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BISHOP'S PROPERTY (β) AND SPECTRAL INCLUSIONS ON BANACH SPACES

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Abstract. Let $T \in L(X)$, $S \in L(Y)$, $A \in L(X,Y)$ and $B \in L(Y,X)$ such that $SA = AT$, $TB = BS$, $AB = S$ and $BA = T$. Then S and T shares the same local spectral properties SVEP, Bishop's property (β), property $(\beta)_\epsilon$, property (δ) and and subscalarity. Moreover, the operators $\lambda I - T$ and $\lambda I - S$ have many basic operator properties in common.

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1. Introduction

Let X be a complex Banach space, and let $L(X)$ be the Banach algebra of all bounded linear operators on X. For an operator $T \in L(X)$, $\sigma(T)$ and $\rho(T)$ denotes the spectrum and resolvent set of T and let $Lat(T)$ stand for the collection of all $T-$invariant closed linear subspaces of X, and for $Y \in Lat(T)$, $T|Y$ denotes the restriction of T on Y. For $T \in L(X)$, we denote by

$$R_T : \lambda \in \rho(T) \to R_T(\lambda) = (T - \lambda I)^{-1} \in L(X)$$

its resolvent map. For an operator $T \in L(X)$ and arbitrary $x \in X$, we define $f : \rho(T) \to X$ by

$$f(\lambda) := R_T(\lambda)x.$$

Then f may have analytic extensions, solutions of the equation $(T-\lambda)f(\lambda) = x$. If for every $x \in X$ any two extensions of $R_T(\lambda)x$ agree on their common domain, $T \in L(X)$ is said to have the *single-valued extension property(abbreviated SVEP)*. In this case, let $\rho_T(x)$ be the maximal domain of such extensions. The

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set $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ is called the *local spectrum* of T at x. Evidently, $\sigma_T(x)$ is closed with $\sigma_T(x) \subseteq \sigma(T)$. The resolvent set $\rho(T)$ is always a subset of $\rho_T(x)$, so the analytic solutions occurring in the definition of the local resolvent set may be thought of as local extensions of the function $(T-\lambda)^{-1}x$.

An operator $T \in L(X)$ is said to have the *single-valued extension property at* λ_0, if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. In fact, the operator T has the SVEP if and only if T has SVEP at every $\lambda \in \mathbb{C}$. It is obvious that T has the SVEP if and only if the zero function is the only analytic function that satisfies $(T-\lambda)f(\lambda) = 0$. By the Liouville theorem, it is clear that T has the SVEP if and only if for any non-zero $x \in X$, we have $\sigma_T(x) \neq \phi$, see [9] and [11] for more details.

Let $\mathcal{E}(U, X)$ be the Fréchet algebra of all infinitely differentiable X-valued functions on $U \subseteq \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of U of all derivtives.

The operator $T \in L(X)$ is said to have *property* $(\beta)_\epsilon$ at $\lambda \in \mathbb{C}$ if there exists U a neighborhood of λ such that for each open set $O \subseteq U$ and for any sequence $(f_n)_n$ of $X-$valued functions in $\mathcal{E}(O, X)$ the convergence of $(T-\mu)f_n(\mu)$ to zero in $\mathcal{E}(O, X)$ yields to the convergence of f_n to zero in $\mathcal{E}(O, X)$. We define by

$$\sigma_{(\beta)_\epsilon}(T) = \{\lambda \in \mathbb{C} : T \text{ fails to satisfy } (\beta)_\epsilon \text{ at } \lambda\}.$$

We say that T satisfies *property* $(\beta)_\epsilon$ provided that $\sigma_{(\beta)_\epsilon}(T) = \phi$.

We shall also need some closely related notions. The operator T is said to satisfy Bishop's property (β) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subseteq D(\lambda, r)$ and for any sequence $(f_n)_n \subseteq \mathcal{E}(U, X)$, if $\lim_{n \longrightarrow \infty}(T-\mu)f_n(\mu) = 0$ in $\mathcal{E}(U, X)$, then $\lim_{n \longrightarrow \infty} f_n(\mu) = 0$ in $\mathcal{E}(U, X)$, where $D(\lambda, r)$ is the open disc centred at $\lambda \in \mathbb{C}$ with radius $r > 0$.

We define by

$$\sigma_\beta(T) = \{\lambda \in \mathbb{C} : T \text{ fails to satisfy}(\beta) \text{ at } \lambda\}.$$

An operator $T \in L(X)$ is said to have *Bishop's property* (β) provided that $\sigma_\beta(T) = \phi$. Obviously, property (β) implies that T has the single valued-extension property. Furthermore, the operator T is said to have the *decomposition property* (δ) if its dual T^* satisfies Bishop's property (β). In [1], Albrecht and Eschmeier proved that thr properties (β) and (δ) are dual to each other in the sense that an operator $T \in L(X)$ satisfies (β) if and only if the adjoint operator T^* on the dual space X^* satisfies (δ).

In this note, we proved that if $T \in L(X)$, $S \in L(Y)$, $A \in L(X, Y)$ and $B \in L(Y, X)$ such that $SA = AT$, $TB = BS$, $AB = S$ and $BA = T$, then S and T shares the same local spectral properties SVEP, Bishop's property (β),

property $(\beta)_\epsilon$, property (δ) and subscalarity. Moreover, the operators $\lambda I - T$ and $\lambda I - S$ have many basic operator properties in common.

2. Local spectral properties of linear operators

For $T \in L(X)$, $S \in L(Y)$ and for positive integer $n \in \mathbb{Z}$, we consider the space

$$\mathcal{I}^n(S,T) := \{A \in L(X,Y) : C(S,T)^n(A) = 0\},$$

where $C(S,T)$ is the n-th composition of the map $C(S,T)(A) := SA - AT$. The interest of the space $\mathcal{I}^n(S,T)$ stems from the fact that it contain many significant classes of maps

Example 1. If $\mathcal{A}$ and $\mathcal{B}$ are semi-simple Banach algebras and $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ an algebra homomorphism then $\theta \in I(\theta(a), a)$ for any $a \in \mathcal{A}$ in the sense that $\theta(a)\theta(x) - \theta(ax) = 0$ for all $x \in \mathcal{A}$, i.e. $\theta L_a = L_{\theta(a)}\theta$ where $L_a x := ax$ for all $x \in \mathcal{A}$.

It is easy to see that if $\mathcal{A}$ is a Banach algebra and $\mathcal{M}$ is a commutatve Banach $\mathcal{A}-$module, that is, $\mathcal{M}$ is an $\mathcal{A}-$bimodule and $ma = am$ for all $a \in \mathcal{A}, m \in \mathcal{M}$ and if $D : \mathcal{A} \longrightarrow \mathcal{M}$ is a module derivation, i.e., D is a linear map obeying the differentiation rule $D(xy) = xDY + D(x)y$ for all $x, y \in \mathcal{A}$ then $C(a,a)D \in I(a,a)$ for every $a \in \mathcal{A}$ as an easy calculation will show.

Example 2. Let $S : X \to Y$ and $R : Y \to X$ be bounded linear operators. Then $S \in I(\lambda I - SR, \lambda I - RS)$ and $R \in I(\lambda I - RS, \lambda I - SR)$ for every complex number $\lambda \in \mathbb{C}$. In particular, $S \in I(SR, RS)$ and $R \in I(RS, SR)$, since $RS \in L(X)$ and $SR \in L(Y)$.

Example 3. Let $T \in L(\mathcal{H})$ be a bounded operator on some Hilbert space $\mathcal{H}$ and $U|T|$ be its polar decomposition, where $|T| := (TT^*)^{\frac{1}{2}}$ and U is the approximate partial isometry. The generalized Aluthge transform associated with T and $s, t \geq 0$, is defined by

$$T(s,t) := |T|^s U |T|^t.$$

In the case $s = t = \frac{1}{2}$, the operator

$$\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

is called the Aluthge transform of T and was first considered by A. Aluthge, see more details [12] and [13]. It is easy to see that for all $0 \leq r \leq t$

$$|T|^s U |T|^{t-r} \in I(T(s,t), T(s+r, t-r))$$

and

$$|T|^r \in I(T(s+r, t-r), T(s,t)).$$

Proposition 2.1. *Let $T \in L(X)$, $S \in L(Y)$, $A \in L(X,Y)$ and $B \in L(Y,X)$ for which $A \in \mathcal{I}(S,T)$ and $B \in \mathcal{I}(T,S)$. Assume that $AB = S$ and $BA = T$. Then T has the Bishop's property (β) at $\lambda \in \mathbb{C}$ if and only if S has the Bishop's property (β) at $\lambda \in \mathbb{C}$. Moreover, T has the Bishop's property (β) if and only if S has the Bishop's property (β).*

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma_\beta(S)$ and let $(f_n)_n$ be a sequence of X-valued analytic functions in a neighborhood of λ such that

$$(1) \qquad \lim_{n\to\infty} (T-\mu) f_n(\mu) = 0 \quad \text{in } \mathcal{E}(V(\lambda), X).$$

Then $0 = \lim_{n\to\infty} A(T-\mu)f_n(\mu) = \lim_{n\to\infty}(S-\mu)Af_n(\mu)$ in $\mathcal{E}(V(\lambda), Y)$. It follows that

$$(2) \qquad \lim_{n\to\infty} Af_n(\mu) = 0 \quad \text{in } \mathcal{E}(V(\lambda), Y),$$

Since $\lambda \notin \sigma_\beta(S)$. From (2), we have

$$0 = \lim_{n\to\infty} BAf_n(\mu) = \lim_{n\to\infty} Tf_n(\mu) \quad \text{in } \mathcal{E}(V(\lambda), X).$$

In equation (1), we deduce that $(\mu f_n(\mu))_n$ converges to 0 on compact sets. Now since f_n are analytic, the maximum modulus priciple implies $(f_n)_n$ converges to 0 on compact sets. Thus $\lambda \in \mathbb{C} \setminus \sigma_\beta(T)$. The reverse implication is obtained by symmetry. $\square$

Corollary 2.2. *Let $T \in L(X)$, $S \in L(Y)$, $A \in L(X,Y)$ and $B \in L(Y,X)$ for which $A \in \mathcal{I}(S,T)$ and $B \in \mathcal{I}(T,S)$. Assume that $AB = S$ and $BA = T$. Then T has the SVEP at $\lambda \in \mathbb{C}$ if and only if S has the SVEP at $\lambda \in \mathbb{C}$.*

Corollary 2.3. *Let $S : X \longrightarrow Y$ and $R : Y \longrightarrow X$ be bounded linear operators. Then for every complex numbers $\lambda \in \mathbb{C}$, $RS - \lambda I$ has the SVEP if and only if $SR - \lambda I$ has SVEP. In particular, RS has the SVEP if and only if SR has SVEP.*

We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. An operator $T \in L(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$, the identity operator on X, and $\Phi(z) = T$ where z denotes the identity function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a spectral distribution for T, see [11], for more details.

An operator $T \in L(X)$ is said to be *subscalar* if T is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces. It follows from [11, Proposition 2.4.9] that all hyponormal and, more generally, all M-hyponormal operators are subscalar.

The following lemma is need.

Lemma 2.4. *Let O be an open set and $(f_n)_n$ be a sequence in $\mathcal{E}(O,X)$ such that $(\mu f_n(\mu))_n$ converges to zero in $\mathcal{E}(O,X)$. Then $(f_n)_n$ converges to zero in $\mathcal{E}(O,X)$.*

Proof. see [3, Lemma 2.1]. □

The following result generalizes [3, Theorem 2.1].

Theorem 2.5. *Let $T \in L(X)$, $S \in L(Y)$, $A \in L(X,Y)$ and $B \in L(Y,X)$ for which $A \in \mathcal{I}(S,T)$ and $B \in \mathcal{I}(T,S)$. Assume that $AB = S$ and $BA = T$. Then $\sigma_{(\beta)_\epsilon}(T) = \sigma_{(\beta)_\epsilon}(S)$. In particular, T is subscalar if and only if S is subscalar.*

Proof. Suppose that $\lambda \in \mathbb{C} \setminus \sigma_{(\beta)_\epsilon}(S)$. Then there exists O a neighborhood of λ such that $O \cap \sigma_{(\beta)_\epsilon}(S) = \phi$. If $(f_n)_n$ is any sequence in $\mathcal{E}(O,X)$ such that $(T-\mu)f_n(\mu)$ converges to zero in $\mathcal{E}(O,X)$, then

$$0 = \lim_{n\to\infty} A(T-\mu)f_n(\mu) = \lim_{n\to\infty} (S-\mu)Af_n(\mu)$$

and hence $\lim_{n\to\infty} Af_n(\mu) = 0$ in $\mathcal{E}(O,Y)$. Thus we have

$$0 = \lim_{n\to\infty} BAf_n(\mu) = \lim_{n\to\infty} Tf_n(\mu).$$

It follows that $(\mu f_n(\mu))_n$ converges to zero in $\mathcal{E}(O,X)$. By Lemma 2.4, we have $(f_n)_n$ converges to zero in $\mathcal{E}(O,X)$. Hence $\lambda \in \mathbb{C} \setminus \sigma_{(\beta)_\epsilon}(T)$. The reverse implication is obtained by symmetry. □

The following corollaries are immediate consequences of Theorem 2.5

Corollary 2.6. *Let $S : X \longrightarrow Y$ and $R : Y \longrightarrow X$ be bounded linear operators. Then $\sigma_{(\beta)_\epsilon}(RS) = \sigma_{(\beta)_\epsilon}(SR)$. In particular, RS is subscalar if and only if SR is subscalar.*

Let $T \in L(\mathcal{H})$ be a bounded operator on some Hilbert space $\mathcal{H}$ and $U|T|$ be its polar decomposition, and $U|T|$ be its polar decomposition, where $|T| := (TT^*)^{\frac{1}{2}}$ and U is the approximate partial isometry. For $r \leq t$, let $R = |T|^r$ and $S = |T|^s U|T|^{t-r}$. Then $SR = T(s,t)$ and $RS = T(s+r,t-r)$. It follows that $T(s,t)$ and $T(s+r,t-r)$ and in particular $\tilde{T}$ and T almost have the same local spectral properties.

Corollary 2.7. *Let $T \in L(\mathcal{H})$, $s \geq 0$ and $0 \leq r \leq t$. Then $T(s,t)$ has the property (β) (resp. (δ) or subscalar) if and only if $T(s+r,t-r)$ has the property (β) (resp. (δ) or subscalar).*

The following result generalizes [3, Proposition 3.1].

Proposition 2.8. *Let $T \in L(X)$, $S \in L(Y)$, $A \in L(X,Y)$ and $B \in L(Y,X)$ for which $A \in \mathcal{I}(S,T)$ and $B \in \mathcal{I}(T,S)$. Assume that $AB = S$ and $BA = T$. Then*

(a) $\sigma_S(Ax) \subseteq \sigma_T(x) \subseteq \sigma_S(Ax) \cup \{0\}$ for every $x \in X$. In particular, if A is an injective then $\sigma_T(x) = \sigma_S(Ax)$ for every $x \in X$.

(b) $\sigma_T(By) \subseteq \sigma_S(y) \subseteq \sigma_T(By) \cup \{0\}$ for every $y \in Y$. In particular, if B is an injective then $\sigma_S(y) = \sigma_T(By)$ for every $y \in Y$.

Proof. (a) Let $\lambda \notin \sigma_T(x)$ and $x(\mu)$ be an X-valued analytic function in a neighborhood O of λ such that $(T-\mu)x(\mu) = x$ for every $\mu \in O$. Then we have

$$Ax = A(T-\mu)x(\mu) = (S-\mu)Ax(\mu) \text{ for all } x \in O,$$

Since $SA = AT$, and hence $\lambda \notin \sigma_S(Ax)$. To show the second inclusion, let $\lambda \notin \sigma_S(Ax) \cup \{0\}$ and $y(\mu)$ be an Y-valued analytic function on an open neighborhood $O (0 \notin O)$ of λ, such that

$$(S-\mu)y(\mu) = Ax$$

for all $\mu \in O$. Thus $Tx = BAx = B(S-\mu)y(\mu) = (T-\mu)By(\mu)$, since $BA = T$ and $TB = BS$, and hence $T(By(\mu) - x) = \mu By(\mu)$. We define $z : O \to X$ by

$$z(\mu) := \frac{1}{\mu}(By(\mu) - x).$$

Then clearly, $x = (T-\mu)z(\mu)$ for every $\mu \in O$, and thus $\lambda \notin \sigma_T(x)$.

(b) It suffices to consider the case $\lambda = 0$, since $\sigma_{T+\lambda I}(x) = \sigma_T(x) + \lambda$ for every $\lambda \in \mathbb{C}$ and every $x \in X$. Suppose that $0 \in \sigma_S(Ax)$. Then, by (a) $\sigma_T(x) = \sigma_S(Ax)$ for every $x \in X$. Suppose that $0 \notin \sigma_S(Ax)$ and let $y(\mu)$ be an $Y-$valued analytic function in a neighborhood U of 0 such that $(S-\mu)y(\mu) = Ax$ for every $\mu \in U$. From the injectivity of A, it follows that $x = By(0)$, since $Ax = Sy(0) = ABy(0)$. Moreover, we have $\mu y(\mu) = Sy(\mu) - Ax = A(By(\mu) - x)$ and so

$$y(\mu) = A[\frac{1}{\mu}(By(\mu) - x)]$$

for every $\mu \in U \setminus \{0\}$. Set $z(\mu) := \frac{1}{\mu}(By(\mu) - x)$ if $\mu \neq 0$, and $z(\mu) := By^{'}(0)$ if $\mu = 0$. It is easily check that $A[x - (T-\mu)z(\mu)] = 0$ for every $\mu \in U$. Since A is injective, we get

$$(T-\mu)z(\mu) = x$$

for every $\mu \in U$, and hence $0 \notin \sigma_T(x)$. The theorem is hence proved. $\square$

We need the following elementary lemma.

Lemma 2.9. *Let $T \in L(X)$, $S \in L(Y)$, $A \in L(X,Y)$ and $B \in L(Y,X)$ for which $A \in \mathcal{I}(S,T)$ and $B \in \mathcal{I}(T,S)$. Assume that $AB = S$ and $BA = T$. Then*

(a) $A(Ker(I-T)) = Ker(I-S)$ *and* $B(Ker(I-S)) = Ker(I-T)$.

(b) $Ker(A) \cap Ker(I-T) = \{0\}$ *and* $Ker(B) \cap Ker(I-S) = \{0\}$.

Proof. (a) If $x \in Ker(I-T)$ then $Tx = x$, Thus $Ax = ATx = SAx$, and hence $Ax \in Ker(I-S)$. To verify the opposite inclusion, suppose $y \in Ker(I-S)$. Arguing as above, we have $B(Ker(I-S)) \subseteq Ker(I-T)$. Therefore $By \in Ker(I-T)$, and thus $y = Sy \in S(Ker(I-T))$. This proves (a).
(b) If $x \in Ker(A) \cap Ker(I-T)$, then $Ax = 0$, and $Tx = x$ and hence $x = Tx = BAx = 0$, since $BA = T$. This proves (b). $\square$

As usual, we use σ_p, σ_{sur}, σ_{com}, σ_{ap}, σ_r and σ_c to denote the *point, surjectivity, compression, approximate point, residual* and *continuous spectrum*, respectively. Thus
$\sigma_p(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\}$,
$\sigma_{sur}(T) := \{\lambda \in \mathbb{C} : (T - \lambda I)X \neq X\}$,
$\sigma_{com}(T) := \{\lambda \in \mathbb{C} : (T - \lambda I)(X) \text{ is not dense in } X\}$,
$\sigma_r(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective and } (T - \lambda I)(X) \text{ is not dense in } X\}$,
$\sigma_c(T) := \sigma(T) \setminus \{\sigma_p(T) \cup \sigma_r(T)\}$.

It is clear that $\sigma_{sur}(T)$ is compact with

$$\partial\sigma(T) \subseteq \sigma_{sur}(T) = \sigma_{ap}(T^*) \subseteq \sigma(T).$$

Furthermore, it is clear that $\sigma_{com}(T) = \sigma_p(T^*)$, $\sigma_r(T) = \sigma_{com}(T) \setminus \sigma_p(T)$ and $\sigma_c(T) = \sigma(T) \setminus \{\sigma_p(T) \cup \sigma_{com}(T)\}$, see [4] and [11] for more details.

The following result generalizes [2, Theorem 3].

Theorem 2.10. *Let $T \in L(X)$, $S \in L(Y)$, $A \in L(X,Y)$ and $B \in L(Y,X)$ for which $A \in \mathcal{I}(S,T)$ and $B \in \mathcal{I}(T,S)$. Assume that $AB = S$ and $BA = T$. Then*

(a) $\sigma_p(T) \setminus \{0\} = \sigma_p(S) \setminus \{0\}$.

(b) $\sigma_{com}(T) \setminus \{0\} = \sigma_{com}(S) \setminus \{0\}$.

(c) $\sigma_{ap}(T) \setminus \{0\} = \sigma_{ap}(S) \setminus \{0\}$.

(d) $\sigma_{sur}(T) \setminus \{0\} = \sigma_{sur}(S) \setminus \{0\}$.

(e) $\sigma_r(T) \setminus \{0\} = \sigma_r(S) \setminus \{0\}$.

(f) $\sigma(T) \setminus \{0\} = \sigma(S) \setminus \{0\}$.

(g) $\sigma_c(T) \setminus \{0\} = \sigma_c(S) \setminus \{0\}$.

Proof. (a) Since $AB = S$ and $BA = T$, we have

$$\begin{aligned}\lambda \notin \sigma_p(T) \setminus \{0\} &\Leftrightarrow \lambda = 0 \text{ or } \lambda I - T \text{ is injective}\\ &\Leftrightarrow \lambda = 0 \text{ or } I - \lambda^{-1}T \text{ is injective}\\ &\Leftrightarrow \lambda = 0 \text{ or } Ker(I - \lambda^{-1}T) = \{0\}\\ &\Leftrightarrow \lambda = 0 \text{ or } A(Ker(I - \lambda^{-1}T)) = Ker(I - \lambda^{-1}S)) = \{0\}\\ &\Leftrightarrow \lambda \notin \sigma_p(S) \setminus \{0\}.\end{aligned}$$

(b) By passing to duals, $\sigma_p(T^*) \setminus \{0\} = \sigma_p(S^*) \setminus \{0\}$. Since $\sigma_{com}(T) = \sigma_p(T^*)$, from this and (a), we obtain

$$\sigma_{com}(T) \setminus \{0\} = \sigma_p(T^*) \setminus \{0\} = \sigma_p(S^*) \setminus \{0\} = \sigma_{com}(S) \setminus \{0\}.$$

(c) Assume that $\lambda \in \sigma_{ap}(T) \setminus \{0\}$. Then there exists $\{x_n\} \subseteq X$, $\|x_n\| = 1$ for all n and $\lim_{n\to\infty} \|(\lambda I - T)x_n\| = 0$. Since $SA = AT$, we have

$$\|(\lambda I - S)Ax_n\| = \|A(\lambda I - T)x_n\| \leq \|A\|\|(\lambda I - T)x_n\|,$$

and hence $\lim_{n\to\infty} \|(\lambda I - S)Ax_n\| = 0$. In fact, $\|Ax_n\|$ is bounded away from zero. For if not, $\|Ax_{n_k}\| \to 0$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then

$$|\lambda| = |\lambda|\|x_{n_k}\| \leq \|(\lambda I - T)x_{n_k}\| + \|Tx_{n_k}\| \longrightarrow 0,$$

a contradiction. This proves $\lambda \in \sigma_{ap}(S)\setminus\{0\}$. The reverse implication is obtained by symmetry.
(d) It is clear from (c) that $\sigma_{ap}(T^*)\setminus\{0\} = \sigma_{ap}(S^*)\setminus\{0\}$. Since $\sigma_{sur}(T) = \sigma_{ap}(T^*)$ for any $T \in L(X)$, we have

$$\begin{aligned}\sigma_{sur}(T) \setminus \{0\} &= \sigma_{ap}(T^*) \setminus \{0\}\\ &= \sigma_{ap}(S^*) \setminus \{0\}\\ &= \sigma_{sur}(S) \setminus \{0\}.\end{aligned}$$

(e) It is clear from the definition $\sigma_r(T)$ that $\sigma_r(T) = \sigma_{com}(T) \setminus \sigma_p(T)$. From (a) and (b), we have

$$\begin{aligned}\sigma_r(T) \setminus \{0\} &= \sigma_{com}(T) \setminus (\sigma_p(T) \cup \{0\})\\ &= (\sigma_{com}(T) \setminus \{0\}) \setminus (\sigma_p(T) \setminus \{0\})\\ &= (\sigma_{com}(S) \setminus \{0\}) \setminus (\sigma_p(S) \setminus \{0\})\\ &= \sigma_{com}(S) \setminus (\sigma_p(S) \cup \{0\})\\ &= \sigma_r(S) \setminus \{0\}.\end{aligned}$$

(f) It is clear that $\sigma(T) = \sigma_p(T) \cup \sigma_{sur}(T)$. From (a) and (d), we have

$$\begin{aligned}\sigma(T) \setminus \{0\} &= (\sigma_p(T) \cup \sigma_{sur}(T)) \setminus \{0\} \\ &= (\sigma_p(T) \setminus \{0\}) \cup (\sigma_{sur}(T) \setminus \{0\}) \\ &= (\sigma_p(S) \setminus \{0\}) \cup (\sigma_{sur}(S) \setminus \{0\}) \\ &= (\sigma_p(S) \cup \sigma_{sur}(S)) \setminus \{0\} \\ &= \sigma(S) \setminus \{0\}.\end{aligned}$$

(g) It is clear from the definition of continuous spectrum that

$$\sigma_c(T) = \sigma(T) \setminus (\sigma_{com}(T) \cup \sigma_p(T)).$$

From (a), (b) and (f), we obtain

$$\begin{aligned}\sigma_c(T) \setminus \{0\} &= \sigma(T) \setminus (\sigma_{com}(T) \cup \sigma_p(T) \cup \{0\}) \\ &= (\sigma(T) \setminus \{0\}) \setminus ((\sigma_{com}(T) \setminus \{0\}) \cup (\sigma_p(T) \setminus \{0\})) \\ &= (\sigma(S) \setminus \{0\}) \setminus ((\sigma_{com}(S) \setminus \{0\}) \cup (\sigma_p(S) \setminus \{0\})) \\ &= \sigma(S) \setminus (\sigma_{com}(S) \cup \sigma_p(S) \cup \{0\}) \\ &= \sigma_c(S) \setminus \{0\}.\end{aligned}$$

This proves theorem. □

The following corollaries are immediate consequences of Theorem 2.5

Corollary 2.11. *Let $S \in L(X, Y)$ and $R \in L(Y, X)$. Then*

$$\sigma(SR) \setminus \{0\} = \sigma(RS) \setminus \{0\} \quad \text{and} \quad \sigma_\Omega(SR) \setminus \{0\} = \sigma_\Omega(RS) \setminus \{0\},$$

where σ_Ω denotes each one of σ, σ_p, σ_{com}, σ_{ap}, σ_{ap}, σ_{sur}, σ_r, and σ_c.

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