everywhere(s-a.e.) if for each $\rho > 0$, $\mu(\{x \in C_{a,b}[0,T] : F(\rho x) \neq G(\rho x)\}) = 0$ [12, 17]. We write that $F \approx G$ if F=G s-a.e..

Let $L^2_{a,b}[0,T]$ be the set of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\}, \tag{2.2}$$

where $|a|(t)$ denotes the total variation of the function $a(\cdot)$ on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)]. \tag{2.3}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore, $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0 & , \ j \neq k \\ 1 & , \ j = k. \end{cases}$$

Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x \rangle$ is defined by the formula

$$\langle v, x \rangle = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_{a,b} \phi_j(t) dx(t) \tag{2.4}$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists.

Remark 2.1. For each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v, x \rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$ and $\langle v, x \rangle$ is a Gaussian random variable on $C_{a,b}[0,T]$ with mean $\int_0^T v(s)da(s)$ and variance $\int_0^T v^2(s)db(s)$. Note that for all $u, v \in L^2_{a,b}[0,T]$,

$$\begin{aligned} &\int_{C_{a,b}[0,T]} \langle u, x \rangle \langle v, x \rangle d\mu(x) \\ &= \int_0^T u(s)v(s)db(s) + \int_0^T u(s)da(s) \int_0^T v(s)da(s). \end{aligned} \tag{2.5}$$

Hence we see that for all $u, v \in L_{a,b}^2[0,T]$, $\int_0^T u(s)v(s)db(s) = 0$ if and only if $\langle u, x\rangle$ and $\langle v, x\rangle$ are independent random variables.

Now, we state the definition of the generalized analytic Feynman integral.

Definition 2.2. Let $\mathbb{C}$ denote the complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a measurable functional such that for each $\lambda > 0$, the function space integral

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)d\mu(x)$$

exists. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}[F] \equiv E_x^{\mathrm{an}_\lambda}[F(x)] = J^*(\lambda). \tag{2.6}$$

Let $q \neq 0$ be a real number and let F be a functional such that $E^{\mathrm{an}_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$E^{\mathrm{anf}_q}[F] \equiv E_x^{\mathrm{anf}_q}[F(x)] = \lim_{\lambda \to -iq} E^{\mathrm{an}_\lambda}[F], \tag{2.7}$$

where $\lambda \to -iq$ through values in $\mathbb{C}_+$.

Next, see [8, 9], we state the definition of the GFFT.

Definition 2.3. Let $q \in \mathbb{R} - \{0\}$. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y + x)]. \tag{2.8}$$

For $p \in (1, 2]$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(p)}(F)(y) = \mathrm{l.i.m.}_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.9}$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left|T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y)\right|^{p'} d\mu(y) = 0,$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.10}$$

if it exists.

We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

Next we give the definition of the CP on $C_{a,b}[0,T]$.

Definition 2.4. Let F and G be measurable functionals on $C_{a,b}[0,T]$. For $\lambda \in \tilde{\mathbb{C}}_+$, we define their CP $(F*G)_\lambda$ (if it exists) by

$$(2.11)\qquad (F*G)_\lambda(y) = \begin{cases} E_x^{\mathrm{an}_\lambda}\big[F\big(\frac{y+x}{\sqrt{2}}\big)G\big(\frac{y-x}{\sqrt{2}}\big)\big], & \lambda \in \mathbb{C}_+ \\ E_x^{\mathrm{anf}_q}\big[F\big(\frac{y+x}{\sqrt{2}}\big)G\big(\frac{y-x}{\sqrt{2}}\big)\big], & \lambda = -iq,\ q \in \mathbb{R},\ q \neq 0. \end{cases}$$

Remark 2.5. (i) When $\lambda = -iq$, we denote $(F*G)_\lambda$ by $(F*G)_q$.

(ii) Our definition of the CP is different than the definition given by Yeh in [20] and used by Yoo in [22]. In [20] and [22], Yeh and Yoo studied relationships between their CP and Fourier-Wiener transform.

The following generalized analytic Feynman integral formula is used several times in this paper.

$$(2.12)\qquad E_x[\exp\{i\lambda^{-1/2}\langle v,x\rangle\}] = \exp\bigg\{-\frac{(v^2,b')}{2\lambda} + i\lambda^{-1/2}(v,a')\bigg\}$$

for all $\lambda \in \tilde{\mathbb{C}}_+$ and $v \in L^2_{a,b}[0,T]$ where

$$(2.13)\qquad (v,a') = \int_0^T v(s)a'(s)ds = \int_0^T v(s)da(s)$$

and

$$(2.14)\qquad (v^2,b') = \int_0^T v^2(s)b'(s)ds = \int_0^T v^2(s)db(s).$$

In this paper, for each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{-\frac{1}{2}}$ or $\lambda^{\frac{1}{2}}$ is chosen to have nonnegative real part.

3. Transforms and convolutions of functionals in a Banach algebra

In this section we introduce a Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ and evaluate the GFFT and CP of functionals belonging to the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$. We then obtain several relationships of the GFFT and CP. First, we give the definition of a Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ which is called the Fresnel type class on $C_{a,b}[0,T]$.

Let

(3.1)
$$C'_{a,b}[0,T] = \Big\{w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T]\Big\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$(3.2)\qquad D_t w = z(t) = \frac{w'(t)}{b'(t)}.$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T D_t w_1 D_t w_2 db(t) \tag{3.3}$$

is a separable Hilbert space. Furthermore, $(C'_{a,b}[0,T], C_{a,b}[0,T], \mu)$ is an abstract Wiener space. For more details, see [19].

Note that for all $w, w_1, w_2 \in C'_{a,b}[0,T]$,

$$((D_t w)^2, b') = \int_0^T (D_t w)^2 db(t) = \|w\|^2_{C'_{a,b}}, \tag{3.4}$$

$$(D_t w, a') = \int_0^T D_t w da(t) = \int_0^T D_t w D_t a db(t) = (w, a)_{C'_{a,b}} \tag{3.5}$$

and

$$\langle D_t w_1, w_2 \rangle = \int_0^T D_t w_1 dw_2(t) = \int_0^T D_t w_1 D_t w_2 db(t) = (w_1, w_2)_{C'_{a,b}}. \tag{3.6}$$

Next, we define a class of functionals on $C_{a,b}[0,T]$ like a Fresnel class of an abstract Wiener space. Note that the linear operator given by the equation (3.2) is an isomorphism. In fact, the inverse operator $D_t^{-1} : L^2_{a,b}[0,T] \to C'_{a,b}[0,T]$ is given by the formula

$$D_t^{-1} z = \int_0^t z(s) db(s) \tag{3.7}$$

and D_t^{-1} is a bounded operator since

$$\begin{aligned} \|D_t^{-1} z\|_{C'_{a,b}} &= \left\| \int_0^t z(s) db(s) \right\|_{C'_{a,b}} = \left(\int_0^T z^2(t) db(t) \right)^{\frac{1}{2}} \\ &\le \left(\int_0^T z^2(t) d[b(t) + |a|(t)] \right)^{\frac{1}{2}} = \|z\|_{a,b}. \end{aligned} \tag{3.8}$$

Thus by the open mapping theorem, D_t is also bounded and there exist positive real numbers α and β such that $\alpha \|w\|_{C'_{a,b}} \le \|D_t w\|_{a,b} \le \beta \|w\|_{C'_{a,b}}$ for all $w \in C'_{a,b}[0,T]$. Hence we see that the Borel σ-algebra on $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$ is given by

$$\mathcal{B}(C'_{a,b}[0,T]) = \{D_t^{-1}(E) : E \in \mathcal{B}(L^2_{a,b}[0,T])\}$$

and that for any complex Borel measure σ on $L^2_{a,b}[0,T]$, $\sigma \circ D_t$ is a complex Borel measure σ on $C'_{a,b}[0,T]$ and for any complex Borel measure f on $C'_{a,b}[0,T]$, $f \circ D_t^{-1}$ is a complex Borel measure σ on $L^2_{a,b}[0,T]$.

Definition 3.1. Let $\mathcal{M}(C'_{a,b}[0,T])$ be the space of complex-valued, countably additive (and hence finite) Boreal measures on $C'_{a,b}[0,T]$. The Banach algebra

$\mathcal{F}(C_{a,b}[0,T])$ consists of those functionals F on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i\langle D_t w, x\rangle\} df(w) \tag{3.9}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where the associated measure f is an element $\mathcal{M}(C'_{a,b}[0, T])$. We call $\mathcal{F}(C_{a,b}[0,T])$ the Fresnel type class of the function space $C_{a,b}[0,T]$.

Remark 3.2. (i) $\mathcal{M}(C'_{a,b}[0,T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(ii) One can show that the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0,T]} |df(w)|.$$

From now on, we will use the notation $(w,x)^\sim$ replaced by $\langle D_t w, x\rangle$. Then we have the following assertions.

(1) For each $w \in C'_{a,b}[0,T]$, the random variable $x \mapsto (w,x)^\sim$ is Gaussian with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$.

(2) $(w,\alpha x)^\sim = \alpha(w,x)^\sim$ for any real number α, $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$.

(3) If $\{w_1, w_2, \dots, w_n\}$ is an orthonormal set in $C'_{a,b}[0,T]$, then the random variables $(w_j,x)^\sim$'s are independent.

We will explain the existence of generalized Feynman integrals of functionals in $\mathcal{F}(C_{a,b}[0,T])$. Let F be an element of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measure f satisfies the condition

$$\int_{C'_{a,b}[0,T]} \exp\big\{|2q_0|^{-\frac{1}{2}}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\big\}|df(w)| < +\infty \tag{3.10}$$

for some $q_0 \in \mathbb{R} - \{0\}$. Using the equation (3.9), Definition 2.2, the Fubini theorem and the equation (2.12), we see that for all real q with $|q| \geq |q_0|$, the generalized analytic Feynman integral $E^{\mathrm{anf}_q}[F]$ of F exists and is given by the formula

$$E^{\mathrm{anf}_q}[F] = \int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{2q}\|w\|^2_{C'_{a,b}} + i\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(w,a)_{C'_{a,b}}\bigg\} df(w).$$

For more detail studies of existence of generalized Feynman integrals, see [7–11].

Throughout this section, for each $f \in \mathcal{M}(C'_{a,b}[0,T])$, we will use the notation

$$df^{\beta a}_{\alpha q}(w) = \exp\bigg\{i\Big(\frac{i}{\alpha q}\Big)^{\frac{1}{2}}(w,\beta a)_{C'_{a,b}}\bigg\} df(w). \tag{3.11}$$

The following theorems are due to Chang and Lee [10, 11].

Theorem 3.3. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measure f satisfies the condition* (3.10) *above. Then for all $p \in [1,2]$ and real q with $|q| \geq |q_0|$, the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, exists and is given by the formula*

$$T_q^{(p)}(F)(y) = \int_{C'_{a,b}[0,T]} \exp\Big\{ i(w,y)^\sim - \frac{i}{2q}\|w\|^2_{C'_{a,b}} \Big\} df_q^a(w) \tag{3.12}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, $T_q^{(p)}(F)$ is an element of $\mathcal{F}(C_{a,b}[0,T])$ with associated measure ϕ defined by

$$\phi(B) = \int_B \exp\Big\{ -\frac{i}{2q}\|w\|^2_{C'_{a,b}} \Big\} df_q^a(w) \tag{3.13}$$

for $B \in \mathcal{B}(C'_{a,b}[0,T])$.

Remark 3.4. In Theorem 3.3 above, for all real q with $|q| \geq |q_0|$ and $y \in C_{a,b}[0,T]$,

$$T_q^{(p)}(F)(y) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(y+x)d\mu(x), \quad 1 \leq p \leq 2. \tag{3.14}$$

In particular,

$$T_q^{(p)}(F)(0) = \int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x)d\mu(x), \quad 1 \leq p \leq 2. \tag{3.15}$$

Theorem 3.5. *Let q_0 and F be as in Theorem 3.3. Then for all $p \in [1,2]$ and all real q with $|q| \geq |q_0|$,*

$$\begin{aligned} T_{-q}^{(p)}\big(T_q^{(p)}(F)\big)(y) &= \int_{C'_{a,b}[0,T]} \exp\Big\{ i(w,y)^\sim + \frac{i}{\sqrt{|q/2|}}(w,a)_{C'_{a,b}} \Big\} df(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\{i(w,y)^\sim\} df^a_{i|q/2|}(w) \end{aligned} \tag{3.16}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, $T_{-q}^{(p)}\big(T_q^{(p)}(F)\big) \in \mathcal{F}(C_{a,b}[0,T])$ and

$$\big\|T_{-q}^{(p)}\big(T_q^{(p)}(F)\big)\big\| = \|F\|. \tag{3.17}$$

In Theorem 3.5 above, let $a(t) \equiv 0$. Then $T_{-q}^{(p)}\big(T_q^{(p)}(F)\big) = F$ for s-a.e. $y \in C_{a,b}[0,T]$, that is, $T_{-q}^{(p)}$ is the inverse transform of $T_q^{(p)}$. For more details for the case $a(t) \equiv 0$, see [3, 13, 14, 16].

In our next theorem we obtain the CP of functionals in $\mathcal{F}(C_{a,b}[0,T])$. The proof is given by a similar method of the proof of Theorem 3.2 in [4].

Theorem 3.6. *Let q_0 be a nonzero real number and let F and G be elements of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measures f and g satisfy the condition*

$$\int_{C'_{a,b}[0,T]} \exp\big\{ |4q_0|^{-\frac{1}{2}} \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}} \big\} \big[|df(w)| + |dg(w)| \big] < +\infty. \tag{3.18}$$

Then their CP $(F*G)_q$ *exists for all real* q *with* $|q|\geq|q_0|$ *and is given by the formula*

$$\begin{aligned}(3.19)\qquad &(F*G)_q(y)\\ &=\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\Big\{\frac{i}{\sqrt{2}}(w_1+w_2,y)^{\sim}\\ &\qquad -\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}}\Big\}df^a_{2q}(w_1)dg^{-a}_{2q}(w_2)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$*. Furthermore,* $(F*G)_q$ *is an element of* $\mathcal{F}(C_{a,b}[0,T])$*.*

In Theorem 3.6 above, $(F*G)_q$ is expressible in the form

$$(3.20)\qquad (F*G)_q(y)=\int_{C'_{a,b}[0,T]}\exp\{i(r,y)^{\sim}\}d(h\circ\psi^{-1})(r)$$

for s-a.e. $y\in C_{a,b}[0,T]$ where $\psi: C'_{a,b}[0,T]\times C'_{a,b}[0,T]\to C'_{a,b}[0,T]$ is given by

$$(3.21)\qquad \psi(w_1+w_2)=\frac{1}{\sqrt{2}}(w_1+w_2)$$

and h is a complex Borel measure on $\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])$ defined by

$$(3.22)\qquad h(B)=\int_B\exp\Big\{-\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}}\Big\}df^a_{2q}(w_1)dg^{-a}_{2q}(w_2)$$

for each $B\in\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])$.

In our next theorem, we obtain the transform of the convolution product.

Theorem 3.7. *Let* q_0 *be a nonzero real number and let* F *and* G *be elements of* $\mathcal{F}(C_{a,b}[0,T])$ *whose associated measures* f *and* g *satisfy the condition*

$$(3.23)\qquad \int_{C'_{a,b}[0,T]}\exp\big\{2|4q_0|^{-\frac{1}{2}}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\big\}\big[|df(w)|+|dg(w)|\big]<+\infty.$$

Then for all $p\in[1,2]$ *and all real* q *with* $|q|\geq|q_0|$*,*

$$(3.24)\qquad T^{(p)}_q\big((F*G)_q\big)(y)=T^{(p)}_{2q}\big(T^{(p)}_{2q}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T^{(p)}_{2q}\big(T^{(p)}_{2q}(G(-\cdot))(-\cdot)\big)\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y\in C_{a,b}[0,T]$*. Also, both of the expressions in* (3.24) *are given by the expression*

$$(3.25)\qquad \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\Big\{\frac{i}{\sqrt{2}}(w_1+w_2,y)^{\sim}-\frac{i}{2q}\Big(\|w_1\|^2_{C'_{a,b}}+\|w_2\|^2_{C'_{a,b}}\Big)\Big\}df^{2a}_{2q}(w_1)dg(w_2).$$

Proof. By using (2.8), (2.11), the Fubini theorem and (2.12), we have for all $\lambda>0$,

$$(3.26)\qquad T_\lambda\big((F*G)_\lambda\big)(y)=T_{2\lambda}\big(T_{2\lambda}(F)\big)\Big(\frac{y}{\sqrt{2}}\Big)T_{2\lambda}\big(T_{2\lambda}(G(-\cdot))(-\cdot)\big)\Big(\frac{y}{\sqrt{2}}\Big)$$

for s-a.e. $y \in C_{a,b}[0,T]$. But both of the expressions on the right-hand side of equation (3.26) are analytic functions of λ throughout $\mathbb{C}_+$, and are continuous functions of λ on $\tilde{\mathbb{C}}_+$ for all $y \in C_{a,b}[0,T]$. Furthermore, it is bounded on the region $\Gamma = \{\lambda \in \tilde{\mathbb{C}}_+ : |\mathrm{Im}(\lambda^{-1/2})| \le 2|4q_0|^{-1/2}\}$ under the condition (3.23). By using (3.23), $T_q^{(p)}((F*G)_q)$ exists for all real q with $|q| \ge |q_0|$ and is given by (3.24) for all desired values of p and q. □

Theorem 3.8. *Let q_0 be a nonzero real number and let F and G be elements of $\mathcal{F}(C_{a,b}[0,T])$ whose associated measures f and g satisfy the condition*

$$\int_{C'_{a,b}[0,T]} \exp\big\{3|4q_0|^{-\frac12}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\big\}\big[|df(w)|+|dg(w)|\big] < +\infty. \tag{3.27}$$

Then for all $p \in [1,2]$ and all real q with $|q| \ge |q_0|$,

$$\begin{aligned} &\int_{C_{a,b}[0,T]}^{anf_{-q}} T_q^{(p)}\big((F*G)_q\big)(y)d\mu(y)\\ \equiv &\int_{C_{a,b}[0,T]}^{anf_{-q}} T_{2q}^{(p)}\big(T_{2q}^{(p)}(F)\big)\Big(\frac{y}{\sqrt2}\Big)T_{2q}^{(p)}\big(T_{2q}^{(p)}(G(-\cdot))(-\cdot)\big)\Big(\frac{y}{\sqrt2}\Big)d\mu(y)\\ = &\int_{C_{a,b}[0,T]}^{anf_{q}} T_{-2q}^{(p)}\big(T_{2q}^{(p)}(F)\big)\Big(\frac{y}{\sqrt2}\Big)T_{-2q}^{(p)}\big(T_{2q}^{(p)}(G)\big)\Big(-\frac{y}{\sqrt2}\Big)d\mu(y). \end{aligned} \tag{3.28}$$

Also, both of the expressions in (3.28) *are given by the expression*

$$\begin{aligned} &\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}}\\ &\qquad + i\Big(\frac{-i}{2q}\Big)^{\frac12}(w_1-w_2,a)_{C'_{a,b}}\bigg\}df_{2q}^{2a}(w_1)dg_{-2q}^{2a}(w_2). \end{aligned} \tag{3.29}$$

Proof. Fix p and q. Then for $\lambda > 0$, using (3.24) and (3.12), we have

$$\begin{aligned} &\int_{C_{a,b}[0,T]} T_q^{(p)}\big((F*G)_q\big)(y/\sqrt\lambda)d\mu(y)\\ = &\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{1}{4\lambda}\|w_1+w_2\|^2_{C'_{a,b}} + \frac{i}{\sqrt{2\lambda}}(w_1+w_2,a)_{C'_{a,b}}\\ &-\frac{i}{2q}\Big(\|w_1\|^2_{C'_{a,b}}+\|w_2\|^2_{C'_{a,b}}\Big) + 2i\Big(\frac{i}{2q}\Big)^{\frac12}(w_1,a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2). \end{aligned} \tag{3.30}$$

But the last expression of (3.30) is analytic through $\mathbb{C}_+$ and is continuous on $\tilde{\mathbb{C}}_+$. Furthermore, it is bounded on the region $\Gamma = \{\lambda \in \tilde{\mathbb{C}}_+ : |\mathrm{Im}(\lambda^{-1/2})| \le$

$3|4q_0|^{-1/2}\}$ under condition (3.27). So letting $\lambda = -i(-q) = iq$, we have

(3.31)

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{-q}} T_q^{(p)}\big((F*G)_q\big)(y)d\mu(y) \\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}} + 2i\Big(\frac{i}{2q}\Big)^{\frac12}(w_1,a)_{C'_{a,b}} \\
&\qquad + i\Big(\frac{-i}{2q}\Big)^{\frac12}(w_1+w_2,a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2) \\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}} \\
&\qquad + i\Big(\frac{-i}{2q}\Big)^{\frac12}(w_1-w_2,a)_{C'_{a,b}}\bigg\}df^{2a}_{2q}(w_1)dg^{2a}_{-2q}(w_2)
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

On the other hand, using (3.12) and the Fubini theorem we have

$$\begin{aligned}
&T_{-2q}^{(p)}\big(T_{2q}^{(p)}(F)\big)\Big(\frac{y}{\sqrt2}\Big) \\
&= \int_{C'_{a,b}[0,T]} \exp\bigg\{\frac{i}{\sqrt2}(w_1,y)^\sim + i\Big(\frac{-i}{2q}\Big)^{\frac12}(w_1,a)_{C'_{a,b}}\bigg\}df^a_{2q}(w_1)
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}
&T_{-2q}^{(p)}\big(T_{2q}^{(p)}(G)\big)\Big(-\frac{y}{\sqrt2}\Big) \\
&= \int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{\sqrt2}(w_2,y)^\sim + i\Big(\frac{i}{2q}\Big)^{\frac12}(w_2,a)_{C'_{a,b}}\bigg\}dg^a_{-2q}(w_2)
\end{aligned} \tag{3.33}$$

for s-a.e. $y \in C_{a,b}[0,T]$. By using (3.32) and (3.33), we have for $\lambda > 0$,

(3.34)

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T_{-2q}^{(p)}\big(T_{2q}^{(p)}(F)\big)\Big(\frac{y}{\sqrt{2\lambda}}\Big)T_{-2q}^{(p)}\big(T_{2q}^{(p)}(G)\big)\Big(-\frac{y}{\sqrt{2\lambda}}\Big)d\mu(y) \\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{1}{4\lambda}\|w_1-w_2\|^2_{C'_{a,b}} + \frac{i}{\sqrt{2\lambda}}(w_1-w_2,a)_{C'_{a,b}} \\
&\qquad + i\Big(\frac{-i}{2q}\Big)^{\frac12}(w_1,a)_{C'_{a,b}} + i\Big(\frac{i}{2q}\Big)^{\frac12}(w_2,a)_{C'_{a,b}}\bigg\}df^a_{2q}(w_1)dg^a_{-2q}(w_2).
\end{aligned}$$

But the last expression above is an analytic function of λ throughout $\mathbb{C}_+$ and is continuous throughout on $\tilde{\mathbb{C}}_+$. Also, it is bounded on the region $\Gamma = \{\lambda \in$

$\tilde{\mathbb{C}}_+ : |\mathrm{Im}(\lambda^{-1/2})| \le 3|4q_0|^{-1/2}\}$. Letting $\lambda = -iq$ we have
(3.35)

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} T_{-2q}^{(p)}(T_{2q}^{(p)}(F))\Big(\frac{y}{\sqrt{2}}\Big)T_{-2q}^{(p)}(T_{2q}^{(p)}(G))\Big(-\frac{y}{\sqrt{2}}\Big)d\mu(y)\\ &= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{-\frac{i}{4q}\|w_1-w_2\|^2_{C'_{a,b}}\\ &\qquad + i\Big(\frac{-i}{2q}\Big)^{\frac{1}{2}}(w_1-w_2,a)_{C'_{a,b}}\bigg\}df^{2a}_{2q}(w_1)dg^{2a}_{-2q}(w_2).\end{aligned}$$

Now (3.31) and (3.35) together yield (3.28) . □

4. Transforms and convolutions of functionals in $\mathcal{F}_{A_1,A_2}$

Let A be a nonnegative self-adjoint operator on $C'_{a,b}[0,T]$ and f any finite complex measure. Then the functional

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i(A^{\frac{1}{2}}w, x)^{\sim}\}df(w)$$

belongs to $\mathcal{F}(C_{a,b}[0,T])$ because it can be rewritten as

$$\int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\}d\nu(w)$$

for $\nu = f\circ(A^{1/2})^{-1}$. Let A be self-adjoint but not nonnegative. Then A has the form

$$A = A^+ - A^- \tag{4.1}$$

and both A^+ and A^- are bounded nonnegative self-adjoint operators.

In this section we will get expressions of the generalized Feynman integral, the GFFT and the CP when A is no longer required to be nonnegative or even self-adjoint. In order to widen the scope of the analytic continuation technique to treat such cases, we will present definitions here in a slightly modified form.

Given two $\mathbb{C}$-valued measurable functions F and G on $C^2_{a,b}[0,T]$, F is said to be equal to G scale almost everywhere(s-a.e.) if for each $\rho_1, \rho_2 > 0$, $\mu(\{(x_1,x_2) \in C^2_{a,b}[0,T] : F(\rho_1x_1,\rho_2x_2) \ne G(\rho_1x_1,\rho_2,x_2)\}) = 0$. We write that $F \approx G$ if F=G s-a.e..

Definition 4.1. Let $\mathbb{C}^2_+ = \{\vec{\lambda} = (\lambda_1,\lambda_2) \in \mathbb{C}^2 : \mathrm{Re}(\lambda_j) > 0 \text{ for } j = 1,2\}$ and let $\tilde{\mathbb{C}}^2_+ = \{\vec{\lambda} = (\lambda_1,\lambda_2) \in \mathbb{C}^2 : \lambda_j \ne 0 \text{ and } \mathrm{Re}(\lambda_j) \ge 0 \text{ for } j = 1,2\}$. Let $F : C^2_{a,b}[0,T] \to \mathbb{C}$ be a measurable functional such that for each $\lambda_1, \lambda_2 > 0$, the function space integral

$$J(\lambda_1,\lambda_2) = \int_{C^2_{a,b}[0,T]} F(\lambda_1^{-1/2}x_1, \lambda_2^{-1/2}x_2)d\mu(x_1,x_2)$$

exists. If there exists a function $J^*(\lambda_1,\lambda_2)$ analytic in $\mathbb{C}_+^2$ such that $J^*(\lambda_1,\lambda_2) = J(\lambda_1,\lambda_2)$ for all $\lambda_1,\lambda_2 > 0$, then $J^*(\lambda_1,\lambda_2)$ is defined to be the analytic function space integral of F over $C_{a,b}^2[0,T]$ with parameter $\vec{\lambda} = (\lambda_1,\lambda_2)$, and for $\vec{\lambda} \in \mathbb{C}_+^2$ we write

(4.2)
$$E^{\mathrm{an}_{\vec{\lambda}}}[F] \equiv E_{\vec{x}}^{\mathrm{an}_{\vec{\lambda}}}[F(x_1,x_2)] \equiv \int_{C_{a,b}^2[0,T]}^{\mathrm{an}_{\vec{\lambda}}} F(x_1,x_2)d(\mu\times\mu)(x_1,x_2) = J^*(\vec{\lambda}).$$

Let q_1 and q_2 be nonzero real numbers. Let F be a functional such that $E^{\mathrm{an}_{\vec{\lambda}}}[F]$ exists for all $\vec{\lambda} \in \mathbb{C}_+^2$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter $\vec{q} = (q_1,q_2)$ and we write

$$\begin{aligned} E^{\mathrm{anf}_{\vec{q}}}[F] &\equiv E_{\vec{x}}^{\mathrm{anf}_{\vec{q}}}[F(x_1,x_2)] \\ &\equiv \int_{C_{a,b}^2[0,T]}^{\mathrm{anf}_{\vec{q}}} F(x_1,x_2)d(\mu\times\mu)(x_1,x_2) = \lim_{\vec{\lambda}\to -i\vec{q}} E^{\mathrm{an}_{\vec{\lambda}}}[F], \end{aligned} \tag{4.3}$$

where $\vec{\lambda} \to -i\vec{q} = (-iq_1,-iq_2)$ through values in $\mathbb{C}_+^2$.

Definition 4.2. Let q_1 and q_2 be nonzero real numbers. For $\vec{\lambda} = (\lambda_1,\lambda_2) \in \mathbb{C}_+^2$ and $(y_1,y_2) \in C_{a,b}^2[0,T]$, let

$$T_{\vec{\lambda}}(F)(y_1,y_2) = E_{\vec{x}}^{\mathrm{an}_{\vec{\lambda}}}[F(y_1+x_1,y_2+x_2)]. \tag{4.4}$$

For $p \in (1,2]$, we define the L_p analytic GFFT, $T_{\vec{q}}^{(p)}(F)$ of F, by the formula ($\vec{\lambda} \in \mathbb{C}_+^2$)

$$T_{\vec{q}}^{(p)}(F)(y_1,y_2) = \text{l.i.m.}_{\vec{\lambda}\to -i\vec{q}} T_{\vec{\lambda}}(F)(y_1,y_2) \tag{4.5}$$

if it exists; i.e., for each $\rho_1,\rho_2 > 0$,

$$\lim_{\vec{\lambda}\to -i\vec{q}} \int_{C_{a,b}^2[0,T]} \left|T_{\vec{\lambda}}(F)(\rho_1 y_1,\rho_2 y_2) - T_{\vec{q}}^{(p)}(F)(\rho_1 y_1,\rho_2 y_2)\right|^{p'} d(\mu\times\mu)(y_1,y_2) = 0,$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_{\vec{q}}^{(1)}(F)$ of F, by the formula ($\vec{\lambda} \in \mathbb{C}_+^2$)

$$T_{\vec{q}}^{(1)}(F)(y_1,y_2) = \lim_{\vec{\lambda}\to -i\vec{q}} T_{\vec{\lambda}}(F)(y_1,y_2) \tag{4.6}$$

if it exists.

We note that for $1 \le p \le 2$, $T_{\vec{q}}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{\vec{q}}^{(p)}(F)$ exists and if $F \approx G$, then $T_{\vec{q}}^{(p)}(G)$ exists and $T_{\vec{q}}^{(p)}(G) \approx T_{\vec{q}}^{(p)}(F)$.

Next we give the definition of the CP on $C_{a,b}^2[0,T]$.

Definition 4.3. Let F and G be measurable functionals on $C^2_{a,b}[0,T]$. For $\vec{\lambda}\in\tilde{\mathbb{C}}^2_+$, we define their CP $(F*G)_{\vec{\lambda}}$ (if it exists) by

$$(4.7)\qquad \begin{aligned}&(F*G)_{\vec{\lambda}}(y_1,y_2)\\ &=\begin{cases} E^{\mathrm{an}_{\vec{\lambda}}}_{\vec{x}}\big[F\big(\frac{y_1+x_1}{\sqrt{2}},\frac{y_2+x_2}{\sqrt{2}}\big)G\big(\frac{y_1-x_1}{\sqrt{2}},\frac{y_2-x_2}{\sqrt{2}}\big)\big]\ , & \vec{\lambda}\in\mathbb{C}_+\\ E^{\mathrm{anf}_{\vec{q}}}_{\vec{x}}\big[F\big(\frac{y_1+x_1}{\sqrt{2}},\frac{y_2+x_2}{\sqrt{2}}\big)G\big(\frac{y_1-x_1}{\sqrt{2}},\frac{y_2-x_2}{\sqrt{2}}\big)\big]\ , & \\ \qquad\qquad \vec{\lambda}=-i\vec{q}=(-iq_1,-iq_2),\ q_1,q_2\in\mathbb{R}-\{0\}. & \end{cases}\end{aligned}$$

Definition 4.4. Let A_1 and A_2 be bounded, nonnegative self-adjoint operators on $C'_{a,b}[0,T]$. The Banach algebra $\mathcal{F}_{A_1,A_2}$ consists of those functionals F on $C^2_{a,b}[0,T]$ expressible in the form

$$(4.8)\qquad F(x_1,x_2)=\int_{C'_{a,b}[0,T]}\exp\big\{i(A_1^{\frac12}w,x_1)^\sim+i(A_2^{\frac12}w,x_2)^\sim\big\}df(w)$$

for s-a.e. $(x_1,x_2)\in C^2_{a,b}[0,T]$, where the associated measure f is an element $\mathcal{M}(C'_{a,b}[0,T])$.

Remark 4.5. In Definition 4.4 above, let A_1 be the identity operator on $C'_{a,b}[0,T]$ and $A_2\equiv 0$. Then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$ which was defined in Section 3, and for real q_j, $j=1,2$,

$$E^{\mathrm{anf}_{\vec{q}}}_{\vec{x}}[F(x_1,x_2)]=\int^{\mathrm{anf}_{q_1}}_{C_{a,b}[0,T]}F_0(x_1)d\mu(x_1)$$

if it exists, where $F_0(x_1)=F(x_1,x_2)$ for all $(x_1,x_2)\in C^2_{a,b}[0,T]$ and

$$\int^{\mathrm{anf}_{q_1}}_{C_{a,b}[0,T]}F_0(x_1)d\mu(x_1)$$

means the generalized analytic Feynman integral on $C_{a,b}[0,T]$ which was defined in Section 2 above.

Let $A_j:C'_{a,b}[0,T]\to C'_{a,b}[0,T]$, $j=1,2$ be nonnegative self-adjoint operators. Throughout this section, for each $f\in\mathcal{M}(C'_{a,b}[0,T])$, we will use the notation

$$df^{\vec{A},\beta a}_{\alpha\vec{q}}(w)=\exp\left\{i\left(\frac{i}{\alpha q_1}\right)^{\frac12}(A_1^{\frac12}w,\beta a)_{C'_{a,b}}+i\left(\frac{i}{\alpha q_2}\right)^{\frac12}(A_2^{\frac12}w,\beta a)_{C'_{a,b}}\right\}df(w).$$

In our next theorem, we obtain the L_p analytic GFFT $T^{(p)}_{\vec{q}}(F)$ of a functional F in $\mathcal{F}_{A_1,A_2}$.

Theorem 4.6. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{F}_{A_1,A_2}$ whose associated measure f satisfies the condition*

$$(4.9)\qquad \int_{C'_{a,b}[0,T]}\exp\left\{|2q_0|^{-\frac12}\left(\|A_1^{\frac12}w\|_{C'_{a,b}}+\|A_2^{\frac12}w\|_{C'_{a,b}}\right)\|a\|_{C'_{a,b}}\right\}df(w)<+\infty.$$

Then for all $p \in [1,2]$ *and all real* q_j *with* $|q_j| \geq |q_0|$, $j = 1,2$, *the* L_p *analytic GFFT,* $T^{(p)}_{\vec{q}}(F)$ *of* F *exists and is given by the formula*

$$\begin{aligned} T^{(p)}_{\vec{q}}(F)(y_1,y_2) = \int_{C'_{a,b}[0,T]} \exp\Big\{ & i(A_1^{\frac{1}{2}}w, y_1)^\sim + i(A_2^{\frac{1}{2}}w, y_2)^\sim \\ & - \frac{i}{2q_1}\|A_1^{\frac{1}{2}}w\|^2_{C'_{a,b}} - \frac{i}{2q_2}\|A_2^{\frac{1}{2}}w\|^2_{C'_{a,b}} \Big\} df^{\vec{A},a}_{\vec{q}}(w) \end{aligned} \tag{4.10}$$

for s-a.e. $(y_1,y_2) \in C^2_{a,b}[0,T]$. *Furthermore,* $T^{(p)}_{\vec{q}}(F)$ *is an element of* $\mathcal{F}_{A_1,A_2}$ *with associated measure* ϕ *defined by*

$$\phi(B) = \int_B \exp\Big\{ - \frac{i}{2q_1}\|A_1^{\frac{1}{2}}w\|^2_{C'_{a,b}} - \frac{i}{2q_2}\|A_2^{\frac{1}{2}}w\|^2_{C'_{a,b}} \Big\} df^{\vec{A},a}_{\vec{q}}(w) \tag{4.11}$$

for $B \in \mathcal{B}(C'_{a,b}[0,T])$.

Proof. For $\lambda_j > 0$, $j = 1,2$ and s-a.e. $(y_1,y_2) \in C^2_{a,b}[0,T]$, using the equation (4.4), the Fubini theorem and the equation (2.12), we have

$$\begin{aligned} & T_{\vec{\lambda}}(F)(y_1,y_2) \\ &= E_{\vec{x}}\big[F(y_1 + \lambda_1^{-\frac{1}{2}}x_1, y_2 + \lambda_2^{-\frac{1}{2}}x_2)\big] \\ &= \int_{C'_{a,b}[0,T]} E_{\vec{x}}\big[\exp\big\{ i(A_1^{\frac{1}{2}}w, y_1)^\sim + i\lambda_1^{-\frac{1}{2}}(A_1^{\frac{1}{2}}w, x_1)^\sim \\ &\qquad + i(A_2^{\frac{1}{2}}w, y_2)^\sim + i\lambda_2^{-\frac{1}{2}}(A_2^{\frac{1}{2}}w, x_2)^\sim)\big\}\big] df(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{ i(A_1^{\frac{1}{2}}w, y_1)^\sim + i(A_2^{\frac{1}{2}}w, y_2)^\sim \\ &\qquad - \frac{1}{2\lambda_1}\|A_1^{\frac{1}{2}}w\|^2_{C'_{a,b}} - \frac{1}{2\lambda_2}\|A_2^{\frac{1}{2}}w\|^2_{C'_{a,b}} \\ &\qquad + \frac{i}{\sqrt{\lambda_1}}(A_1^{\frac{1}{2}}w, a)_{C'_{a,b}} + \frac{i}{\sqrt{\lambda_2}}(A_2^{\frac{1}{2}}w, a)_{C'_{a,b}} \Big\} df(w). \end{aligned} \tag{4.12}$$

But the last expression above is analytic through $\mathbb{C}^2_+$ and is continuous on $\tilde{\mathbb{C}}^2_+$. Also, it is bounded on the region $\Gamma = \{\vec{\lambda} = (\lambda_1,\lambda_2) \in \tilde{\mathbb{C}}^2_+ : |\mathrm{Im}(\lambda_j^{-1/2})| \leq |2q_0|^{-1/2}, j = 1,2\}$. Thus the equation (4.10) is established.

Let ϕ be a set function on $\mathcal{B}(C'_{a,b}[0,T])$ defined by the equation (4.11). By using the condition (4.9) we see that

$$\begin{aligned} \|\phi\| &= \int_{C'_{a,b}[0,T]} |df^{\vec{A},a}_{\vec{q}}(w)| \\ &\leq \int_{C'_{a,b}[0,T]} \exp\Big\{ \frac{1}{\sqrt{|2q_0|}}\|A_1^{\frac{1}{2}}w\|_{C'_{a,b}}\|a\|_{C'_{a,b}} \\ &\qquad + \frac{1}{\sqrt{|2q_0|}}\|A_2^{\frac{1}{2}}w\|_{C'_{a,b}}\|a\|_{C'_{a,b}} \Big\} |df(w)| < +\infty. \end{aligned} \tag{4.13}$$

Hence we have the desired result. □

Let A be self-adjoint but not nonnegative. Then A has the form (4.1). Let $F \in \mathcal{F}_{A_+,A_-}$. Suppose that the associated measure f of F satisfies condition (4.9) with A_1 and A_2 replaced with A_+ and A_-, respectively. Then for $\vec{q} = (q, -q)$ with $q \in \mathbb{R} - \{0\}$ and $|q| \geq |q_0|$,

(4.14)
$$T_{\vec{q}}^{(p)}(F)(y_1, y_2) = \int_{C'_{a,b}[0,T]} \exp\Big\{ i(A_+^{\frac{1}{2}} w, y_1)^\sim + i(A_-^{\frac{1}{2}} w, y_2)^\sim - \frac{i}{2q} \|A^{\frac{1}{2}} w\|^2_{C'_{a,b}} \Big\} df_{\vec{q}}^{(A_+,A_-),a}(w)$$

and

(4.15)
$$E^{\mathrm{anf}_{\vec{q}}}[F] = T_{\vec{q}}^{(p)}(F)(0,0) = \int_{C'_{a,b}[0,T]} \exp\Big\{ -\frac{i}{2q} \|A^{\frac{1}{2}} w\|^2_{C'_{a,b}} \Big\} df_{\vec{q}}^{(A_+,A_-),a}(w).$$

Moreover, if $a(t) \equiv 0$, then

$$T_{\vec{q}}^{(p)}(F)(y_1, y_2) = \int_{C'_{a,b}[0,T]} \exp\Big\{ i(A_+^{\frac{1}{2}} w, y_1)^\sim + i(A_-^{\frac{1}{2}} w, y_2)^\sim - \frac{i}{2q} \|A^{\frac{1}{2}} w\|^2_{C'_{a,b}} \Big\} df(w) \tag{4.16}$$

and

$$E^{\mathrm{anf}_{\vec{q}}}[F] = T_{\vec{q}}^{(p)}(F)(0,0) = \int_{C'_{a,b}[0,T]} \exp\Big\{ -\frac{i}{2q} \|A^{\frac{1}{2}} w\|^2_{C'_{a,b}} \Big\} df(w). \tag{4.17}$$

In our next theorem, we obtain the CP of functionals in $\mathcal{F}_{A_1,A_2}$.

Theorem 4.7. *Let q_0 be a nonzero real number and let F and G be elements of $\mathcal{F}_{A_1,A_2}$ whose associated measures f and g satisfy the condition*

$$\int_{C'_{a,b}[0,T]} \exp\Big\{ |4q_0|^{-\frac{1}{2}} \Big(\|A_1^{\frac{1}{2}} w\|_{C'_{a,b}} + \|A_2^{\frac{1}{2}} w\|_{C'_{a,b}} \Big) \|a\|_{C'_{a,b}} \Big\} \big[|df(w)| + |dg(w)| \big] < +\infty. \tag{4.18}$$

Then their CP $(F * G)_{\vec{q}}$ exists for all real q_j with $|q_j| \geq |q_0|$, $j = 1, 2$ and is given by the formula

(4.19)
$$(F * G)_{\vec{q}}(y_1, y_2) = \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp\Big\{ \frac{i}{\sqrt{2}} (A_1^{\frac{1}{2}}(w_1 + w_2), y_1)^\sim + \frac{i}{\sqrt{2}} (A_2^{\frac{1}{2}}(w_1 + w_2), y_2)^\sim - \frac{i}{4q_1} \|A_1^{\frac{1}{2}}(w_1 - w_2)\|^2_{C'_{a,b}}$$

$$- \frac{i}{4q_2}\|A_2^{\frac{1}{2}}(w_1-w_2)\|_{C'_{a,b}}^2\bigg\}df_{2\vec{q}}^{\vec{A},a}(w_1)dg_{2\vec{q}}^{\vec{A},-a}(w_2)$$

for s-a.e. $(y_1,y_2)\in C_{a,b}^2[0,T]$. *Furthermore,* $(F*G)_{\vec{q}}$ *is an element of* $\mathcal{F}_{A_1,A_2}$.

Proof. By using (4.7), the Fubini theorem and (2.12), we have for $\lambda_j>0$, $j=1,2$

(4.20)

$$\begin{aligned}
&(F*G)_{\vec{\lambda}}(y_1,y_2)\\
&=\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\bigg\{\frac{i}{\sqrt{2}}(A_1^{\frac{1}{2}}(w_1+w_2),y_1)^\sim\\
&\qquad+\frac{i}{\sqrt{2}}(A_2^{\frac{1}{2}}(w_1+w_2),y_2)^\sim-\frac{1}{4\lambda_1}\|A_1^{\frac{1}{2}}(w_1-w_2)\|_{C'_{a,b}}^2\\
&\qquad-\frac{1}{4\lambda_2}\|A_2^{\frac{1}{2}}(w_1-w_2)\|_{C'_{a,b}}^2+i\bigg(\frac{1}{2\lambda_1}\bigg)^{\frac{1}{2}}(A_1^{\frac{1}{2}}(w_1-w_2),a)_{C'_{a,b}}\\
&\qquad+i\bigg(\frac{1}{2\lambda_2}\bigg)^{\frac{1}{2}}(A_2^{\frac{1}{2}}(w_1-w_2),a)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2)
\end{aligned}$$

for s-a.e. $(y_1,y_2)\in C_{a,b}^2[0,T]$. But the last expression above is analytic throughout $\mathbb{C}_+$, is continuous on $\tilde{\mathbb{C}}_+$, and is bounded on the region $\Gamma=\{\vec{\lambda}=(\lambda_1,\lambda_2)\in\tilde{\mathbb{C}}_+^2:|\mathrm{Im}(\lambda_j^{-1/2})|\le|4q_0|^{-1/2}, j=1,2\}$. Thus letting $\vec{\lambda}=-i\vec{q}$ and using a simple calculation, we have the equation (4.19) above.

Let a set function $h:\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])\to\mathbb{C}$ be defined by

$$\begin{aligned}
h(B)=\int_B\exp\bigg\{&-\frac{i}{4q_1}\|A_1^{\frac{1}{2}}(w_1-w_2)\|_{C'_{a,b}}^2\\
&-\frac{i}{4q_2}\|A_2^{\frac{1}{2}}(w_1-w_2)\|_{C'_{a,b}}^2\bigg\}df_{2\vec{q}}^{\vec{A},a}(w_1)dg_{2\vec{q}}^{\vec{A},-a}(w_2)
\end{aligned}\tag{4.21}$$

for each $B\in\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])$. Then h is a complex Borel measure on $\mathcal{B}(C'_{a,b}[0,T]\times C'_{a,b}[0,T])$. Now we define a function $\psi:C'_{a,b}[0,T]\times C'_{a,b}[0,T]\to C'_{a,b}[0,T]$ by $\psi(w_1,w_2)=(w_1+w_2)/\sqrt{2}$. Then ψ is continuous and so it is Borel measurable. Let $\tilde{h}=h\circ\psi^{-1}$. By the condition (4.18) above, we see that for real q_j with $|q_j|\ge|q_0|$, $j=1,2$,

(4.22)

$$\begin{aligned}
\|\tilde{h}\|&=\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}|dh(w_1,w_2)|\\
&\le\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\bigg|\exp\bigg\{-\frac{i}{4q_1}\|A_1^{\frac{1}{2}}(w_1-w_2)\|_{C'_{a,b}}^2\\
&\qquad-\frac{i}{4q_2}\|A_2^{\frac{1}{2}}(w_1-w_2)\|_{C'_{a,b}}^2+i\bigg(\frac{i}{2q_1}\bigg)^{\frac{1}{2}}(A_1^{\frac{1}{2}}(w_1-w_2),a)_{C'_{a,b}}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + i\left(\frac{i}{2q_2}\right)^{\frac{1}{2}}(A_2^{\frac{1}{2}}(w_1-w_2),a)_{C'_{a,b}}\Bigg\}\Bigg||df(w_1)||dg(w_2)| \\
&\le \int_{C'_{a,b}[0,T]}\exp\left\{\frac{1}{\sqrt{|4q_0|}}\Big(\|A_1^{\frac{1}{2}}w_1\|_{C'_{a,b}}+\|A_2^{\frac{1}{2}}w_1\|_{C'_{a,b}}\Big)\|a\|_{C'_{a,b}}\right\}|df(w_1)| \\
&\quad\cdot\int_{C'_{a,b}[0,T]}\exp\left\{\frac{1}{\sqrt{|4q_0|}}\Big(\|A_1^{\frac{1}{2}}w_2\|_{C'_{a,b}}+\|A_2^{\frac{1}{2}}w_2\|_{C'_{a,b}}\Big)\|a\|_{C'_{a,b}}\right\}|dg(w_2)| \\
&< \infty.
\end{aligned}
$$

Hence $\tilde{h} = h\circ\psi^{-1}$ belongs to $\mathcal{M}(C'_{a,b}[0,T])$ and

$$
(4.23)\qquad (F*G)_{\vec{q}}(y_1,y_2) = \int_{C'_{a,b}[0,T]}\exp\left\{i(A_1^{\frac{1}{2}}r,y_1)^{\sim}+i(A_2^{\frac{1}{2}}r,y_2)^{\sim}\right\}d\tilde{h}(r)
$$

for s-a.e. $(y_1,y_2)\in C^2_{a,b}[0,T]$. Hence $(F*G)_{\vec{q}}$ exists and is given by (4.19) for all real q_j with $|q_j|\ge|q_0|$ and it belongs to $\mathcal{F}_{A_1,A_2}$. □

In next two theorems, we also give some relationships of the GFFT and the CP of functionals in $\mathcal{F}_{A_1,A_2}$ without proofs.

Theorem 4.8. *Let q_0 be a nonzero real number and let F and G be elements of $\mathcal{F}_{A_1,A_2}$ whose associated measures f and g satisfy the condition*

(4.24)

$$
\begin{aligned}
&\int_{C'_{a,b}[0,T]}\exp\Bigg\{2|4q_0|^{-\frac{1}{2}}\Big(\|A_1^{\frac{1}{2}}w\|_{C'_{a,b}} \\
&\qquad\qquad + \|A_2^{\frac{1}{2}}w\|_{C'_{a,b}}\Big)\|a\|_{C'_{a,b}}\Bigg\}\big[|df(w)|+|dg(w)|\big] < +\infty.
\end{aligned}
$$

Then for all $p\in[1,2]$ and all real q_j with $|q_j|\ge|q_0|$, $j=1,2$,

(4.25)

$$
\begin{aligned}
&T_{\vec{q}}^{(p)}\big((F*G)_{\vec{q}}\big)(y_1,y_2) \\
&= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\Bigg\{\frac{i}{\sqrt{2}}(A_1^{\frac{1}{2}}(w_1+w_2),y_1)^{\sim}+\frac{i}{\sqrt{2}}(A_2^{\frac{1}{2}}(w_1+w_2),y_2)^{\sim} \\
&\qquad -\frac{i}{2q_1}\Big[\|A_1^{\frac{1}{2}}w_1\|^2_{C'_{a,b}}+\|A_1^{\frac{1}{2}}w_2\|^2_{C'_{a,b}}\Big] \\
&\qquad -\frac{i}{2q_2}\Big[\|A_2^{\frac{1}{2}}w_1\|^2_{C'_{a,b}}+\|A_2^{\frac{1}{2}}w_2\|^2_{C'_{a,b}}\Big]\Bigg\}df_{2\vec{q}}^{\vec{A},2a}(w_1)dg(w_2) \\
&= T_{2\vec{q}}^{(p)}\big(T_{2\vec{q}}^{(p)}(F)\big)\left(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\right)\cdot T_{2\vec{q}}^{(p)}\big(T_{2\vec{q}}^{(p)}(G(-\cdot,-\cdot))(-\cdot,-\cdot)\big)\left(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\right)
\end{aligned}
$$

for s-a.e. $(y_1,y_2)\in C^2_{a,b}[0,T]$.

Theorem 4.9. *Let q_0 be a nonzero real number and let F and G be elements of $\mathcal{F}_{A_1,A_2}$ whose associated measures f and g satisfy the condition*

(4.26)
$$\int_{C'_{a,b}[0,T]} \exp\Big\{3|4q_0|^{-\frac{1}{2}}\Big(\|A_1^{\frac{1}{2}}w\|_{C'_{a,b}} + \|A_2^{\frac{1}{2}}w\|_{C'_{a,b}}\Big)\|a\|_{C'_{a,b}}\Big\}\big[|df(w)|+|dg(w)|\big] < +\infty.$$

Then for all $p\in[1,2]$ and all real q_j with $|q_j|\ge|q_0|$, $j=1,2$,

(4.27)
$$\begin{aligned}
&\int^{anf_{-\vec{q}}}_{C^2_{a,b}[0,T]} T^{(p)}_{\vec{q}}\big((F*G)_{\vec{q}}\big)(y_1,y_2)d(\mu\times\mu)(y_1,y_2)\\
&\equiv \int^{anf_{-\vec{q}}}_{C^2_{a,b}[0,T]} T^{(p)}_{2\vec{q}}\big(T^{(p)}_{2\vec{q}}(F)\big)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)\\
&\qquad\cdot T^{(p)}_{2\vec{q}}\big(T^{(p)}_{2\vec{q}}\big(G(-\cdot,-\cdot)\big)(-\cdot,-\cdot)\big)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)d(\mu\times\mu)(y_1,y_2)\\
&= \int^{anf_{\vec{q}}}_{C^2_{a,b}[0,T]} T^{(p)}_{-2\vec{q}}\big(T^{(p)}_{2\vec{q}}(F)\big)\Big(\frac{y_1}{\sqrt{2}},\frac{y_2}{\sqrt{2}}\Big)\\
&\qquad\cdot T^{(p)}_{-2\vec{q}}\big(T^{(p)}_{2\vec{q}}(G)\big)\Big(-\frac{y_1}{\sqrt{2}},-\frac{y_2}{\sqrt{2}}\Big)d(\mu\times\mu)(y_1,y_2)
\end{aligned}$$

for s-a.e. $(y_1,y_2)\in C^2_{a,b}[0,T]$. Also, both of the expressions in (4.27) *are given by the expression*

(4.28)
$$\begin{aligned}
&\int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\Big\{-\frac{i}{4q_1}\|A_1^{\frac{1}{2}}(w_1-w_2)\|^2_{C'_{a,b}}\\
&\qquad -\frac{i}{4q_2}\|A_2^{\frac{1}{2}}(w_1-w_2)\|^2_{C'_{a,b}} + i\Big(\frac{-i}{2q_1}\Big)^{\frac{1}{2}}(A_1^{\frac{1}{2}}(w_1-w_2),a)_{C'_{a,b}}\\
&\qquad + i\Big(\frac{-i}{2q_2}\Big)^{\frac{1}{2}}(A_2^{\frac{1}{2}}(w_1-w_2),a)_{C'_{a,b}}\Big\}df^{\vec{A},2a}_{2\vec{q}}(w_1)dg^{\vec{A},2a}_{-2\vec{q}}(w_2).
\end{aligned}$$

5. Example

In this section we apply the results obtained in Section 4 to a specific linear operator A on $C'_{a,b}[0,T]$.

Let $S : C'_{a,b}[0,T] \longrightarrow C'_{a,b}[0,T]$ be the linear operator defined by

$$Sw(t) = \int_0^t w(s)db(s).$$

Then, we see that the adjoint operator S^* of S is given by

$$S^*w(t) = w(T)b(t) - \int_0^t w(s)db(s) = \int_0^t [w(T) - w(s)]db(s),$$

and the linear operator $B = S^*S$ is given by

$$Bw(t) = \int_0^T \min\{b(s), b(t)\}w(s)db(s).$$

Furthermore, we see that B is a self-adjoint operator on $C'_{a,b}[0,T]$ and that

$$(w_1, Bw_2)_{C'_{a,b}} = (Sw_1, Sw_2)_{C'_{a,b}} = \int_0^T w_1(s)w_2(s)db(s)$$

for all $w_1, w_2 \in C'_{a,b}[0,T]$. Hence B is a positive definite operator, that is, $(w, Bw)_{C'_{a,b}} \geq 0$ for all $w \in C'_{a,b}[0,T]$.

One can show that the orthonormal eigenfunctions $\{e_m\}$ of B are given by

$$e_m(t) = \frac{\sqrt{2b(T)}}{(m-\frac{1}{2})\pi} \sin\left(\frac{(m-\frac{1}{2})\pi}{b(T)} b(t)\right) \tag{5.1}$$

with corresponding eigenvalues β_m given by

$$\beta_m = \left(\frac{b(T)}{(m-\frac{1}{2})\pi}\right)^2. \tag{5.2}$$

Furthermore, it can be shown that $\{e_m\}$ is a basis of $C'_{a,b}[0,T]$ and that B is a trace class operator and so S is a Hilbert-Schmidt operator on $C'_{a,b}[0,T]$.

Define a self-adjoint operator on $C'_{a,b}[0,T]$ by

$$Aw = \sum_{m=1}^{\infty} \gamma_m (w, e_m)_{C'_{a,b}} e_m, \tag{5.3}$$

where

$$\gamma_m = \begin{cases} \beta_m, & m : \text{even} \\ -\beta_m, & m : \text{odd}. \end{cases}$$

Then

$$Aw = \sum_{m=1}^{\infty} (-1)^m \beta_m (w, e_m)_{C'_{a,b}} e_m,$$

$$A_+^{\frac{1}{2}} w = \sum_{m:\text{even}} \sqrt{\beta_m} (w, e_m)_{C'_{a,b}} e_m, \tag{5.4}$$

and

$$A_-^{\frac{1}{2}} w = \sum_{m:\text{odd}} \sqrt{\beta_m} (w, e_m)_{C'_{a,b}} e_m. \tag{5.5}$$

In this case, we see that A_+ is the positive part of A and A_- is the negative part of A. One can show that $A_+^{\frac{1}{2}}$ and $A_-^{\frac{1}{2}}$ are trace class operators with $\mathrm{Tr}A_+^{\frac{1}{2}} = \frac{b^2(T)}{8}$ and $TrA_-^{\frac{1}{2}} = \frac{3b^2(T)}{8}$.

Let $F \in \mathcal{F}_{A_+,A_-}$. Then

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\left\{i(A_+^{\frac{1}{2}}w, x)^\sim + i(A_-^{\frac{1}{2}}w, x)^\sim\right\}df(w)$$

for s-a.e. $(y_1, y_2) \in C^2_{a,b}[0,T]$. Suppose that the associated measure f of F satisfies the condition (4.9) with A_1 and A_2 replaced with A_+ and A_-, respectively. Then for all $\vec{q} = (q, -q)$ with $q \in \mathbb{R} - \{0\}$ and $|q| \geq |q_0|$, using the equations (4.10) and (5.1)-(5.5), we have

$$\begin{aligned} T_{\vec{q}}^{(p)}(F)(y_1, y_2) = \int_{C'_{a,b}[0,T]} &\exp\bigg\{i(A_+^{\frac{1}{2}}w, y_1)^\sim + i(A_-^{\frac{1}{2}}w, y_2)^\sim \\ &- \frac{i}{2q}\sum_{m=1}^{\infty}(-1)^m\left(\frac{b(T)}{(m-\frac{1}{2})\pi}\right)^2 (w, e_m)^2_{C'_{a,b}} \\ &+ i\left(\frac{i}{q}\right)^{\frac{1}{2}} \sum_{m:\text{ even}} \frac{b(T)}{(m-\frac{1}{2})\pi}(w, e_m)_{C'_{a,b}}(a, e_m)_{C'_{a,b}} \\ &+ i\left(\frac{-i}{q}\right)^{\frac{1}{2}} \sum_{m:\text{ odd}} \frac{b(T)}{(m-\frac{1}{2})\pi}(w, e_m)_{C'_{a,b}}(a, e_m)_{C'_{a,b}}\bigg\}df(w) \end{aligned}$$

for s-a.e. $y \in C^2_{a,b}[0,T]$.

Also, for all $\vec{q} = (q, -q)$ with $q \in \mathbb{R} - \{0\}$ and $|q| \geq |q_0|$, using the equations (4.19), (5.1)-(5.5), we have

$$\begin{aligned} &(F * G)_{\vec{q}}(y_1, y_2) \\ &= \int_{C'_{a,b}[0,T]}\int_{C'_{a,b}[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}(A_+^{\frac{1}{2}}(w_1+w_2), y_1)^\sim + \frac{i}{\sqrt{2}}(A_-^{\frac{1}{2}}(w_1+w_2), y_1)^\sim \\ &\quad - \frac{i}{4q}\sum_{m=1}^{\infty}(-1)^m\left(\frac{b(t)}{(m-\frac{1}{2})}\right)^2 (w_1 - w_2, e_m)^2_{C'_{a,b}} \\ &\quad + i\left(\frac{i}{2q}\right)^{\frac{1}{2}} \sum_{m:\text{ even}} \frac{b(T)}{(m-\frac{1}{2})\pi}(w_1 - w_2, e_m)_{C'_{a,b}}(a, e_m)_{C'_{a,b}} \\ &\quad + i\left(\frac{-i}{2q}\right)^{\frac{1}{2}} \sum_{m:\text{odd}} \frac{b(T)}{(m-\frac{1}{2})\pi}(w_1 - w_2, e_m)_{C'_{a,b}}(a, e_m)_{C'_{a,b}}\bigg\}df(w_1)dg(w_2) \end{aligned}$$

for s-a.e. $y \in C^2_{a,b}[0,T]$.

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