

Some Theoretical Considerations in Body Tide Calculation

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고체지구조석계산에 있어 몇 가지 이론적 고찰

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Abstract: The largest terms in the solid Earth body tide calculation are second degree spherical harmonic components due to the moon or the sun, and they compose about 98 percent of total contribution. Each degree harmonics of the tidal perturbation should be evaluated through multiplication with distinct Love numbers or their combinations. Correct evaluation of these terms in gravity tide is considered with re-calculated Love numbers. Frequency dependence of Love numbers for spherical harmonic tide upon the order number is discussed. Tidal displacement and tidally induced deviation of the vertical are also evaluated. Essential concepts underlying the body tide calculation are briefly summarized.

Key words: Earth tide, Love number, Spherical harmonic analysis

요약: 지구조석을 계산하는데 있어 가장 주요한 항은 달이나 태양에 의한 구면조화차수가 2인 항으로서 지구조석의 약 98%를 이루며, 나머지는 3차 이상의 고차 항들이다. 그런데 차수 3항과 그 이상의 고차수의 항들의 값들은 차수 2항과는 다른 상수가 각각 곱하여져서 계산되어야 한다. 본 연구에서는 조석에 의한 중력변화, 변위 및 수직선의 변화에 대한 바른 계산을 검토하였으며, 필요한 관련상수들을 다시 구하였다. 동일차수내의 다른 계수의 조화함수적 조석주기의존성을 토의하였다. 한편, 지구조석을 계산하는데 관련된 중요한 개념들을 요약 소개하였다.

주요어: 지구조석, Love상수, 구면조화해석

Introduction

Body tide is the largest periodic perturbation in Earth's gravity field. There are numerous processes in Earth system science, where body tide does the key role. Melchior (1978) gave a thorough compilation of tides from both observational and theoretical points of view. Na and Moon (2010) recently reported another body tide calculation scheme, which includes tidal Love number calculation. Correct calculation of body tide requires two kinds of information; first, 'reliable physical properties inside Earth', and, second, 'position of the moon and the sun at the given epoch'. Effects from other planets are much smaller and negligible in practical applica-

tions. For example, gravity tide on Earth due to Jupiter is only about 0.0002 μ Gal. Contributions from other planets are usually smaller. So, tidal perturbations on Earth due to planets can be ignored, although their gravitational attractions do affect the orbit of Earth around the sun. This is due to the fact that gravitational pull has range dependence of $1/r^2$, where as tidal force has its as $1/r^3$.

Numerous periodicities exist in the time series of Earth tide due to the fact that both lunar orbit and orbit of Earth around the sun deviate from circular ones, and also that three planes – Earth's equatorial plane and the two orbital planes differ from each other. M_2 and S_2 tides are largest among all the tidal constituents. Next large ones are K_1 , O_1 , and N_2 tides. Other constituents are of smaller amplitudes. Kudryavtsev (2004) elaborated the calculation of the tidal potentials of these periodic components with highest precision.

Decomposition of the tide of second degree harmonic into three distinct terms as diurnal, semidiurnal, and permanent,

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has been known from early days of tidal study. Permanent tide is of comparable amplitude with diurnal tide, but is often neglected simply because it exists without any variation in time. Semidiurnal and diurnal tides are the essential ones. One may adopt a quasi-static approach, in which tide raising bodies are treated as stationary at each epoch of body tide calculation (Na and Moon, 2010). In this approach, a set of spherical harmonics is calculated for each tide raising body without considering numerous frequency decompositions via infinite series expansion in terms of orbital parameters such as inclination and eccentricity. So the calculation of any degree tidal perturbation can be achieved with just few terms only. This can be justified because of the much smaller apparent angular velocities of the sun and the moon on the celestial sphere, compared to Earth's spin rotation.

The relative magnitudes of leading terms of tidal perturbations are given in Table 1. The numeral values in Table 1 are of the tidal potential. Degree number should be multiplied for comparison of gravity, while order number should be multiplied for deflection of the vertical (that refers to harmonic degree n and order m in $P_{nm}(\cos\theta)e^{im\phi}$). It is readily seen that the lunar and solar tides of degree 2 compose the major part of Earth tide. The amplitude of lunar tidal gravity of degree 2 in most urban areas of the world is about 100 μGal . Terms needed to evaluate the tidal gravity with precision of 2 μGal are the lunar and solar tides of degree 2. For the precision better than 2 μGal , the lunar tide of degree 3 must be included. For the precision of 0.05 μGal , lunar tide of degree 4 should be considered as well. Solar tide of degree 3 and lunar tide of degree 5 result in the gravity tide of 0.003 μGal and 0.001 μGal respectively. From the theoretical point of view, terms of higher degree harmonics can be added without any difficulty.

Before the advent of space geodetic technique, tidal displacement could not be measured, while reduction of gravity tide has been an important routine of the gravity prospecting. But nowadays, the assessment of tidal displacement is included in procedures of precise GNSS positioning. Tidally induced deviation of the vertical is quite small ($\sim 10^{-7}$ rad),

Table 1. Relative Amplitude of Tidal Perturbation.

	Lunar tide	Solar tide
degree 2	1	0.459
degree 3	0.0166	0.0000195
degree 4	0.00027	0.000000008
degree 5	0.0000045	-

but its prediction can be useful in high accuracy measurements. In this study, gravity tide and deviation of the vertical as well as tidal displacement are re-analyzed to evaluate most accurately within the precision of present state of the art of gravity measurement, using quasi-static approach.

Tidal Potential

Tidal potential can be expressed as follow:

$$W = \sum_{n=2}^{\infty} W_n = GM \sum_{n=2}^{\infty} \frac{r^n}{(r')^{n+1}} P_n(\cos\psi) \quad (1)$$

where ψ is zenith angle of tide-raising body, G is the universal constant of gravitation, M is mass of the body, r is geocentric distance of field point (in some cases, may be approximated as a - the average radius of the Earth), r' is the distance between the tide-raising body and Earth, and P_n is Legendre polynomial of degree n . As aforementioned, second degree harmonic components due to the moon and the sun make the largest contribution, approximately 98 percent, and third degree harmonic potential term due to the moon again makes the most contribution of the remaining part.

Essential information needed in evaluation of the tidal potential are the position of the moon and the sun at the given epoch. Either numerical calculation or analytical approximation can be used to acquire the ephemerides of the moon around Earth and those of Earth around the sun. Accurate numerical integration of ephemerides should be done simultaneously for all the planetary bodies together. There are software packages available from the International Astronomical Union or from the United States Naval Observatory. DE 405 is one such product originally from Jet Propulsion Laboratory (NASA) for those ephemerides. Meeus (1988) provided trigonometric series for the positions of the moon and the sun. His formulae are known to be accurate within 20 arc seconds in direction and with relative range error of about 1×10^{-5} for the position of the sun and the moon.

When the position of any planetary body is calculated, it is normally referenced to Terrestrial Time (TT), which was formerly termed as Terrestrial Dynamical Time (TDT) or Ephemeris Time (ET). TT is 32.184 second ahead of the International Atomic Time (TAI). Universal time (UT or UT1) is defined with the reference to the actual spin rotation of Earth. Due to slow decrease of Earth's spin, the 'leap second' has been introduced to compensate the delay of Universal Time. For convenience, Coordinated Universal Time (UTC),

which is basically an atomic time, is maintained to be close to UT1 within 0.9 second (Seidelmann, 1992). In 2010, UTC is defined to be 34 seconds behind of TAI. In summary the relation between TT, TAI and UTC can be expressed as follows.

$$TT = TAI + 32.184s = UTC + dAT + 32.184s, \quad (2)$$

where dAT is the sum of leap seconds.

Love Numbers

Set of three kinds of parameters, usually called Love numbers, is used to represent the tidal deformation of Earth due to exterior harmonic perturbation. Love numbers k_n , h_n and l_n are defined in the following relations.

$$V = \sum_{n=2}^{\infty} k_n W_n, \quad u_r = \sum_{n=2}^{\infty} h_n \frac{W_n}{g}, \quad u_\theta = \sum_{n=2}^{\infty} \frac{l_n}{g} \frac{\partial W_n}{\partial \theta},$$

$$u_\phi = \sum_{n=2}^{\infty} \frac{l_n}{g \sin \theta} \frac{\partial W_n}{\partial \phi}, \quad (3)$$

where W_n is the perturbing potential of spherical harmonic degree n , and V is the change of gravity potential caused by the deformation, and (u_r, u_θ, u_ϕ) are the radial and tangential components of displacement, and g is the gravity at the earth's surface. In many instances of planetary tide study, only the second degree spherical harmonics is considered, but higher degree terms are needed for better accuracy.

Love numbers can be calculated through integration of the converted differential equation of motion for whole Earth oscillation with reliable Earth model. In this study, all the tidal Love numbers of second and higher degrees have been re-calculated using the IASPEI Earth model (Kennett and Engdahl, 1991). Discrepancies in Love numbers calculated for different Earth models are minute for low degree Love numbers. As for k_2 of lunar diurnal tide, one former estimate (Na and Moon, 2010) was 0.30140, while its value for IASPEI Earth model is acquired as 0.29927 differing by 0.7 percent. Likewise the discrepancies in h_2 and l_2 of lunar diurnal tide are 0.7 and 1.0 percent. In Table 2, Love numbers of degree 2, 3, 4 and 5 calculated for IASPEI Earth model are given. In that table, 'terdiurnal' or 'quatdiurnal' simply refers to harmonic tidal frequency, and neither has any relation with localized features of ocean tide observed in certain areas due to nonlinear fluid dynamic interaction. Slightly different values of Love numbers of the same degree are ascribed to difference in tidal frequencies. One may take diurnal Love number of a certain degree and assign it to oth-

Table 2. Tidal Love Numbers of degrees up to five for the IASPEI Earth model.

Degree	Frequency	k_n	h_n	l_n
$n = 2$	Lunar diurnal	0.29927	0.60618	0.08397
	Solar diurnal	0.29930	0.60623	0.08397
	Lunar semidiurnal	0.30036	0.60845	0.08408
	Solar semidiurnal	0.30046	0.60867	0.08409
$n = 3$	Lunar diurnal	0.09240	0.28933	0.01456
	Solar diurnal	0.09241	0.28934	0.01456
	Lunar semidiurnal	0.09253	0.28975	0.01453
	Solar semidiurnal	0.09254	0.28979	0.01453
$n = 4$	Lunar diurnal	0.09274	0.29044	0.01448
	Lunar diurnal	0.04158	0.17570	0.01003
	Lunar semidiurnal	0.04160	0.17581	0.01002
	Lunar terdiurnal	0.04165	0.17601	0.01001
$n = 5$	Lunar diurnal	0.04170	0.17628	0.00999
	Lunar diurnal	0.02440	0.12927	0.00832

ers of different order and same degree, such as semidiurnal tide. The theoretical ambiguity here is due to the fact that the equation of motion for harmonic body tide has quite often been solved for non-rotating Earth. But the approach taken in this paper is physically meaningful, in the fact that all the tidal standing waves of different spherical harmonic degree and order would maintain the same phase velocity as the true tidal perturbation propagating speed in Earth. For Love numbers of degree 5, only the diurnal tide is listed for simplicity. Likewise, lunar tidal Love numbers only are listed for degree 4.

Tidal Gravity and Other Perturbations

On rigid Earth, tidal perturbations in gravity or the vertical direction would simply be given as $\Delta g = -\sum_{n=2}^{\infty} \frac{\partial W_n}{\partial r}$ and $\frac{1}{g\alpha} \left(\sum_{n=2}^{\infty} \frac{\partial W_n}{\partial \theta}, \frac{1}{\sin \theta} \sum_{n=2}^{\infty} \frac{\partial W_n}{\partial \phi} \right)$ without any deformation. But Earth has certain elasticity, and so, as noted before, the total gravity potential perturbation is expressed as $\sum_{n=2}^{\infty} (1 + k_n) W_n$, while the tidal deformation \bar{u} is acquired as $\left(\sum_{n=2}^{\infty} h_n \frac{W_n}{g} \sum_{n=2}^{\infty} \frac{l_n}{g} \frac{\partial W_n}{\partial \theta} \sum_{n=2}^{\infty} \frac{l_n}{g \sin \theta} \frac{\partial W_n}{\partial \phi} \right)$. The height change of the mean ocean surface would be evaluated as $\sum_{n=2}^{\infty} (1 + k_n - h_n) \frac{W_n}{g}$, because the ocean bottom would rise by an amount of $u_r = \sum_{n=2}^{\infty} h_n \frac{W_n}{g}$. Following Melchior (1978), with separate r

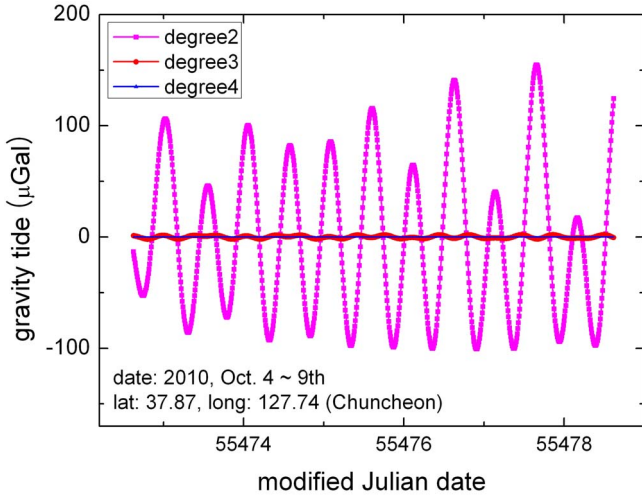


Fig. 1. Sample plot of gravity tide. Each degree components are separately shown [unit: μGal]. Values were calculated for six days in 10 minute interval during the week of the Korean Society of Earth and Exploration Geophysicists 2010 Fall Meeting at Chuncheon.

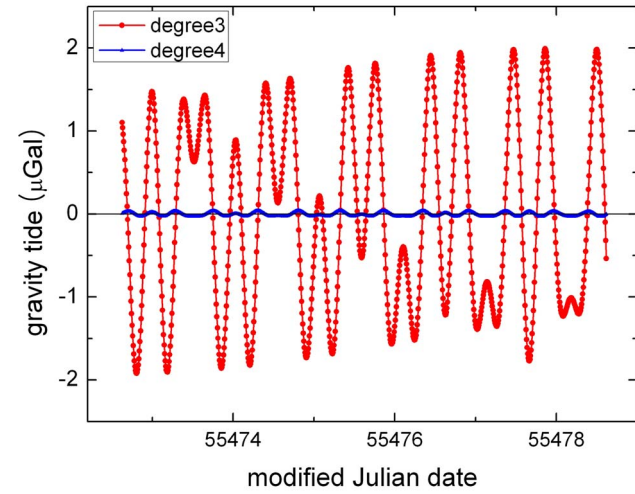


Fig. 2. Gravity tides of third and fourth degrees in Fig. 1 are re-drawn in finer scale [unit: μGal].

dependence for tidal potential and the potential due to deformation, one can evaluate the gravity tide as follows.

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n = - \sum_{n=2}^{\infty} \left(1 + \frac{2}{n} h_n - \frac{n+1}{n} k_n \right) \frac{\partial W_n}{\partial r} \quad (4)$$

Also, tidally induced deviation of the vertical can be estimated as follows.

$$(\Delta \xi, \Delta \eta) = \left(\sum_{n=2}^{\infty} \frac{1+k_n-h_n}{ga} \frac{\partial W_n}{\partial \theta}, - \sum_{n=2}^{\infty} \frac{1+k_n-h_n}{ga \sin \theta} \frac{\partial W_n}{\partial \phi} \right) \quad (5)$$

Detailed derivation of eq. (4) and (5) is given in Appendix. A sample plot of gravity tide at a selected location is given in Fig. 1, where the terms of second and third degrees are

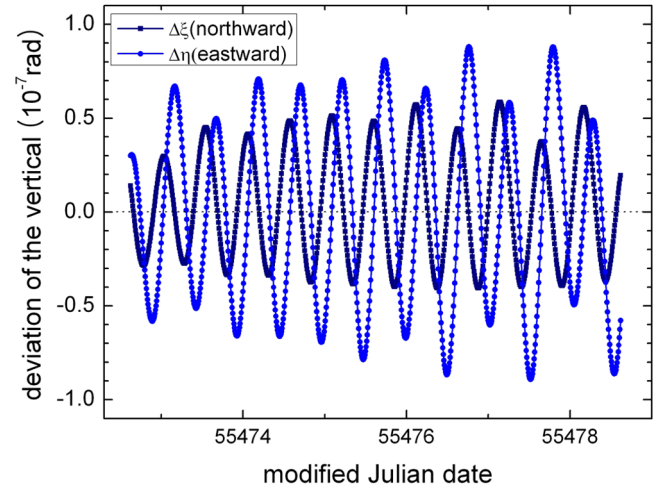


Fig. 3. Deviation of the vertical due to tidal perturbation [unit: 10^{-7} rad]. Both the northward and eastward components are illustrated together. Only degree 2 terms are shown. The location and time span are the same as former figures.

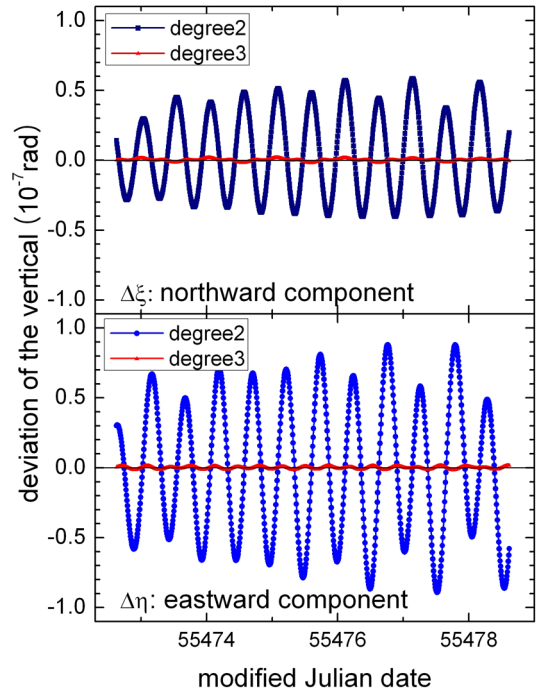


Fig. 4. Deviation of the vertical due to tidal perturbation of degree 2 and 3 [unit: 10^{-7} rad]. The northward and eastward components are separately shown. Same location and time span as former figures.

shown separately (at Chuncheon during Oct. 4 ~ 9, 2010: the week of 2010 KSEG convention). Mean (r.m.s.) values of the second and third degree of gravity tide in Fig. 1, are $68.4 \mu\text{Gal}$ and $1.15 \mu\text{Gal}$ respectively. In fact we have found that the third and fourth degree terms have been evaluated incorrectly (multiplied with the Love numbers of second degree) in a body tide FORTRAN program (Tamura, 1982), which has been widely used in Korea for more than a

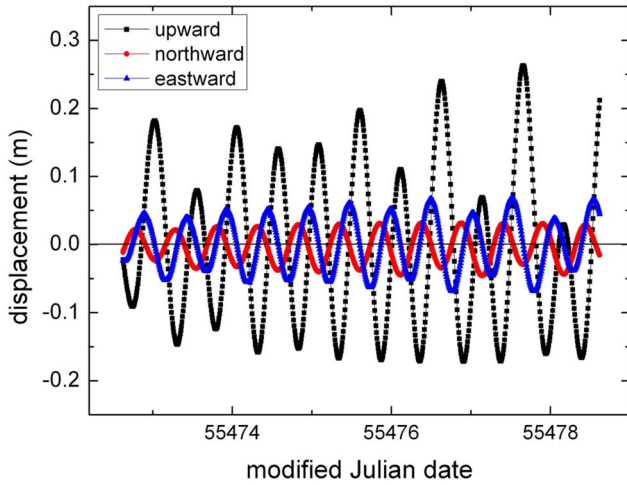


Fig. 5. Three directional components of the tidal deformation. Degree 2 and 3 terms are evaluated. Same location and time span as former figures.

decade, and this has been the motive of this report. In Fig. 2, third and fourth degree terms of gravity tide calculated for the same place and same time span are shown (lunar tide only). Mean value of the fourth degree gravity tide in Fig. 2 is $0.023 \mu\text{Gal}$. It should be noted that the gravity tide illustrated in Fig. 1 and Fig. 2 are to be added with the observed gravity to attain the unperturbed gravity value (this sign convention is the same as Tamura's). The corresponding deviations of the vertical due to lunar and solar tides of degree 2 are shown in Fig. 3. Degree 3 components of the deviation of the vertical are illustrated with degree 2 components in Fig. 4. Mean values of the deviation of the vertical of degree 2 in Fig. 3 are 3.01 and $4.97 \times 10^{-8} \text{ rad}$ for northward and eastward directions, and those of degree 3 in Fig. 4 are 8.31 and $6.98 \times 10^{-10} \text{ rad}$. Vertical and two horizontal components of tidal displacement are illustrated in Fig. 5. A new FORTRAN code has been written for all the computations described above, and is named as 'K-Tide.for'. Another set of test output is given in Fig. 6, where the gravity tides at three locations (north pole, equator, and Chuncheon) for the time span of Oct 4 ~ 19 in 2010 are illustrated. Corresponding output for the deviation of the vertical is given in Fig. 7.

Conclusions

It has been clarified that correct Love numbers should be assigned for different harmonic degree tide. Set of Love numbers for the IASPEI earth model is reported. A new computer code (K-Tide.for) has been written to predict grav-

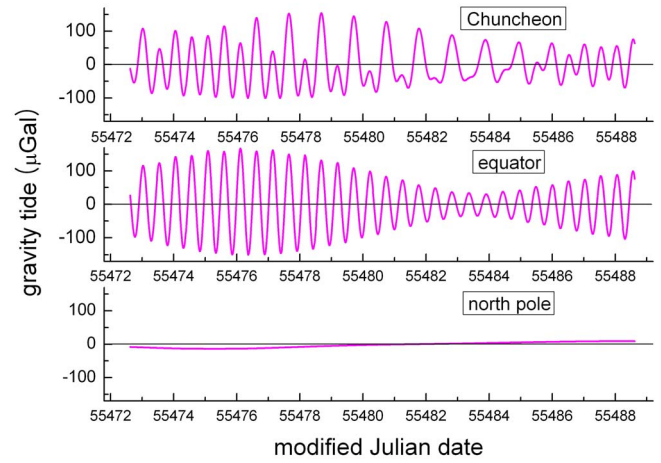


Fig. 6. Gravity tide at three locations (north pole, equator, and Chuncheon, Korea) for the time span of Oct 4 ~ 19 in 2010 (longitude of the position at equator is the same as that of Chuncheon).

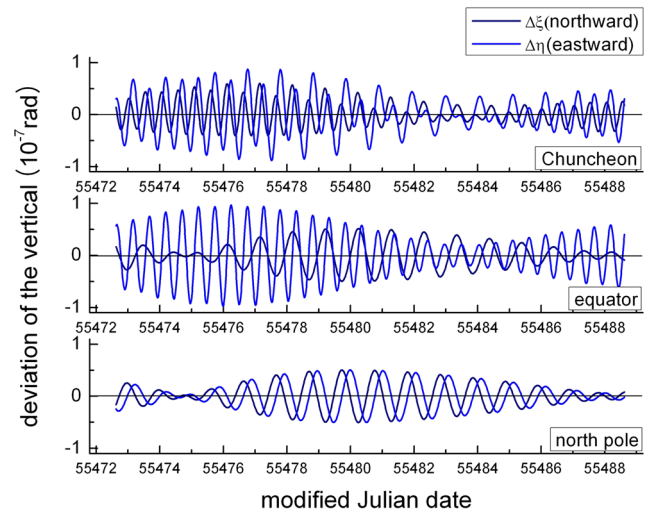


Fig. 7. Tidal deviation of the vertical at three locations (north pole, equator, and Chuncheon, Korea) for the time span of Oct 4 ~ 19 in 2010.

ity tide in terms of spherical harmonics up to degree 4. It also predicts tidal displacement and deviation of the vertical in terms up to degree 3. Distinct Love numbers can be assigned for tidal perturbations of different spherical harmonic order. Sample outputs are presented.

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APPENDIX

Derivations for tidal perturbations in gravity and deviation of the vertical on the surface of Earth are given here. Our approach is equivalent to that of Melchior (1978). The spherical approximation here is valid with error of the same order of Earth's flattening $f \approx 1/298$. Since tidal deformation is less than one meter, the error of spherical approximation should be less than 3 millimeter.

Let U be the gravity potential of unperturbed Earth, while W stands for the tide raising potential and V is the gravity potential due to tidal deformation. The gravity potential U is defined as $G(\bar{r}) = G \int \frac{dm}{|\bar{r} - \bar{r}'|}$, and the gravity vector \bar{g} is defined as $\bar{g} = \vec{\nabla} U \approx \hat{r} \frac{\partial U}{\partial r} = -\hat{r}g$, where g is the magnitude of \bar{g} . This sign convention is common in geophysics and geodesy. Then we have $g = g(a) = -\frac{\partial U}{\partial r} \Big|_{r=a} \approx \frac{GM_e}{a^2}$ and its derivative $\frac{dg}{dr} \Big|_{r=a} = -\frac{\partial^2 U}{\partial r^2} \Big|_{r=a} \approx \frac{2GM_e}{a^3} = -\frac{2g}{a}$, that is normal gravity gradient - also called free-air correction factor.

Change in \bar{g} due to tidal deformation is consisted of three terms. The first one is normal gravity change due to the tidal displacement Δr . Since the amount of vertical tidal displacement is $\sum_{n=2}^{\infty} h_n \frac{W_n}{g}$, the normal gravity change due to the tide induced surface rise would be $\Delta \bar{g} = \frac{d\bar{g}}{dr} \Delta r = \hat{r} \left(-\frac{d\bar{g}}{dr} \right) \Delta r = \hat{r} \frac{2}{a} \sum_{n=2}^{\infty} h_n W_n$.

$$\Delta \bar{g}_{NG} = \hat{r} \frac{2}{a} \sum_{n=2}^{\infty} h_n W_n(a) \quad (A1)$$

Series expansion of the potential of Earth's gravity field is valid, when r dependence is denoted as $\frac{r^n}{r^{n+1}}$ (such as in Jackson (1999)), *i.e.* the potential has field range r dependence as r^n for outside source and as r^{-n-1} for inside source,

where n stands for the spherical harmonic degree.

Since tidal gravity vector is $\vec{\nabla} W$, where tidal potential W is expressed as $W = \sum_{n=2}^{\infty} W_n = GM \sum_{n=2}^{\infty} \frac{r^n}{(r')^{n+1}} P_n(\cos \psi)$. Its vertical component is expressed as $\frac{\partial W}{\partial r} = \sum_{n=2}^{\infty} \frac{\partial W_n}{\partial r} = GM \sum_{n=2}^{\infty} \frac{nr^{n-1}}{(r')^{n+1}} P_n(\cos \psi) = \sum_{n=2}^{\infty} \frac{n}{r} W_n$. Then the vertical component of the tidal gravity vector at the surface of Earth is expressed as follows.

$$\Delta \bar{g}_{TG} = \hat{r} \frac{\partial W}{\partial r} \Big|_{r=a} = \hat{r} \sum_{n=2}^{\infty} \frac{n}{a} W_n(a) \quad (A2)$$

Gravity potential change associated with the tidal deformation itself can be expressed as $V = \sum_{n=2}^{\infty} \left(\frac{a}{r} \right)^{n+1} k_n W_n(a)$ outside of Earth, and so is acquired its derivative as $\frac{\partial V}{\partial r} = \sum_{n=2}^{\infty} (-n-1) \frac{a^{n+1}}{r^{n+2}} k_n W_n(a)$. The gravity change due to the tidal deformation in Earth should be given as

$$\Delta \bar{g}_{TD} = \hat{r} \frac{\partial V}{\partial r} \Big|_{r=a} = -\hat{r} \sum_{n=2}^{\infty} \frac{n+1}{a} k_n W_n(a) \quad (A3)$$

By summing these three terms as $\Delta \bar{g} = \Delta \bar{g}_{NG} + \Delta \bar{g}_{TG} + \Delta \bar{g}_{TD}$, we find the gravity vector change at the surface of Earth due to tide.

$$\begin{aligned} \Delta \bar{g} &= \hat{r} \frac{2}{a} \sum_{n=2}^{\infty} h_n W_n(a) + \hat{r} \frac{1}{a} \sum_{n=2}^{\infty} n W_n(a) - \hat{r} \frac{1}{a} \sum_{n=2}^{\infty} (n+1) k_n W_n(a) \\ &= \hat{r} \sum_{n=2}^{\infty} [2h_n + n - (n+1)k_n] \frac{W_n(a)}{a} \end{aligned} \quad (A4)$$

By using eq. (A2), this can be readily rewritten as eq. (4).

Due to tidal force of harmonic degree n , the direction of gravity would change by an amount of $\frac{1+k_n}{ga} \left(\frac{\partial W_n}{\partial \theta}, \frac{1}{\sin \theta} \frac{\partial W_n}{\partial \phi} \right)$. While the degree n tidal deformation of the surface of Earth is defined as $r = a + h_n \frac{W_n}{g}$, and so the angle between the unperturbed surface and \hat{g} the deformed surface would be $\sum_{n=2}^{\infty} \frac{h_n}{ga} \left(\frac{\partial W_n}{\partial \theta}, \frac{1}{\sin \theta} \frac{\partial W_n}{\partial \phi} \right)$. Combining these two terms, the tidal tilt of the plumb line is expressed as follows.

$$\bar{\tau} = \frac{1}{ga} \sum_{n=2}^{\infty} (1+k_n-h_n) \frac{\partial W_n}{\partial \theta} \hat{\theta} + \frac{1}{ga} \sum_{n=2}^{\infty} (1+k_n-h_n) \frac{1}{\sin \theta} \frac{\partial W_n}{\partial \phi} \hat{\phi} \quad (\text{A5})$$

The deviation of the vertical should be the opposite of the tilt. Since (ξ, η) are defined as northward and eastward components of deflection of the vertical, while $\hat{\theta}$ and $\hat{\phi}$ are southward and eastward unit vectors, tidally induced deviation of the vertical is acquired as follows.

$$(\Delta \xi, \Delta \eta) = \frac{1}{ga} \left(\sum_{n=2}^{\infty} (1+k_n-h_n) \frac{\partial W_n}{\partial \theta}, - \sum_{n=2}^{\infty} (1+k_n-h_n) \frac{1}{\sin \theta} \frac{\partial W_n}{\partial \phi} \right) \quad (\text{A6})$$