J. Appl. Math. & Informatics Vol. **29***(2011), No. 1 - 2, pp. 475-483*
Website: http://www.kcam.biz

OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SECOND ORDER NONLINEAR DIFFERENTIAL INEQUALITY WITH PERTURBATION

QUANXIN ZHANG*, XIA SONG

Abstract. In this paper,we study the oscillatory and asymptotic behavior of a class of second order nonlinear differential inequality with perturbation and establish several theorems by using classification and analysis, which develop and generalize some known results.

AMS Mathematics Subject Classification: 34C10.
Key words and phrases : Differential inequality, Oscillation criterion, Asymptotic behavior.

1. Introduction

The oscillation for the following second order nonlinear differential equation with damping

$$(a(t)\psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad ' = \frac{d}{dt}$$

has been studied in [1,2], and several theorems about the oscillation have been established. In this paper, we discuss the oscillatory and asymptotic behavior of the following second order nonlinear differential inequality with perturbation

$$x(t)\{(a(t)\psi(x(t))x'(t))' + Q(t, x(t)) + P(t, x(t), x'(t))\} \le 0. \qquad (1)$$

Under some conditions, by using classification and analysis, we establish four oscillatory and asymptotic theorems, which generalize and develop the results of [1-3].

For Iq.(1), assume that:

(A_1) $a : [t_0, +\infty) \to (0, +\infty)$ is continuously differentiable;

(A_2) $\psi : R \to R$ is continuously differentiable, and $\psi(u) > 0$ for $u \neq 0$;

Received June 24, 2010. Revised July 19, 2010. Accepted July 26, 2010. *Corresponding author.

(A_3) $Q : [t_0, +\infty) \times R \to R$ is continuous function, and there exist continuous function q and continuously differentiable function f: where $q : [t_0, +\infty) \to (0, +\infty)$, $f : R \to R$, $uf(u) > 0$, $f'(u) > 0$, $u \neq 0$, such that $Q(t, x)/f(x) \geq q(t)$, $x \neq 0$;

(A_4) $P : [t_0, +\infty) \times R^2 \to R$ is continuous function, and there exists continuous function $p : [t_0, +\infty) \to R$ such that $x(t)P(t, x(t), x'(t)) \geq x(t)p(t)x'(t)$, $x \neq 0$.

In this paper, we assume that each solution x of Iq.(1) can be extended to $[t_0, +\infty)$. A solution is said to be *regular* if there exists t on arbitrary interval $[T, +\infty)$, such that $x(t) \neq 0$. A regular solution is said to be *oscillatory*, if it has arbitrarily large zeros; otherwise it is said to be *nonoscillatory*. A nonoscillatory solution x of Iq.(1) is said to be *weakly oscillatory* if $x'(t)$ changes sign for arbitrarily large values of t. Iq.(1) is called *oscillatory* if all its regular solutions are oscillatory.

With respect to their asymptotic behavior, all the regular solutions of Iq.(1) can be divided into the following classes:

$S^+ = \{x = x(t)$: regular solution of Iq.(1): there exists $t_x \geq t_0$ such that $x(t)x'(t) > 0$ for $t \geq t_x\}$;

$S^- = \{x = x(t)$: regular solution of Iq.(1): there exists $t_x \geq t_0$ such that $x(t)x'(t) \leq 0$ for $t \geq t_x\}$;

$S^O = \{x = x(t)$: regular solution of Iq.(1): there exists $\{t_n\}, t_n \to +\infty$, such that $x(t_n) = 0\}$;

$S^{WO} = \{x = x(t)$: regular solution of Iq.(1): $x(t) \neq 0$ for t sufficiently large, and for all $t_\alpha > t_0$ there exists $t_{\alpha_1} > t_\alpha$, $t_{\alpha_2} > t_\alpha$ such that $x'(t_{\alpha_1})x'(t_{\alpha_2}) < 0\}$.

It is easy to prove that S^+, S^-, S^O, S^{WO} are mutually disjoint. By the above definitions, it turns out that solutions in the class S^+ are eventually either positive increasing or negative decreasing, solutions in the class S^- are eventually either positive nonincreasing or negative nondecreasing, solutions in the class S^O are oscillatory, and finally, solutions in the class S^{WO} are weakly oscillatory.

2. Main Results

Lemma 1. Assume that $p(t) \leq 0, t \geq t_0; \psi(x) \geq c > 0, f'(x)/\psi(x) \geq \alpha > 0$, $x \neq 0$. Suppose that there exists a differentiable function $\rho : [t_0, +\infty) \to (0, +\infty)$ such that $\rho'(t) \geq 0$, and for sufficiently large T,

$$\liminf_{t\to+\infty} \int_T^t \rho(s) \left[q(s) - \frac{a(s)}{4\alpha} \left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)} \right)^2 \right] \mathrm{d}s \geq 0 \tag{2}$$

holds. Then $S^{WO} = \emptyset$ for Iq.(1).

Proof. Suppose that Iq.(1) has a solution $x \in S^{WO}$, without loss of generality, we may assume that there exists $t_1 \geq t_0$, such that $x(t) > 0$ for $t \geq t_1$ (for $x(t) < 0$, the proof is similar). Then for all $t_\alpha > t_1$, there exists $t_{\alpha_1} > t_\alpha$, $t_{\alpha_2} > t_\alpha$, such that $x'(t_{\alpha_1})x'(t_{\alpha_2}) < 0$. Hence there exists a sequence $\{C_n\} \to$

$+\infty$, such that $x'(C_n) < 0$. Choosing sufficiently large N, such that C_N satisfies condition (2), i.e.

$$\liminf_{t\to+\infty} \int_{C_N}^{t} \rho(s) \left[q(s) - \frac{a(s)}{4\alpha} \left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)} \right)^2 \right] \mathrm{d}s \geq 0.$$

Consider the function

$$W(t) = \rho(t) \frac{a(t)\psi(x(t))x'(t)}{f(x(t))}, \ t \geq t_1,$$

then it follows from Iq.(1) that

$$\begin{aligned}
W'(t) &= \rho(t)\frac{(a(t)\psi(x(t))x'(t))'}{f(x(t))} + \rho'(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \\
&\quad - \frac{\rho(t)a(t)\psi(x(t))f'(x(t))x'^2(t)}{f^2(x(t))} \\
&\leq -\rho(t)\frac{P(t,x(t),x'(t)) + Q(t,x(t))}{f(x(t))} + \rho'(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \\
&\quad - \frac{\rho(t)a(t)\psi(x(t))f'(x(t))x'^2(t)}{f^2(x(t))} \\
&\leq -\rho(t)\left[\frac{p(t)x'(t)}{f(x(t))} + q(t)\right] + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{f'(x(t))}{\rho(t)a(t)\psi(x(t))}W^2(t) \\
&= -\rho(t)q(t) + \left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{a(t)\psi(x(t))}\right]W(t) - \frac{f'(x(t))}{a(t)\rho(t)\psi(x(t))}W^2(t) \\
&= -\rho(t)q(t) + \frac{a(t)\rho(t)\psi(x(t))}{4f'(x(t))}\left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{a(t)\psi(x(t))}\right]^2 \\
&\quad - \left[\left(\frac{f'(x(t))}{a(t)\rho(t)\psi(x(t))}\right)^{\frac{1}{2}} W(t) - \frac{\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{a(t)\psi(x(t))}}{2\left(\frac{f'(x(t))}{a(t)\rho(t)\psi(x(t))}\right)^{\frac{1}{2}}}\right]^2 \\
&\leq -\rho(t)q(t) + \frac{a(t)\rho(t)}{4\alpha}\left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{ca(t)}\right]^2.
\end{aligned}$$

For arbitrarily $b \geq t_1$, integrating the above inequality from b to $t (t \geq b)$, we have

$$\begin{aligned}
\rho(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} &\leq \rho(b)\frac{a(b)\psi(x(b))x'(b)}{f(x(b))} \\
&\quad - \int_b^t \rho(s)\left[q(s) - \frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)}\right)^2\right]\mathrm{d}s. \quad (3)
\end{aligned}$$

For the above C_N, if $t \geq C_N$, we have

$$\begin{aligned}\rho(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \ \leq\ & \rho(C_N)\frac{a(C_N)\psi(x(C_N))x'(C_N)}{f(x(C_N))} \\ & -\int_{C_N}^{t}\rho(s)\left[q(s)-\frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{ca(s)}\right)^2\right]\mathrm{d}s.\end{aligned}$$

Hence

$$\begin{aligned}\limsup_{t\to+\infty}\rho(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} &\leq \rho(C_N)\frac{a(C_N)\psi(x(C_N))x'(C_N)}{f(x(C_N))} \\ +\limsup_{t\to+\infty}\left\{-\int_{C_N}^{t}\rho(s)\left[q(s)-\frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{ca(s)}\right)^2\right]\mathrm{d}s\right\} &< 0.\end{aligned}$$

Then we obtain $x'(t) < 0 (t \geq C_N)$, which contradicts with $x'(t_{\alpha_1})x'(t_{\alpha_2}) < 0$. The proof is complete. $\square$

Lemma 2. Assume that $p(t) \leq 0, t \geq t_0; \psi(x) \geq c > 0, f'(x)/\psi(x) \geq \alpha > 0, x \neq 0$. Suppose that there exists a differentiable function $\rho : [t_0, +\infty) \to (0, +\infty)$ such that $\rho'(t) \geq 0$, and

$$\int_{t_0}^{+\infty}\rho(s)\left[q(s)-\frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{ca(s)}\right)^2\right]\mathrm{d}s < +\infty, \tag{4}$$

$$\lim_{t\to+\infty}\int_{t_0}^{t}\frac{1}{a(s)\rho(s)}\int_{s}^{+\infty}\rho(\tau)\left[q(\tau)-\frac{a(\tau)}{4\alpha}\left(\frac{\rho'(\tau)}{\rho(\tau)}-\frac{p(\tau)}{ca(\tau)}\right)^2\right]\mathrm{d}\tau\mathrm{d}s = +\infty. \tag{5}$$

If $f(u)/\psi(u)$ is strongly superlinear, that is for arbitrarily $\varepsilon > 0$,

$$\int_{\varepsilon}^{+\infty}\frac{\psi(u)}{f(u)}du < +\infty, \quad \int_{-\infty}^{-\varepsilon}\frac{\psi(u)}{f(u)}du > -\infty \tag{6}$$

holds. Then $S^+ = \emptyset$ for Iq.(1).

Proof. Suppose that Iq.(1) has a solution $x \in S^+$, without loss of generality, assume that there exists $t_1 \geq t_0$ such that $x(t) > 0, x'(t) > 0$ for $t \geq t_1$ (for $x(t) < 0, x'(t) < 0$, the proof is similar). As in the proof of Lemma1, we obtain (3). Noting that $x'(t) > 0$ for $t \geq b$, from (4), we have

$$0 < \rho(b)\frac{a(b)\psi(x(b))x'(b)}{f(x(b))} - \int_{b}^{+\infty}\rho(s)\left[q(s)-\frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{ca(s)}\right)^2\right]\mathrm{d}s,$$

for arbitrarily b. For all $t \geq b$, we have

$$\int_{t}^{+\infty}\rho(s)\left[q(s)-\frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{ca(s)}\right)^2\right]\mathrm{d}s < \rho(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))},$$

and we can obtain

$$\int_b^t \frac{1}{a(s)\rho(s)} \int_s^{+\infty} \rho(\tau)\left[q(\tau) - \frac{a(\tau)}{4\alpha}\left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)}\right)^2\right] \mathrm{d}\tau \mathrm{d}s$$

$$< \int_b^t \frac{\psi(x(s))x'(s)}{f(x(s))} \mathrm{d}s.$$

Letting $t \to +\infty$, from (5) and (6), we obtain a contradiction. The proof is complete. □

Theorem 1. Assume that $p(t) \le 0, t \ge t_0; \psi(x) \ge c > 0, f'(x)/\psi(x) \ge \alpha > 0, x \ne 0$. Suppose that there exists a differentiable function $\rho : [t_0, +\infty) \to (0, +\infty)$, such that $\rho'(t) \ge 0$. If (2), (4)-(6) hold, and

$$\lim_{t \to +\infty} \int_{t_0}^t \frac{1}{a(s)} ds = +\infty \tag{7}$$

also holds, then Iq.(1) is oscillatory.

Proof. From Lemma 1 and Lemma 2, $S^+ = S^{WO} = \emptyset$ for Iq.(1). Therefore, it suffices to show that $S^- = \emptyset$ for Iq.(1). Suppose that Iq.(1) has a solution $x \in S^-$. Without loss of generality, we may assume that there exists $t_1 \ge t_0$, such that $x(t) > 0, x'(t) \le 0$ for $t \ge t_1$ (for $x(t) < 0, x'(t) \ge 0$, the proof is similar). By the assumption of Iq.(1), there exists $t \ge t_1$, such that $x'(t) \ne 0$, then there exists $t_2 \ge t_1$, such that $x'(t_2) < 0$. From Iq.(1), for $t \ge t_2$,

$$(a(t)\psi(x(t))x'(t))' \le -p(t)x'(t) - q(t)f(x(t)) < 0.$$

Hence

$$a(t)\psi(x(t))x'(t) < a(t_2)\psi(x(t_2))x'(t_2) = k \quad (k < 0).$$

Therefore, for $t \ge t_2$, we have

$$\int_{x(t_2)}^{x(t)} \psi(u) du \le k \int_{t_2}^t \frac{1}{a(s)} ds,$$

noting condition (7), for $t \to +\infty$(noting $0 < x(t) \le x(t_2)$), the left of the above inequality is lower bounded while the right is eventually minus infinity, which gives a contradiction. The proof is complete. □

Theorem 2. Assume that $p(t) \le 0, t \ge t_0; \psi(x) \ge c > 0, f'(x)/\psi(x) \ge \alpha > 0, x \ne 0$. Suppose that there exists a differentiable function $\rho : [t_0, +\infty) \to (0, +\infty)$, such that $\rho'(t) \ge 0$ and

$$\int^{+\infty} \rho(s)\left[q(s) - \frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)}\right)^2\right] \mathrm{d}s = +\infty, \tag{8}$$

then all the nonoscillatory solution of Iq.(1) can be divided into the following two types:

A_c: $x(t) \to C$(constant)$\ne 0 (t \to +\infty)$;
A_0: $x(t) \to 0 (t \to +\infty)$.

Proof. Let x be a nonoscillatory solution of Iq.(1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_1 \geq t_0$. Considering the function

$$W(t) = \rho(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))}, \quad t \geq t_1.$$

It follows from Iq.(1) that

$$\begin{aligned}
W'(t) &= \rho(t)\frac{(a(t)\psi(x(t))x'(t))'}{f(x(t))} + \rho'(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \\
&\quad - \frac{\rho(t)a(t)\psi(x(t))f'(x(t))x'^2(t)}{f^2(x(t))} \\
&\leq -\rho(t)\frac{P(t,x(t),x'(t)) + Q(t,x(t))}{f(x(t))} + \rho'(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \\
&\quad - \frac{\rho(t)a(t)\psi(x(t))f'(x(t))x'^2(t)}{f^2(x(t))} \\
&\leq -\rho(t)\left[\frac{p(t)x'(t)}{f(x(t))} + q(t)\right] + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{f'(x(t))}{\rho(t)a(t)\psi(x(t))}W^2(t) \\
&= -\rho(t)q(t) + \left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{a(t)\psi(x(t))}\right]W(t) - \frac{f'(x(t))}{a(t)\rho(t)\psi(x(t))}W^2(t) \\
&= -\rho(t)q(t) + \frac{a(t)\rho(t)\psi(x(t))}{4f'(x(t))}\left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{a(t)\psi(x(t))}\right]^2 \\
&\quad - \left[\left(\frac{f'(x(t))}{a(t)\rho(t)\psi(x(t))}\right)^{\frac{1}{2}}W(t) - \frac{\dfrac{\rho'(t)}{\rho(t)} - \dfrac{p(t)}{a(t)\psi(x(t))}}{2\left(\dfrac{f'(x(t))}{a(t)\rho(t)\psi(x(t))}\right)^{\frac{1}{2}}}\right]^2 \\
&\leq -\rho(t)q(t) + \frac{a(t)\rho(t)}{4\alpha}\left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{ca(t)}\right]^2. \qquad (9)
\end{aligned}$$

Integrating the above inequality from t_1 to t

$$\rho(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} \leq L - \int_{t_1}^{t}\rho(s)\left[q(s) - \frac{a(s)}{4\alpha}\left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)}\right)^2\right]\mathrm{d}s,$$

where $L = \rho(t_1)\dfrac{a(t_1)\psi(x(t_1))x'(t_1)}{f(x(t_1))}$. Noting condition (8) and the symbols of a, ψ and f, then there exists $T_0 \geq t_1$, such that $x'(t) < 0$ for $t \geq T_0$, i.e., $x'(t)$ is eventually minus. Hence x is monotone decreasing. Noting $x(t) > 0$, then x is monotone decreasing and lower bounded for $t \geq T_0$. Therefore, $\lim_{t\to+\infty} x(t)$ exists, and it is also limited. It's easily to obtain $\lim_{t\to+\infty} x(t) = C \geq 0$.

For $x(t) < 0 (t \geq t_1)$, similarly, we obtain $\lim_{t\to+\infty} x(t) = C \leq 0$. The proof is complete. □

Theorem 3. Assume that $p(t) \leq 0, t \geq t_0; \psi(x) \geq c > 0, f'(x)/\psi(x) \geq \alpha > 0, x \neq 0$. Suppose that there exists a differentiable function $\rho : [t_0, +\infty) \to (0, +\infty)$, such that $\rho'(t) \geq 0$. If condition (8) holds, then Iq.(1) has a nonoscillatory solution x of type A_c (i.e., $\lim_{t\to+\infty} x(t) = C \neq 0$) if and only if

$$\int_T^{+\infty} \frac{1}{a(s)\rho(s)} \left(\int_T^s \rho(\tau) \left[q(\tau) - \frac{a(\tau)}{4\alpha} \left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)} \right)^2 \right] \mathrm{d}\tau \right) \mathrm{d}s < +\infty, \tag{10}$$

for sufficiently large $T \geq t_0$.

Proof. Let x be a nonoscillatory solution of type A_c of Iq.(1). Without loss of generality, we assume that $C > 0$, hence, x is eventually plus. As in the proof of Theorem 1, $x'(t)$ is eventually minus. Noting (8), then there exists $T \geq t_0$, such that $x'(t) < 0$ and

$$\int_T^t \rho(s) \left[q(s) - \frac{a(s)}{4\alpha} \left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)} \right)^2 \right] \mathrm{d}s \geq 0,$$

for $t \geq T$. Integrating (9) from T to $t(t \geq T)$, we have

$$\begin{aligned} \rho(t)\frac{a(t)\psi(x(t))x'(t)}{f(x(t))} &\leq M - \int_T^t \rho(s) \left[q(s) - \frac{a(s)}{4\alpha} \left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)} \right)^2 \right] \mathrm{d}s \\ &\leq - \int_T^t \rho(s) \left[q(s) - \frac{a(s)}{4\alpha} \left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)} \right)^2 \right] \mathrm{d}s, \end{aligned}$$

where $M = \rho(T)a(T)\psi(x(T))x'(T)/f(x(T))$. Hence

$$\frac{\psi(x(t))x'(t)}{f(x(t))} \leq -\frac{1}{a(t)\rho(t)} \int_T^t \rho(s) \left[q(s) - \frac{a(s)}{4\alpha} \left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{ca(s)} \right)^2 \right] \mathrm{d}s.$$

Integrating the above inequality from T to t, then

$$\int_{x(T)}^{x(t)} \frac{\psi(u)}{f(u)} \mathrm{d}u \leq - \int_T^t \frac{1}{a(s)\rho(s)} \int_T^s \rho(\tau) \left[q(\tau) - \frac{a(\tau)}{4\alpha} \left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)} \right)^2 \right] \mathrm{d}\tau\, \mathrm{d}s.$$

Letting $t \to +\infty$, then

$$\begin{aligned} &\int_{x(T)}^{C} \frac{\psi(u)}{f(u)} \mathrm{d}u \\ &\leq - \int_T^{+\infty} \frac{1}{a(s)\rho(s)} \int_T^s \rho(\tau) \left[q(\tau) - \frac{a(\tau)}{4\alpha} \left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)} \right)^2 \right] \mathrm{d}\tau\, \mathrm{d}s. \end{aligned}$$

Noting that $x(T) > C > 0, \psi(u)/f(u) > 0$, we have

$$\begin{aligned} &\int_T^{+\infty} \frac{1}{a(s)\rho(s)} \int_T^s \rho(\tau) \left[q(\tau) - \frac{a(\tau)}{4\alpha} \left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)} \right)^2 \right] \mathrm{d}\tau\, \mathrm{d}s \\ &\leq \int_C^{x(T)} \frac{\psi(u)}{f(u)} \mathrm{d}u < +\infty, \end{aligned}$$

then (10) holds. For $C < 0$, the proof is similar. The proof is complete. □

Theorem 4. Assume that $p(t) \le 0, t \ge t_0; \psi(x) \ge c > 0, f'(x)/\psi(x) \ge \alpha > 0, x \ne 0$. Suppose that there exists a differentiable function $\rho : [t_0, +\infty) \to (0, +\infty)$, such that $\rho'(t) \ge 0$. If the condition (8) holds, and for arbitrarily $\varepsilon > 0$,

$$\int_0^{\varepsilon} \frac{\psi(u)}{f(u)} \,\mathrm{d}u < +\infty, \quad \int_0^{-\varepsilon} \frac{\psi(u)}{f(u)} \,\mathrm{d}u < +\infty. \tag{11}$$

Then Iq.(1) has a nonoscillatory solution x of type A_0 (i.e., $\lim_{t\to+\infty} x(t) = 0$) if and only if (10) holds for sufficiently large $T \ge t_0$.

Proof. Let x be a nonoscillatory solution of type A_0 of Iq.(1). Without loss of generality, we may assume that x is eventually plus. As in the proof of Theorem 3, we have

$$\int_{x(T)}^{x(t)} \frac{\psi(u)}{f(u)} \,\mathrm{d}u \le -\int_T^t \frac{1}{a(s)\rho(s)} \int_T^s \rho(\tau) \left[q(\tau) - \frac{a(\tau)}{4\alpha} \left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)} \right)^2 \right] \mathrm{d}\tau \,\mathrm{d}s.$$

Letting $t \to +\infty$, from $x(t) \to 0$, $x(T) > 0$ and condition (11), then

$$\int_T^{+\infty} \frac{1}{a(s)\rho(s)} \int_T^s \rho(\tau) \left[q(\tau) - \frac{a(\tau)}{4\alpha} \left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)} \right)^2 \right] \mathrm{d}\tau \,\mathrm{d}s$$
$$\le \int_0^{x(T)} \frac{\psi(u)}{f(u)} \,\mathrm{d}u < +\infty.$$

Thus (10) holds. For x is eventually minus, the proof is similar. The proof is complete. □

From the above three theorems, we obtain the following corollary.

Corollary. Assume that $p(t) \le 0, t \ge t_0; \psi(x) \ge c > 0, f'(x)/\psi(x) \ge \alpha > 0, x \ne 0$. Suppose that there exists a differentiable function $\rho : [t_0, +\infty) \to (0, +\infty)$, such that $\rho'(t) \ge 0$. If condition (8) and (11) hold, and

$$\int_T^{+\infty} \frac{1}{a(s)\rho(s)} \left(\int_T^s \rho(\tau) \left[q(\tau) - \frac{a(\tau)}{4\alpha} \left(\frac{\rho'(\tau)}{\rho(\tau)} - \frac{p(\tau)}{ca(\tau)} \right)^2 \right] \mathrm{d}\tau \right) \mathrm{d}s = +\infty, \tag{12}$$

then Iq.(1) is oscillatory.

Remark. The corollary develops and generalizes the results of [1], especially for $P(t, x(t), x'(t)) = -p(t)x'(t)$, $Q(t, x(t)) = q(t)f(x(t))$, and $\rho(t) = 1$, the corollary will be the Theorem 1 in [1].

Acknowledgements

The authors are grateful to reviewers for their comments and suggestions.

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Quanxin Zhang was made professor in 1997. He is the incumbent Dean of the Department of Mathematics in Binzhou University, he was selected as the "Excellent Well-known Teacher of Shandong Province". His current research interests include differential equations and dynamical systems.

Department of Mathematics and Information Science, Binzhou University, Shandong 256603, P.R. China.
e-mail: 3314744@163.com

Xia Song received her MS degree in Applied Mathematics from Shandong University, China, in 2009. She is now an assistant of Department of Mathematics and Information in Binzhou University, Shandong, China. Her current research interests include differential equations and dynamical systems.

Department of Mathematics and Information Science, Binzhou University, Shandong 256603, P.R. China.
e-mail: songxia119@163.com