J. Appl. Math. & Informatics Vol. 29(2011), No. 1 - 2, pp. 417 - 426
Website: http://www.kcam.biz

EXISTENCE AND ITERATION OF MONOTONE POSITIVE SOLUTIONS FOR THIRD-ORDER THREE-POINT BVPS†

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Abstract. This paper is concerned with the existence of monotone positive solutions for a class of nonlinear third-order three-point boundary value problem. By applying iterative techniques, we not only obtain the existence of monotone positive solutions, but also establish iterative schemes for approximating the solutions. An example is also included to illustrate the importance of the results obtained.

AMS Mathematics Subject Classification : 34B10, 34B15.
Key words and phrases : third-order three-point boundary value problem, monotone positive solution, existence,iterative method.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [6].

Third-order three-point boundary value problems (BVPs for short) have been studied extensively. For example, in 2008, Guo, Sun and Zhao [7] considered the third-order three-point BVP

$$\begin{cases} u'''(t) + a(t) f(u(t)) = 0, \ t \in (0,1), \\ u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta), \end{cases} \tag{1}$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. The existence of at least one positive solution for the BVP (1) was proved when f was superlinear or sublinear. The main tool used was the well-known Guo-Krasnoselskii fixed point theorem. For other related results, one can refer to [2], [4]-[5], [8], [12]-[14], [16] and the references therein. However, almost all of the papers we mentioned focused attention on the existence of positive solutions and there are few papers concerned with

Received April 24, 2010. Revised May 31, 2010. Accepted June 24, 2010. *Corresponding author. †This work was supported by the National Natural Science Foundation of China (10801068).
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the computation of positive solutions. Recently, iterative methods have been successfully employed to prove the existence of positive solutions of nonlinear boundary value problems for ordinary differential equations, see [1], [9]-[11], [15].

In this paper, we consider the following nonlinear third-order three-point BVP

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \ t \in (0,1), \\ u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta), \end{cases} \tag{2}$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. By applying iterative methods, we not only obtain the existence of monotone positive solutions, but also establish iterative schemes for approximating the solutions. Here, monotone positive solutions mean nondecreasing, nonnegative and nontrivial solutions. Our main tool is the following theorem.

Theorem 1. *[3] Let K be a normal cone of a Banach space E and $v_0 \leq w_0$. Suppose that*

(a_1) *$T : [v_0, w_0] \to E$ is completely continuous;*

(a_2) *T is monotone increasing on $[v_0, w_0]$;*

(a_3) *v_0 is a lower solution of T, that is, $v_0 \leq Tv_0$;*

(a_4) *w_0 is an upper solution of T, that is, $Tw_0 \leq w_0$.*

Then the iterative sequences

$$v_n = Tv_{n-1} \text{ and } w_n = Tw_{n-1} \ (\ n = 1, 2, 3 \cdots)$$

satisfy

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0$$

and converge to, respectively, v and $w \in [v_0, w_0]$, which are fixed points of T.

2. Preliminary

In this section, we present several important lemmas.

Lemma 1. *[7] Let $\alpha\eta \neq 1$. Then for any $h \in C[0,1]$, the BVP*

$$\begin{cases} u'''(t) + h(t) = 0, \ t \in (0,1), \\ u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta) \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (2ts - s^2)(1-\alpha\eta) + t^2 s(\alpha - 1), & s \leq \min\{\eta, t\}, \\ t^2(1-\alpha\eta) + t^2 s(\alpha - 1), & t \leq s \leq \eta, \\ (2ts - s^2)(1-\alpha\eta) + t^2(\alpha\eta - s), & \eta \leq s \leq t, \\ t^2(1-s), & \max\{\eta, t\} \leq s \end{cases}$$

is called the Green's function.

For convenience, we define

$$g(s)=\frac{1+\alpha}{1-\alpha\eta}s(1-s),\ \ s\in[0,1].$$

Lemma 2. *[7] Let $0<\eta<1$ and $1<\alpha<\frac{1}{\eta}$. Then*

$$0\le G(t,s)\le tg(s)\ \text{ and } 0\le G_t(t,s)\le g(s)\ \text{ for } (t,s)\in[0,1]\times[0,1].$$

Lemma 3. *[7] Let $0<\eta<1$ and $1<\alpha<\frac{1}{\eta}$. Then*

$$G(t,s)\ge\gamma g(s)\ \text{ for } (t,s)\in\left[\frac{\eta}{\alpha},\eta\right]\times[0,1],$$

where $0<\gamma=\frac{\eta^2}{2\alpha^2(1+\alpha)}\min\{\alpha-1,1\}<1$.

3. Main results

In the remainder of this paper, we always assume that $0<\eta<1$ and $1<\alpha<\frac{1}{\eta}$. If we denote $\Lambda=\frac{1}{\int_0^1 g(s)ds}$, then $\Lambda>0$.

Theorem 2. *Assume that $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous, $f(t,0,0)$ is not identically zero on $[0,1]$ and there exists a constant $R>0$ such that*

$$f(t,u_1,v_1)\le f(t,u_2,v_2)\le\Lambda R,\ 0\le t\le 1,\ 0\le u_1\le u_2\le R,\ 0\le v_1\le v_2\le R, \tag{3}$$

then the BVP (2) has monotone positive solutions.

Proof. Let $E=C^1[0,1]$ be equipped with the norm

$$\|u\|=\max\left\{\max_{t\in[0,1]}|u(t)|,\max_{t\in[0,1]}|u'(t)|\right\}$$

and

$$K=\{u\in E: u(t)\ge 0 \text{ and } u'(t)\ge 0 \text{ for } t\in[0,1]\}.$$

Then K is a normal cone in Banach space E. Note that this induces an order relation $\le$ in E by defining $u\le v$ if and only if $v-u\in K$. If we define an operator $T:K\to E$ by

$$(Tu)(t)=\int_0^1 G(t,s)f(s,u(s),u'(s))ds,\ t\in[0,1],$$

then

$$(Tu)'(t)=\int_0^1 G_t(t,s)f(s,u(s),u'(s))ds,\ t\in[0,1],$$

which together with Lemma 2 implies that $T:K\to K$. Obviously, fixed points of T are monotone solutions of the BVP (2).

Let $v_0(t)=0$ and $w_0(t)=Rt$, $t\in[0,1]$. We divide our proof into the following steps:

Step 1. We verify that $T : [v_0, w_0] \to K$ is completely continuous.

First, we prove that T is a compact operator. Let D be a bounded set in $[v_0, w_0]$. We will prove that $T(D)$ is relatively compact in K.

For any $\{w_k\}_{k=1}^{\infty} \subset T(D)$, there exist $\{u_k\}_{k=1}^{\infty} \subset D$ such that $w_k = Tu_k$. Obviously, $0 \leq u_k(t) \leq R$ and $0 \leq u_k'(t) \leq R$ for $t \in [0,1]$. It follows from Lemma 2 and (3) that

$$\begin{aligned} |w_k(t)| &= |(Tu_k)(t)| \\ &= \int_0^1 G(t,s) f(s, u_k(s), u_k'(s)) ds \\ &\leq \Lambda R \int_0^1 t g(s) ds \\ &\leq R, \ t \in [0,1], \end{aligned}$$

which indicates that $\{w_k\}_{k=1}^{\infty}$ is uniformly bounded. Similarly, we have

$$\begin{aligned} |w_k'(t)| &= |(Tu_k)'(t)| \\ &= \int_0^1 G_t(t,s) f(s, u_k(s), u_k'(s)) ds \\ &\leq \Lambda R \int_0^1 g(s) ds \\ &= R, \ t \in [0,1]. \end{aligned}$$

This shows that $\{w_k'\}_{k=1}^{\infty}$ is uniformly bounded, which implies that $\{w_k\}_{k=1}^{\infty}$ is equicontinuous. By Arzela-Ascoli theorem, we know that $\{w_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Without loss of generality, we may assume that $\{w_k\}_{k=1}^{\infty}$ converges in $C[0,1]$.

On the other hand, for any $\epsilon > 0$, by the uniform continuity of $G_t(t,s)$, we know that there exists a $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, $|G_t(t_1, s) - G_t(t_2, s)| < \frac{\epsilon}{\Lambda R}$, $s \in [0,1]$. So,

$$\begin{aligned} |w_k'(t_1) - w_k'(t_2)| &= |(Tu_k)'(t_1) - (Tu_k)'(t_2)| \\ &= \left| \int_0^1 (G_t(t_1,s) - G_t(t_2,s)) f(s, u_k(s), u_k'(s)) ds \right| \\ &\leq \int_0^1 |G_t(t_1,s) - G_t(t_2,s)| \, f(s, u_k(s), u_k'(s)) ds \\ &< \epsilon, \end{aligned}$$

which shows that $\{w_k'\}_{k=1}^{\infty}$is equicontinuous. Again, it follows from Arzela-Ascoli theorem that $\{w_k'\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Therefore, $\{w_k\}_{k=1}^{\infty}$ has a convergent subsequence in K.

Next, we prove that $T : [v_0, w_0] \to K$ is continuous.

Suppose that $u_m,\ u \in [v_0, w_0]$ and $\|u_m - u\| \to 0$ $(m \to \infty)$. In view of Lemma 2 and (3), for all m, we have

$$\begin{aligned} G(t,s)f(s,u_m(s),u'_m(s)) &\leq tg(s)f(s,u_m(s),u'_m(s)) \\ &\leq \Lambda R g(s),\ (t,s) \in [0,1] \times [0,1] \end{aligned}$$

and

$$\begin{aligned} G_t(t,s)f(s,u_m(s),u'_m(s)) &\leq g(s)f(s,u_m(s),u'_m(s)) \\ &\leq \Lambda R g(s),\ (t,s) \in [0,1] \times [0,1]. \end{aligned}$$

According to Lebesgue Dominated Convergence theorem, we get that

$$\begin{aligned} \lim_{m\to\infty}(Tu_m)(t) &= \lim_{m\to\infty}\int_0^1 G(t,s)f(s,u_m(s),u'_m(s))ds \\ &= \int_0^1 G(t,s)f(s,\lim_{m\to\infty}u_m(s),\lim_{m\to\infty}u'_m(s))ds \\ &= \int_0^1 G(t,s)f(s,u(s),u'(s))ds \\ &= (Tu)(t),\ t \in [0,1] \end{aligned}$$

and

$$\begin{aligned} \lim_{m\to\infty}(Tu_m)'(t) &= \lim_{m\to\infty}\int_0^1 G_t(t,s)f(s,u_m(s),u'_m(s))ds \\ &= \int_0^1 G_t(t,s)f(s,\lim_{m\to\infty}u_m(s),\lim_{m\to\infty}u'_m(s))ds \\ &= \int_0^1 G_t(t,s)f(s,u(s),u'(s))ds \\ &= (Tu)'(t),\ t \in [0,1], \end{aligned}$$

which indicates that $T : [v_0, w_0] \to K$ is continuous.

To sum up, $T : [v_0, w_0] \to K$ is completely continuous.

Step 2. We assert that T is monotone increasing on $[v_0, w_0]$.

Suppose that $u,\ v \in [v_0, w_0]$ and $u \leq v$. Then $0 \leq u(t) \leq v(t) \leq R$ and $0 \leq u'(t) \leq v'(t) \leq R$ for $t \in [0,1]$. By (3), we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t,s)f(s,u(s),u'(s))ds \\ &\leq \int_0^1 G(t,s)f(s,v(s),v'(s))ds \\ &= (Tv)(t),\ t \in [0,1] \end{aligned}$$

and

$$\begin{aligned}(Tu)'(t) &= \int_0^1 G_t(t,s)f(s,u(s),u'(s))ds \\ &\leq \int_0^1 G_t(t,s)f(s,v(s),v'(s))ds \\ &= (Tv)'(t),\ t \in [0,1]\,,\end{aligned}$$

which shows that $Tu \leq Tv$.

Step 3. We prove that v_0 is a lower solution of T.

For any $t \in [0,1]$, we know that

$$(Tv_0)(t) = \int_0^1 G(t,s)f(s,0,0)ds \geq 0 = v_0(t)$$

and

$$(Tv_0)'(t) = \int_0^1 G_t(t,s)f(s,0,0)ds \geq 0 = v_0'(t),$$

which implies that $v_0 \leq Tv_0$.

Step 4. We show that w_0 is an upper solution of T.

It follows from Lemma 2 and (3) that

$$\begin{aligned}(Tw_0)(t) &= \int_0^1 G(t,s)f(s,w_0(s),w_0'(s))ds \\ &\leq \Lambda Rt\int_0^1 g(s)ds \\ &= w_0(t),\ t \in [0,1]\end{aligned}$$

and

$$\begin{aligned}(Tw_0)'(t) &= \int_0^1 G_t(t,s)f(s,w_0(s),w_0'(s))ds \\ &\leq \Lambda R\int_0^1 g(s)ds \\ &= w_0'(t),\ t \in [0,1],\end{aligned}$$

which indicates that $Tw_0 \leq w_0$.

Step 5. We claim that the BVP (2) has monotone positive solutions.

In fact, if we construct sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as follows:

$$v_n = Tv_{n-1} \text{ and } w_n = Tw_{n-1},\ n = 1,2,3\cdots,$$

then it follows from Theorem 1 that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0,$$

and $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ converge to, respectively, v and $w \in [v_0, w_0]$, which are monotone solutions of the BVP (2). Moreover, for any $t \in [\frac{\eta}{\alpha}, \eta]$, by Lemma

3, we know that

$$\begin{aligned}(Tv_0)(t) &= \int_0^1 G(t,s)f(s,0,0)ds \\ &\geq \gamma\int_0^1 g(s)f(s,0,0)ds \\ &> 0,\end{aligned}$$

and so,

$$0<(Tv_0)(t)\leq (Tv)(t)=v(t)\leq w(t),\ t\in[\frac{\eta}{\alpha},\eta],$$

which shows that v and w are positive solutions of the BVP (2). □

4. An example

In this section, an example is given to illustrate the main results of this paper.

Example 1. *Consider the following BVP*

$$\begin{cases} u'''(t)+t+\frac{1}{4}u^2(t)+\frac{1}{10}u'^2(t)=0,\ t\in(0,1), \\ u(0)=u'(0)=0,\ u'(1)=\frac{3}{2}u'(\frac{1}{3}). \end{cases} \tag{4}$$

Since $\alpha=\dfrac{3}{2}$ and $\eta=\dfrac{1}{3}$, a simple calculation shows that $\Lambda=\frac{6}{5}$. Thus, if we choose $R=2$, then all the conditions of Theorem 2 are fulfilled. It follows from Theorem 2 that the BVP (4) has monotone positive solutions v and w. Furthermore, if we let $v_0(t)=0$ and $w_0(t)=2t$ for $t\in[0,1]$, then for $n=0,1,2\cdots$, the two iterative schemes are

$$v_{n+1}(t)=\begin{bmatrix} \int_0^1 t^2(1-s)\left(t+\frac{1}{4}v_n^2(s)+\frac{1}{10}v_n'^2(s)\right)ds \\ -\frac{1}{2}\int_0^t (t-s)^2\left(t+\frac{1}{4}v_n^2(s)+\frac{1}{10}v_n'^2(s)\right)ds \\ -\frac{3}{2}\int_0^{\frac{1}{3}} t^2\left(\frac{1}{3}-s\right)\left(t+\frac{1}{4}v_n^2(s)+\frac{1}{10}v_n'^2(s)\right)ds \end{bmatrix},\ t\in[0,1]$$

and

$$w_{n+1}(t)=\begin{bmatrix} \int_0^1 t^2(1-s)\left(t+\frac{1}{4}w_n^2(s)+\frac{1}{10}w_n'^2(s)\right)ds \\ -\frac{1}{2}\int_0^t (t-s)^2\left(t+\frac{1}{4}w_n^2(s)+\frac{1}{10}w_n'^2(s)\right)ds \\ -\frac{3}{2}\int_0^{\frac{1}{3}} t^2\left(\frac{1}{3}-s\right)\left(t+\frac{1}{4}w_n^2(s)+\frac{1}{10}w_n'^2(s)\right)ds \end{bmatrix},\ t\in[0,1].$$

The first, second, third and fourth terms of the two schemes are as follows:

$$v_0(t)=0,$$

$$v_1(t)=\frac{5}{12}t^3-\frac{1}{6}t^4,$$

$$v_2(t)=\frac{5}{12}t^3-\frac{1}{6}t^4+\frac{25}{384}t^6-\frac{55}{576}t^7+\frac{445}{6912}t^8-\frac{503}{17280}t^9+\frac{5}{576}t^{10}-\frac{1}{864}t^{11},$$

$$v_3(t) = \frac{5}{12}t^3 - \frac{1}{6}t^4 + \frac{25}{384}t^6 - \frac{55}{576}t^7 + \frac{445}{6912}t^8 + \frac{1601}{138240}t^9 - \frac{2735}{27648}t^{10} + \frac{4859}{36864}t^{11} - \frac{785663}{7962624}t^{12} + \frac{57697}{1658880}t^{13} + \frac{10942943}{530841600}t^{14} - \frac{4395781}{95551488}t^{15} + \frac{25402181}{573308928}t^{16} - \frac{43121441}{1433272320}t^{17} + \frac{898651277}{57330892800}t^{18} - \frac{305268569}{47775744000}t^{19} + \frac{1937567}{955514880}t^{20} - \frac{3498011}{7166361600}t^{21} + \frac{12287}{143327232}t^{22} - \frac{1283}{119439360}t^{23} + \frac{35}{35831808}t^{24} - \frac{1}{17915904}t^{25};$$

$$w_0(t) = 2t,$$

$$w_1(t) = \frac{1}{6}t^2 + \frac{7}{20}t^3 + \frac{1}{4}t^4 - \frac{1}{6}t^5,$$

$$w_2(t) = \frac{5}{12}t^3 - \frac{35}{216}t^4 + \frac{59}{2160}t^5 + \frac{5611}{86400}t^6 + \frac{1651}{36000}t^7 - \frac{6983}{172800}t^8 - \frac{2293}{43200}t^9 + \frac{59}{1280}t^{10} - \frac{311}{17280}t^{11} + \frac{11}{1728}t^{12} - \frac{1}{864}t^{13},$$

$$w_3(t) = \frac{5}{12}t^3 - \frac{1}{6}t^4 + \frac{25}{384}t^6 - \frac{485}{5184}t^7 + \frac{21505}{279936}t^8 - \frac{8729}{11197440}t^9 + \frac{396353}{37324800}t^{10} - \frac{27532421}{559872000}t^{11} - \frac{166052891}{44789760000}t^{12} + \frac{147064543}{1749600000}t^{13} - \frac{161335996543}{1791590400000}t^{14} + \frac{667371689963}{22394880000000}t^{15} - \frac{67365024401}{8957952000000}t^{16} + \frac{20454301621}{895795200000}t^{17} - \frac{102742235323}{7166361600000}t^{18} - \frac{3316750297}{398131200000}t^{19} + \frac{13173971261}{716636160000}t^{20} - \frac{2900165131}{179159040000}t^{21} + \frac{558586477}{57330892800}t^{22} - \frac{122701391}{28665446400}t^{23} + \frac{20462977}{14332723200}t^{24} - \frac{2654267}{7166361600}t^{25} + \frac{16793}{238878720}t^{26} - \frac{1043}{119439360}t^{27} + \frac{1}{1327104}t^{28} - \frac{1}{17915904}t^{29}.$$

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