J. Appl. Math. & Informatics Vol. **29**(2011), No. 1 - 2, pp. 485 - 495
Website: http://www.kcam.biz

INVOLUTORY AND S+1-POTENCY OF LINEAR COMBINATIONS OF A TRIPOTENT MATRIX AND AN ARBITRARY MATRIX†

CHANGJIANG BU* AND YIXIN ZHOU

Abstract. Let A_1 and A_2 be $n \times n$ nonzero complex matrices, denote a linear combination of the two matrices by $A = c_1A_1 + c_2A_2$, where c_1, c_2 are nonzero complex numbers. In this paper, we research the problem of the linear combinations in the general case. We give a sufficient and necessary condition for A is an involutive matrix and $s+1-$potent matrix, respectively, where A_1 is a tripotent matrix, with $A_1A_2 = A_2A_1$. Then, using the results, we also give the sufficient and necessary conditions for the involutory of the linear combination A, where A_1 is a tripotent matrix, anti-idempotent matrix, and involutive matrix, respectively, and A_2 is a tripotent matrix, idempotent matrix, and involutive matrix, respectively, with $A_1A_2 = A_2A_1$..

AMS Mathematics Subject Classification : 15A27, 15A57, 62H10.
Key words and phrases : Tripotent matrix, idempotent matrix, involutive matrix, $s+1-$potent matrix, linear combination.

1. Introduction

The symbols C, $C\backslash\{0\}$ and $C^{n\times n}$ denote the sets of complex numbers, nonzero complex numbers and $n \times n$ complex matrices, respectively. Let I_n and X^t be $n \times n$ identity matrix and the transpose of X. Let $c_1,\ c_2 \in C\backslash\{0\}$, nonzero matrices $A_1,\ A_2 \in C^{n\times n}$, A is a linear combination of A_1 and A_2, i.e. $A = c_1A_1 + c_2A_2$.

Idempotent matrix, tripotent matrix and involutive matrix have important applications in statistical theory: if A is an $n\times n$ real symmetric matrix, X is an $n \times 1$ real vector and X satisfied the multivariate normal distribution $N_n(0, I)$, where I denotes the identity matrix, then a sufficient and necessary condition for the quadratic form X^tAX (1) to be distributed as a chi-square is $A^2 = A$;

Received March 7, 2010. Revised June 9, 2010. Accepted June 16, 2010. *Corresponding author. †This work was supported by the NSFH, No. 159110120002.

(2) to be distributed as a difference of two independent chi-square variables is $A^3 = A$ (see[1]-[4]). Consequently, the idempotency (or tripotency) of the linear combination $A = c_1A_1 + c_2A_2$, where A_1, A_2 are two commuting real symmetric idempotent (or tripotent) matrices, is related to the linear combination of two quadratic form X^tA_1X, X^tA_2X, it is a chi-square distribution (or as difference of two independent chi-square variables), where X^tA_1X and X^tA_2X are satisfied chi-square distribution (or a difference of two independent chi-square variables). Obviously, if A is an involutive matrix, then there exist two idempotent matrices P_1 and P_2 such that $A = P_1 - P_2$, $I = P_1 + P_2$ and $P_1P_2 = 0$ (see[5]). However, the matrices, which neither real nor symmetric, are also used in many branches of applied sciences. For example the matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, which is a member of the class of matrices known as the Pauli spin matrices and the Dirac spin matrices is neither real nor symmetric but involutive, and they are widely used in quantum mechanics (see[6]-[10]). The problem of the linear combinations has applications in control theory, too. Let us consider a singular control linear system: $E\dot{x} = Ax + Bu$, where $x \in R^n$ is the descriptor variable, $u \in R^m$ is the control input, and $E,\ A \in R^{n\times n}$, $B \in R^{n\times m}$ are constant matrices, E is a singular matrix. The singular system $E\dot{x} = Ax + Bu$ satisfies the regularity condition if there exists $\lambda \in C$ such that $det(\lambda E + A) \neq 0$ (see[11]-[12]). Actually, this problem, when $\lambda \neq 0$, is the invertibility of the linear combination $P = c_1E + c_2A$. Therefore, it is significant to research on this kind of issues.

In 2000, J.K. Baksalary and O.M. Baksalary gave the sufficient and necessary conditions of the idempotency of linear combinations of two idempotent matrices (see[13]); In 2002, J.K. Baksalary, O.M. Baksalary and G.P.H. Styan gave the sufficient and necessary conditions of the idempotency of linear combinations of an idempotent matrix and a tripotent matrix (see[14]); In 2007, [15] give the idempotency of the linear combination $A = c_1A_1 + c_2A_2 + c_3A_3$, where $A_i^2 = A_i$, and $A_iA_j = A_jA_i$, $i \neq j$, $i,\ j = 1, 2, 3$; In 2008, M. Sarduvan and H. Özdemir gave the sufficient and necessary conditions of tripotency, idempotency and involutory of the linear combinations $A = c_1A_1 + c_2A_2$, where A_1 and A_2 are two commuting tripotent, idempotent or involutive matrices, respectively, i.e. (1) the sufficient and necessary conditions of A is a tripotent or idempotent matrix when A_1 and A_2 are commuting involutive matrices, (2) the sufficient and necessary conditions of A is an involutive matrix when A_1 and A_2 are commuting tripotent or idempotent matrices (see[16]); In 2009, H. Özdemir, M. Sarduvan, A.Y. Özban and N. Güler gave the sufficient and necessary conditions of the idempotency and tripotency of linear combinations of two commuting tripotent matrices (see[17]).

In this paper, we research the problem of the linear combinations $A = c_1A_1 + c_2A_2$ in the more general cases. We give, when A_1 is a tripotent matrix, $A = c_1A_1 + c_2A_2$ is an involutive or an $s+1-$potent matrix, with $A_1A_2 = A_2A_1$, all the forms of the arbitrary matrix A_2, and when $A_1^2 = -A_1$ or $A_1^2 = I$, we also give all the forms of A_2, respectively. From the forms of A_2, we give the sufficient and necessary conditions for the involutory of the linear combination A, when A_2 is a tripotent matrix, idempotent matrix, or involutive matrix, with $A_1A_2 = A_2A_1$. Then, Theorem 2.1, 2.2, 2.3 and 2.5 in [16] can be obtained for the special cases of this paper's results.

2. Some Lemmas

Lemma 1. ([18]) *Let $A \in C^{n\times n}$ be a tripotent matrix. Then there exists a nonsingular matrix $P \in C^{n\times n}$ such that $A = P(I_p \oplus -I_q \oplus O)P^{-1}$, where* $p + q = \mathrm{rankA}$.

Lemma 2. ([19]) *Let $A \in C^{n\times n}$ be an involutive matrix. Then there exists a nonsingular matrix $P \in C^{n\times n}$ such that $A = P(I_r \oplus -I_s)P^{-1}$, where $r + s = n$.*

Lemma 3. ([18]) *Let $A \in C^{n\times n}$ be an idempotent matrix. Then there exists a nonsingular matrix $P \in C^{n\times n}$ such that $A = P(I_r \oplus O)P^{-1}$, where* $r = \mathrm{rankA}$.

3. Main Results

In this section, we give all the forms of the arbitrary matrix A_2, when $A = c_1A_1 + c_2A_2$ is an involutive or $s+1-$potent matrix, with $A_1A_2 = A_2A_1$, where A_1 is a tripotent matrix.

3.1.On involutory of linear combinations of a tripoent matrix and an arbitrary matrix.

Theorem 1. *Let $A_1,\ A_2 \in C^{n\times n}$ be two nonzero matrices, and $A_1^3 = A_1$, $A_1A_2 = A_2A_1$, $A = c_1A_1 + c_2A_2$, where $c_1,\ c_2 \in C\backslash\{0\}$. Then the sufficient and necessary conditions for $A^2 = I$ is existing a nonsingular matrix $Q \in C^{n\times n}$ such that*

$$A_1 = Q(I_p \oplus -I_q \oplus O)Q^{-1},$$

$$A_2 = Q[(\frac{1-c_1}{c_2}I_r \oplus \frac{-1-c_1}{c_2}I_s) \oplus (\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t) \oplus (\frac{1}{c_2}I_h \oplus \frac{-1}{c_2}I_l)]Q^{-1},$$

where $p + q = \mathrm{rankA_1}$, $r + s = p$, $m + t = q$, $h + l = n - p - q$, and $p,\ q,\ r,\ s,\ m,\ t,\ h,\ l$ are nonnegative integers.

Proof. The sufficiency is obvious. Now we only prove the necessity.

Since $A_1^3 = A_1$, from Lemma 1, there exists a nonsingular matrix $P \in C^{n\times n}$ such that

$$A_1 = P(I_p \oplus -I_q \oplus O)P^{-1},$$

where $p + q = \mathrm{rankA_1}$.

And from $A_1A_2 = A_2A_1$. Let

$$A_2 = P(X_1 \oplus X_2 \oplus X_3)P^{-1},$$

where $X_1 \in C^{p\times p},\ X_2 \in C^{q\times q},\ X_3 \in C^{(n-p-q)\times(n-p-q)}$.

Then

$$A = P[(c_1I_p + c_2X_1) \oplus (-c_1I_q + c_2X_2) \oplus c_2X_3]P^{-1}.$$

From $A^2 = I$, we have

$$(c_1 I_p + c_2 X_1)^2 = I,\ (-c_1 I_q + c_2 X_2)^2 = I \ and \ (c_2 X_3)^2 = I.. \tag{1}$$

From Lemma 2 and (1), there exist nonsingular matrices $Q_1 \in C^{p\times p},\ Q_2 \in C^{q\times q}$ and $Q_3 \in C^{(n-p-q)\times(n-p-q)}$ such that

$$c_1 I_p + c_2 X_1 = Q_1(I_r \oplus -I_s)Q_1^{-1},$$

$$-c_1 I_q + c_2 X_2 = Q_2(I_m \oplus -I_t)Q_2^{-1},$$

$$c_2 X_3 = Q_3(I_h \oplus -I_l)Q_3^{-1},$$

i.e.

$$X_1 = Q_1(\frac{1-c_1}{c_2} I_r \oplus \frac{-1-c_1}{c_2} I_s)Q_1^{-1},$$

$$X_2 = Q_2(\frac{1+c_1}{c_2} I_m \oplus \frac{-1+c_1}{c_2} I_t)Q_2^{-1},$$

$$X_3 = Q_3(\frac{1}{c_2} I_h \oplus -\frac{1}{c_2} I_l)Q_3^{-1},$$

where $r+s=p,\ m+t=q,\ h+l=n-p-q$.

Therefore,

$$\begin{aligned}
A_2 &= P(X_1 \oplus X_2 \oplus X_3)P^{-1} \\
&= P[Q_1(\frac{1-c_1}{c_2} I_r \oplus \frac{-1-c_1}{c_2} I_s)Q_1^{-1} \oplus Q_2(\frac{1+c_1}{c_2} I_m \oplus \frac{-1+c_1}{c_2} I_t)Q_2^{-1} \\
&\oplus Q_3(\frac{1}{c_2} I_h \oplus \frac{-1}{c_2} I_l)Q_3^{-1}]P^{-1} \\
&= P(Q_1 \oplus Q_2 \oplus Q_3)[(\frac{1-c_1}{c_2} I_r \oplus \frac{-1-c_1}{c_2} I_s) \oplus (\frac{1+c_1}{c_2} I_m \oplus \frac{-1+c_1}{c_2} I_t) \\
&\oplus (\frac{1}{c_2} I_h \oplus \frac{-1}{c_2} I_l)](Q_1^{-1} \oplus Q_2^{-1} \oplus Q_3^{-1})P^{-1}.
\end{aligned}$$

Let $Q = P(Q_1 \oplus Q_2 \oplus Q_3)$. Then

$$A_2 = Q[(\frac{1-c_1}{c_2} I_r \oplus \frac{-1-c_1}{c_2} I_s) \oplus (\frac{1+c_1}{c_2} I_m \oplus \frac{-1+c_1}{c_2} I_t) \oplus (\frac{1}{c_2} I_h \oplus \frac{-1}{c_2} I_l)]Q^{-1},$$

and

$$\begin{aligned}
A_1 &= P(Q_1 \oplus Q_2 \oplus Q_3)(I_p \oplus -I_q \oplus O)(Q_1^{-1} \oplus Q_2^{-1} \oplus Q_3^{-1})P^{-1} \\
&= Q(I_p \oplus -I_q \oplus O)Q^{-1}.
\end{aligned}$$

□

Theorem 2.3 in [16] gave the sufficient and necessary conditions of involutory of two commuting tripotent matrices. Let $A_2^3 = A_2$. Then Theorem 2.3 in [16] can be obtained from the Theorem 1 of this paper, see Corollary 1.

Corollary 1. *Let* $A_1,\ A_2 \in C^{n\times n}$ *be two nonzero matrices, and* $A_1 \neq \pm A_2,\ A_1^3 = A_1,\ A_2^3 = A_2,\ A_1A_2 = A_2A_1,\ A = c_1A_1 + c_2A_2,$ *where* $c_1,\ c_2 \in C\backslash\{0\}$*. Then we have the following situations for which* A *is an involutive matrix:*
(*a*) $(c_1, c_2) = (1, 1)$ *or* $(c_1, c_2) = (-1, -1)$*, and* $A_1^2 + 2A_1A_2 + A_2^2 = I,\ A_1^2 \neq I,\ A_2^2 \neq I;$
(*b*) $(c_1, c_2) = (1, -1)$ *or* $(c_1, c_2) = (-1, 1)$*, and* $A_1^2 - 2A_1A_2 + A_2^2 = I,\ A_1^2 \neq I,\ A_2^2 \neq I;$
(*c*) $(c_1, c_2) = (2, 1)$ *or* $(c_1, c_2) = (-2, -1)$*, and* $4A_1^2 + 4A_1A_2 + A_2^2 = I,\ A_1^2 \neq I;$
(*d*) $(c_1, c_2) = (2, -1)$ *or* $(c_1, c_2) = (-2, 1)$*, and* $4A_1^2 - 4A_1A_2 + A_2^2 = I,\ A_1^2 \neq I;$
(*e*) $(c_1, c_2) = (1, 2)$ *or* $(c_1, c_2) = (-1, -2)$*, and* $A_1^2 + 4A_1A_2 + 4A_2^2 = I,\ A_2^2 \neq I;$
(*f*) $(c_1, c_2) = (1, -2)$ *or* $(c_1, c_2) = (-1, 2)$*, and* $A_1^2 - 4A_1A_2 + 4A_2^2 = I,\ A_2^2 \neq I.$

Proof. The sufficiency is obvious. Now we only prove the necessity.

From Theorem 1, there exists a nonsingular matrix $Q \in C^{n\times n}$ such that

$$A_1 = Q(I_p \oplus -I_q \oplus O)Q^{-1},$$

$$A_2 = Q[(\frac{1-c_1}{c_2}I_r \oplus \frac{-1-c_1}{c_2}I_s) \oplus (\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t) \oplus (\frac{1}{c_2}I_h \oplus \frac{-1}{c_2}I_l)]Q^{-1},$$

where $p + q = \mathrm{rankA_1},\ r + s = p,\ m + t = q,\ h + l = n - p - q,$ and $p,\ q,\ r,\ s,\ m,\ t,\ h,\ l$ are nonnegative integers.

From $A_2^3 = A_2$, we know the following relations:

$$(\frac{1-c_1}{c_2}I_r \oplus \frac{-1-c_1}{c_2}I_s)^3 = (\frac{1-c_1}{c_2}I_r \oplus \frac{-1-c_1}{c_2}I_s),$$

$$(\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t)^3 = (\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t),$$

$$(\frac{1}{c_2}I_h \oplus \frac{-1}{c_2}I_l)^3 = (\frac{1}{c_2}I_h \oplus \frac{-1}{c_2}I_l).$$

Case 1.

$$(\frac{1-c_1}{c_2}I_r \oplus \frac{-1-c_1}{c_2}I_s)^3 = (\frac{1-c_1}{c_2}I_r \oplus \frac{-1-c_1}{c_2}I_s).$$

If $r = 0,\ s = p \neq 0,$

$$c_1 - c_2 = -1 \text{ or } c_1 + c_2 = -1 \text{ or } c_1 = -1. \tag{2}$$

If $s = 0,\ r = p \neq 0,$

$$c_1 - c_2 = 1 \text{ or } c_1 + c_2 = 1 \text{ or } c_1 = 1. \tag{3}$$

If $p = 0,$

$$c_1,\ c_2 \in C\backslash\{0\}. \tag{4}$$

Case 2.

$$(\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t)^3 = (\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t).$$

If $m = 0,\ t = q \neq 0$,

$$c_1 - c_2 = 1 \text{ or } c_1 + c_2 = 1 \text{ or } c_1 = 1. \tag{5}$$

If $t = 0,\ m = q \neq 0$,

$$c_1 - c_2 = -1 \text{ or } c_1 + c_2 = -1 \text{ or } c_1 = -1. \tag{6}$$

If $q = 0$,

$$c_1,\ c_2 \in C\backslash\{0\}. \tag{7}$$

Case 3.

$$(\frac{1}{c_2} I_h \oplus \frac{-1}{c_2} I_l)^3 = (\frac{1}{c_2} I_h \oplus \frac{-1}{c_2} I_l).$$

If $h = 0,\ l = n - p - q \neq 0$,

$$c_2 = \pm 1. \tag{8}$$

If $l = 0,\ h = n - p - q \neq 0$,

$$c_2 = \pm 1. \tag{9}$$

If $h \neq 0,\ l \neq 0$,

$$c_2 = \pm 1. \tag{10}$$

If $n - p - q = 0$,

$$c_1,\ c_2 \in C\backslash\{0\}. \tag{11}$$

Combination of all the results of Case 1, 2 and 3.

Combination of (2), (6) and (8), we get

$$c_1 = -2,\ c_2 = 1,\ 4A_1^2 - 4A_1A_2 + A_2^2 = I,\ A_1^2 \neq I,$$

or

$$c_1 = -1,\ c_2 = 1,\ A_1^2 - 2A_1A_2 + A_2^2 = I,\ A_1^2 \neq I,\ A_2^2 \neq I,$$

or

$$c_1 = -2,\ c_2 = -1,\ 4A_1^2 + 4A_1A_2 + A_2^2 = I,\ A_1^2 \neq I,$$

or

$$c_1 = -1,\ c_2 = -1,\ A_1^2 + 2A_1A_2 + A_2^2 = I,\ A_1^2 \neq I,\ A_2^2 \neq I.$$

In the similar way, combination of $(2) - (11)$, we get the results $(a) - (f)$. □

Theorem 2.1 in [16] gave the sufficient and necessary conditions of tripotency of two commuting involutive matrices. Let $A_2^2 = I$. Then Theorem 2..1 in [16] can be obtained from the Theorem 1 of this paper, too, see Corollary 2.

Corollary 2. *Let $A_1,\ A_2 \in C^{n\times n}$ be two nonzero matrices, $A_1 \neq \pm A_2$, and $A_1^3 = A_1,\ A_2^2 = I,\ A_1A_2 = A_2A_1,\ A = c_1A_1 + c_2A_2$, where $c_1,\ c_2 \in C\backslash\{0\}$. Then we have the following situations for which A is an involutive matrix:*
(a) $(c_1, c_2) = (2, -1)$ or $(c_1, c_2) = (-2, 1)$, and $A_1^2 = A_1A_2$;
(b) $(c_1, c_2) = (2, 1)$ or $(c_1, c_2) = (-2, -1)$, and $A_1^2 = -A_1A_2$.

Proof. The proof is similar to Corollary 1. □

In the Theorem 1, $A_1 = Q(I_p \oplus -I_q \oplus O)Q^{-1}$, A_1 is degenerated anti-idempotent matrix, when $p = 0$, i.e. $A_1^2 = -A_1$ and $A_1A_2 = A_2A_1$. We get the form of arbitrary matrix A_2, when A is an involutive matrix.

Theorem 2. *Let $A_1,\ A_2 \in C^{n\times n}$ be two nonzero matrices, $A_1^2 = -A_1,\ A_1A_2 = A_2A_1$, $A = c_1A_1+c_2A_2$, where $c_1,\ c_2 \in C\backslash\{0\}$. Then the sufficient and necessary conditions for $A^2 = I$ is existing a nonsingular matrix $Q \in C^{n\times n}$ such that*

$$A_1 = Q(-I_q \oplus O)Q^{-1},$$

$$A_2 = Q[(\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t) \oplus (\frac{1}{c_2}I_h \oplus \frac{-1}{c_2}I_l)]Q^{-1},$$

where $q = \mathrm{rankA}_1$, $m+t = q,\ h+l = n-q$, and $m,\ t,\ h,\ l$ are nonnegative integers.

Proof. The proof is similar to Theorem 1. □

Theorem 2.5 in [16] gave the sufficient and necessary conditions of involutory of two commuting idempotent matrices. Let $A_2^2 = A_2$. Then Theorem 2.5 in [16] can be obtained from the Theorem 2 of this paper, see Corollary 3.

Corollary 3. *Let $A_1,\ A_2 \in C^{n\times n}$ be two nonzero matrices, $A_1 \neq \pm A_2$, and $A_1^2 = -A_1,\ A_2^2 = A_2,\ A_1A_2 = A_2A_1$, $A = c_1A_1 + c_2A_2$, where $c_1,\ c_2 \in C\backslash\{0\}$. Then we have the following situations for which A is an involutive matrix:*
(a) $(c_1, c_2) = (1, -1)$ or $(c_1, c_2) = (-1, 1)$, and $-A_1 + A_2 = I$;
(b) $(c_1, c_2) = (1, 2)$ or $(c_1, c_2) = (-1, -2)$, and $A_1 = -I$;
(c) $(c_1, c_2) = (2, 1)$ or $(c_1, c_2) = (-2, -1)$, and $A_2 = I$.

Proof. The proof is similar to Corollary 1. □

In the Theorem 1, $A_1 = Q(I_p \oplus -I_q \oplus O)Q^{-1}$, A_1 is degenerated an involutive matrix, when $p + q = n$, i.e. $A_1^2 = I$ and $A_1A_2 = A_2A_1$. We get the form of arbitrary matrix A_2, when A is an involutive matrix.

Theorem 3. *Let $A_1,\ A_2 \in C^{n\times n}$ be two nonzero matrices, and $A_1^2 = I,\ A_1A_2 = A_2A_1$, $A = c_1A_1+c_2A_2$, where $c_1,\ c_2 \in C\backslash\{0\}$. Then the sufficient and necessary conditions for $A^2 = I$ holds is existing a nonsingular matrix $Q \in C^{n\times n}$ such that*

$$A_1 = Q(I_p \oplus -I_q)Q^{-1},$$

$$A_2 = Q[(\frac{1-c_1}{c_2}I_r \oplus \frac{-1-c_1}{c_2}I_s) \oplus (\frac{1+c_1}{c_2}I_m \oplus \frac{-1+c_1}{c_2}I_t)]Q^{-1},$$

where $p+q = n,\ r+s = p,\ m+t = q$, and $p,\ q,\ r,\ s,\ m,\ t$ are nonnegative integers.

Proof. The proof is similar to Theorem 1. □

Theorem 2.2 in [16] gave the sufficient and necessary conditions of idempotency of two commuting involutive matrices. Let $A_2^2 = A_2$. Then the Theorem 2.2 in [16] can be obtained from the Theorem 3 of this paper, see Corollary 4.

Corollary 4. *Let* $A_1,\ A_2 \in C^{n\times n}$ *be two nonzero matrices, and* $A_1 \neq \pm A_2,\ A_1^2 = I,\ A_2^2 = A_2,\ A_1A_2 = A_2A_1,\ A = c_1A_1 + c_2A_2,$ *where* $c_1,\ c_2 \in C\backslash\{0\}$. *Then we have the following situations for which* A *is an involutive matrix:*
(a) $(c_1, c_2) = (1, 2)$ *or* $(c_1, c_2) = (-1, -2)$, *and* $A_1A_2 + A_2 = O$;
(b) $(c_1, c_2) = (1, -2)$ *or* $(c_1, c_2) = (-1, 2)$, *and* $A_1A_2 - A_2 = O$..

Proof. The proof is similar to Corollary 1. □

If the matrix A_1 in Theorem 1 is taken as $s+1-$potent matrix, and let A_2 be an arbitrary matrix, the following results are obtained.

3.2.On s+1-potency of linear combinations of a tripoent matrix and an arbitrary matrix.

Theorem 4. *Let* $A_1,\ A_2 \in C^{n\times n}$ *be two nonzero matrices, and* $A_1^3 = A_1$, $A_1A_2 = A_2A_1$, $A = c_1A_1 + c_2A_2$, *where* $c_1,\ c_2 \in C\backslash\{0\}$. *Then the sufficient and necessary conditions for* $A^{s+1} = A$ *is existing a nonsingular matrix* $Q \in C^{n\times n}$ *such that*

$$A_1 = Q(I_p \oplus -I_q \oplus O)Q^{-1},$$

$$A_2 = Q(\frac{X_1 - c_1I_p}{c_2} \oplus \frac{X_2 + c_1I_q}{c_2} \oplus \frac{X_3}{c_2})Q^{-1},$$

where $p + q = \mathrm{rankA_1}$, $X_1 = \mathrm{diag}(\beta_1, \dots, \beta_p)$, $X_2 = \mathrm{diag}(\beta_{p+1}, \dots, \beta_{p+q})$, $X_3 = \mathrm{diag}(\beta_{p+q+1}, \dots, \beta_n)$, $\beta_i \in \{0\} \cup V$ $(i = 1, \dots, n,\ V = \{x|x^s = 1\})$, *and* $p,\ q$ *are nonnegative integers,* s *is a positive integer.*

Proof. The sufficiency is obvious. Now we only prove the necessity.

Since $A_1^3 = A_1$, from Lemma 1, there exists a nonsingular matrix $P \in C^{n\times n}$ such that

$$A_1 = P(I_p \oplus -I_q \oplus O)P^{-1},$$

where $p + q = \mathrm{rankA_1}$.

And from $A_1A_2 = A_2A_1$. Let

$$A_2 = P(Y_1 \oplus Y_2 \oplus Y_3)P^{-1},$$

where $Y_1 \in C^{p\times p},\ Y_2 \in C^{q\times q},\ Y_3 \in C^{(n-p-q)\times(n-p-q)}$.

Then

$$A = P[(c_1I_p + c_2Y_1) \oplus (-c_1I_q + c_2Y_2) \oplus c_2Y_3]P^{-1}.$$

From $A^{s+1} = A$, we have

$$(c_1I_p + c_2Y_1)^{s+1} = (c_1I_p + c_2Y_1),\ (-c_1I_q + c_2Y_2)^{s+1} = (-c_1I_q + c_2Y_2)$$

and

$$(c_2Y_3)^{s+1} = (c_2Y_3).$$

An $s+1-$potent matrix is diagonalizable, so, from $(c_1I_p+c_2Y_1)^{s+1} = (c_1I_p+c_2Y_1)$, there exists a nonsingular matrix $Q_1 \in C^{p\times p}$ such that

$$c_1I_p + c_2Y_1 = Q_1(\beta_1 \oplus \cdots \oplus \beta_p)Q_1^{-1}.$$

Let $X_1 = \mathrm{diag}(\beta_1, \ldots, \beta_p)$, where $\beta_i \in \{0\}\cup V$ $(i = 1, \ldots, p, V = \{x|x^s = 1\})$. Then

$$Y_1 = Q_1(\frac{X_1 - c_1I_p}{c_2})Q_1^{-1}.$$

In the same way, there exist nonsingular matrices $Q_2 \in C^{q\times q},\ Q_3 \in C^{(n-p-q)\times(n-p-q)}$ such that

$$Y_2 = Q_2(\frac{X_2 + c_1I_q}{c_2})Q_2^{-1},\ Y_3 = Q_3(\frac{X_3}{c_2})Q_3^{-1},$$

where $X_2 = \mathrm{diag}(\beta_{p+1}, \ldots, \beta_{p+q}),\ X_3 = \mathrm{diag}(\beta_{p+q+1}, \ldots, \beta_n),\ \beta_i \in \{0\} \cup V$ $(i = p+1, \ldots, n,\ V = \{x|x^s = 1\})$.

Therefore,

$$A_2 = P(Q_1 \oplus Q_2 \oplus Q_3)(\frac{X_1 - c_1I_p}{c_2} \oplus \frac{X_2 + c_1I_q}{c_2} \oplus \frac{X_3}{c_2})(Q_1^{-1} \oplus Q_2^{-1} \oplus Q_3^{-1})P^{-1}.$$

Let $Q = P(Q_1 \oplus Q_2 \oplus Q_3)$. Then

$$A_2 = Q(\frac{X_1 - c_1I_p}{c_2} \oplus \frac{X_2 + c_1I_q}{c_2} \oplus \frac{X_3}{c_2})Q^{-1},$$

and

$$\begin{aligned} A_1 &= P(Q_1 \oplus Q_2 \oplus Q_3)(I_p \oplus -I_q \oplus O)(Q_1^{-1} \oplus Q_2^{-1} \oplus Q_3^{-1})P^{-1} \\ &= Q(I_p \oplus -I_q \oplus O)Q^{-1}. \end{aligned}$$

□

In the Theorem 4, $A_1 = Q(I_p \oplus -I_q \oplus O)Q^{-1}$, A_1 is degenerated an anti-idempotent matrix, when $p = 0$, i.e. $A_1^2 = -A_1$ and $A_1A_2 = A_2A_1$. We get the form of arbitrary matrix A_2, when A is an $s+1-$potent matrix.

Theorem 5. *Let $A_1,\ A_2 \in C^{n\times n}$ be two nonzero matrices, $A_1^2 = -A_1,\ A_1A_2 = A_2A_1,\ A = c_1A_1+c_2A_2$, where $c_1,\ c_2 \in C\backslash\{0\}$. Then the sufficient and necessary conditions for $A^{s+1} = A$ is existing a nonsingular matrix $Q \in C^{n\times n}$ such that*

$$A_1 = Q(-I_q \oplus O)Q^{-1},$$

$$A_2 = Q(\frac{X_1 + c_1I_q}{c_2} \oplus \frac{X_2}{c_2})Q^{-1},$$

where $q = \mathrm{rankA_1},\ X_1 = \mathrm{diag}(\beta_1, \ldots, \beta_q),\ X_2 = \mathrm{diag}(\beta_{q+1}, \ldots, \beta_n),\ \beta_i \in \{0\} \cup V$ $(i = 1, \ldots, n,\ V = \{x|x^s = 1\})$, and s is a positive integer.

Proof. The proof is similar to Theorem 4. □

In the Theorem 4, $A_1 = Q(I_p \oplus -I_q \oplus O)Q^{-1}$, A_1 is degenerated an involutive matrix, when $p + q = n$, i.e. $A_1^2 = I$ and $A_1A_2 = A_2A_1$. We get the form of arbitrary matrix A_2, when A is an $s+1-$potent matrix.

Theorem 6. *Let $A_1,\ A_2 \in C^{n\times n}$ be two nonzero matrices, and $A_1^2 = I,\ A_1A_2 = A_2A_1$, $A = c_1A_1 + c_2A_2$, where $c_1,\ c_2 \in C\backslash\{0\}$. Then the sufficient and necessary conditions for $A^{s+1} = A$ is existing a nonsingular matrix $Q \in C^{n\times n}$ such that*

$$A_1 = Q(I_p \oplus -I_q)Q^{-1},$$

$$A_2 = Q(\frac{X_1 - c_1I_p}{c_2} \oplus \frac{X_2 + c_1I_q}{c_2})Q^{-1},$$

where $p + q = \mathrm{rankA_1} = \mathrm{n}$, $X_1 = \mathrm{diag}(\beta_1, \dots, \beta_p)$, $X_2 = \mathrm{diag}(\beta_{p+1}, \dots, \beta_{p+q})$, $\beta_i \in \{0\} \cup V$ $(i = 1, \dots, n,\ V = \{x|x^s = 1\})$, and $p,\ q$ are nonnegative integers, s is a positive integer.

Proof. The proof is similar to Theorem 4. □

Remark. Let $A = c_1A_1 + \cdots + c_kA_k$, $A_i \in C^{n\times n}$, $c_i \in C\backslash\{0\}$, $i = 1, 2, \cdots, k$ $(k \geq 2)$, where $A_i^2 = A_i$ or $A_i^3 = A_i$ or $A_i^{s+1} = A_i$ or $A_i^2 = I$ $(i \in \{1, 2, \cdots, k-1\})$, and $A_iA_j = A_jA_i$, $i \neq j$; $i, j = 1, \cdots, k$, A_k is an arbitrary matrix. With the method in this paper, we can get all the forms of the arbitrary matrix A_k, when $A^2 = A$ or $A^3 = A$ or $A^{s+1} = A$ or $A^2 = I$.

References

1. C.R. Rao and S.K. Mitra, *Generalized Inverse of matrices and its applications*, John Wiley, New York, 1971.
2. G.A.F. Seber, *Linear regression analysis*, John Wiley, New York, 1977.
3. F.A. Graybill, *Introduction to matrices with applications in statistics*, Wadsworth Publishing Company Inc., California, 1969.
4. B. Baldessari, *The distribution of a quadratic form of normal random variables*, Ann. Math. Statist **38** (1967), 1700-1704.
5. C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.
6. N.A. Gromov, *The matrix quantum unitary Cayley-Klein groups*, J. Phys. A **26** (1993), 5-8.
7. M.A. Ovchinnikov, *Properties of Viro-Turaev representations of the mapping class group of a Torus*, J. Math. Sci. (NY) **113** (2003), 856-867.
8. M.M. Mestechkin, *Restricted Hartree-Fock method instability*, Int. J. Quant. Chem. **13** (1978), 469-481.
9. H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One-and Two-electron Atom*, Plenum Pub. Co., New York, 1997.
10. S.L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press Inc., New York, 1995.
11. D. Cobb, *Descriptor variable systems and optimal state regulation*, IEEE Trans. Automatic Control. **AC-28** (1983), 601-611.
12. F.L. Lewis, *Preliminary notes on optimal control for singular system*, Proc. 24th IEEE Conference on Decision and Control (1985), 266-272.
13. J.K. Baksalary and O.M. Baksalary, *Idempotency of linear combinations of two idempotent matrices*, Linear Algebra Appl. **321** (2000), 3-7.
14. J.K. Baksalary, O.M. Baksalary and G.P.H. Styan, *Idempoentcy of linear combinations of an idempotent matrix and a tripotent matrix*, Linear Algebra Appl. **354** (2002), 21-34.
15. O.M. Baksalary and J. Benítez, *Idempotency of linear combinations of three idempotent matrices, two of which are commuting*, Linear Algebra Appl. **424** (2007), 320-337.

16. M. Sarduvan and H. Özdemir, *On linear combinations of two tripotent, idempotent, and involutive matrices*, Appl. Math. Comp. **200** (2008), 401-406.
17. H. Özdemir, M. Sarduvan, A.Y. Özban and N. Güler, *On idempotency and tripotency of linear combinations of two commuting tripotent matrices*, Appl. Math. Comp. **207** (2009), 197-201.
18. C. Bu, *Linear maps preserving Drazin inverses of matrices over fields*, Linear Algebra Appl. **396** (2005), 159-173.
19. X. Zhang and C. Cao, *Additive operators presserving identity matrices*, Harbin Press, 2001.

Changjiang Bu received his BS from Heilongjiang University. Since 1985 he has been teached at Harbin Engineering University. In September of 2005, he received a professor from Heilongjiang Education Committee. His research interests focus on the theory of matrix algebra and the theory of graph.

Department of Applied Mathematics, College of Science, Harbin Engineering University, Harbin 150001, China.
e-mail: `buchangjiang@hrbeu.edu.cn`

Yixin Zhou

Department of Applied Mathematics, College of Science, Harbin Engineering University, Harbin 150001, China.
e-mail: `zhouyixin0@163.com`