J. Appl. Math. & Informatics Vol. **29**(2011), No. 1 - 2, pp. 369-380
Website: http://www.kcam.biz

SOLVABILITY OF LUIKOV'S SYSTEM OF HEAT AND MASS DIFFUSION IN ONE-DIMENSIONAL CASE

LAZHAR BOUGOFFA* AND HIND K. AL-JEAID

Abstract. This paper studies a boundary value problem for a linear coupled Luikov's system of heat and mass diffusion in one-dimensional case. Using an a priori estimate, we prove the uniqueness of the solution. Also, some traveling wave solutions and explicit solutions are obtained by using the transformation $\xi = x - ct$ and separation method respectively.

AMS Mathematics Subject Classification : 35K40, 35K45, 35K50.
Key words and phrases : Coupled system, Luikov's system, A priori estimate, traveling solutions.

1. Introduction

Consider the following linear coupled Luikov's system of heat and mass diffusion in one-dimensional case [1] and [2]

$$\begin{cases} \frac{\partial w}{\partial t} - \alpha \frac{\partial^2 w}{\partial x^2} - \beta \frac{\partial \theta}{\partial t} & = & 0, \\ \frac{\partial \theta}{\partial t} - \gamma \frac{\partial^2 \theta}{\partial x^2} - \delta \frac{\partial^2 w}{\partial x^2} & = & 0, \end{cases} \tag{1}$$

where w is the temperature, θ is the moisture potential, t the time and $\alpha,\ \beta,\ \gamma$ and δ are assumed to be positive constants. Various initial and boundary value problems associated with this system have been investigated in [2]. In [3] Luikov and Mikhailov used the Laplace transform technique to obtain their solutions. These same problems were also treated by Mikhailov and Özisik [2] using the finite integral transform technique. They obtained the same solutions as those of Luikov and Mikhailov. Also, Liu and Cheng [4] developed an analytical approach to obtain complete and satisfactory solutions of these equations subject to specified initial and boundary conditions and numerical results are compared with the solutions obtained by Thomas [5] and Keylwerth [6].
One of the main problem of mathematics appears when $\alpha(t, x),\ \beta(t, x),\ \gamma(t, x)$

Received November 23, 2009. Accepted June 30, 2010. *Corresponding author.

and $\delta(t,x)$ are analytic functions and are added to the system with a heat supply. The new problem, incorporating the above functions, is

$$\begin{cases} \frac{\partial w}{\partial t} - \alpha(t,x)\frac{\partial^2 w}{\partial x^2} - \beta(t,x)\frac{\partial \theta}{\partial t} &= f(t,x), \\ \frac{\partial \theta}{\partial t} - \gamma(t,x)\frac{\partial^2 \theta}{\partial x^2} - \delta(t,x)\frac{\partial^2 w}{\partial x^2} &= g(t,x). \end{cases} \tag{2}$$

A question which arises naturally is under what conditions on the functions $\alpha(t,x)$, $\beta(t,x)$, $\gamma(t,x)$ and $\delta(t,x)$ does the problem have solutions?
An analytical method to obtain explicit solutions of these equations is still lacking in the literature.
In the bounded domain

$$\Omega = \{(t,\ x):\ 0 \le t \le T,\ 0 \le x \le 1\},$$

we attach the following initial conditions

$$\begin{cases} w(0,x) &= w_0(x),\ 0 < x < 1, \\ \theta(0,x) &= \theta_0(x),\ 0 < x < 1 \end{cases} \tag{3}$$

and the boundary conditions

$$\begin{cases} \frac{\partial w}{\partial x} &= 0,\ 0 < t \le T \\ \\ \frac{\partial \theta}{\partial x} &= 0,\ 0 < t \le T. \end{cases} \tag{4}$$

We assume that the following requirements:

$$\begin{cases} \alpha_1 \le \alpha(t,x) \le \alpha_2, \\ \beta_1 \le \beta(t,x) \le \beta_2, \\ \gamma_1 \le \gamma(t,x) \le \gamma_1, \\ \delta_1 \le \delta(t,x) \le \delta_2, \\ \frac{\partial \alpha}{\partial t} \le -\alpha_3 < 0, \\ \frac{\partial \gamma}{\partial t} \le -\gamma_3 < 0, \end{cases} \tag{5}$$

$\forall (t,x) \in \Omega$.
Here, we shall prove a result on the uniqueness of the solutions for the given system with initial and boundary conditions by using an a priori estimate. Some traveling wave solutions to systems (2) are obtained. The basic key of this method is to transform (2) into integrable systems of ODEs by using the transformation $\xi = x - ct$. Also, explicit solutions are presented in the form of series by separation method.

2. Preliminaries

We reformulate the given system (2)-(4) as the problem of solving the operator equation

$$LU = F,$$

where $U,\ LU$ and F are respectively the pairs:

$$U = (w,\ \theta),$$

$$LU = (L_1,\ L_2)$$

and

$$F = (F_1,\ F_2),$$

where

$$L_1 = \{\ell_1(w,\theta),\ l_o w\},$$

$$L_2 = \{\ell_2(w,\theta),\ l_1\theta\},$$

$$\ell_1(w,\ \theta) = \frac{\partial w}{\partial t} - \alpha(t,x)\frac{\partial^2 w}{\partial x^2} - \beta(t,x)\frac{\partial \theta}{\partial t},$$

$$\ell_2(w,\theta) = \frac{\partial \theta}{\partial t} - \gamma(t,x)\frac{\partial^2 \theta}{\partial x^2} - \delta(t,x)\frac{\partial^2 w}{\partial x^2},$$

$$l_o w = w_0(x),\ l_1\theta = \theta_0(x)$$

and

$$F_1 = \{f, w_0\},\ F_2 = \{g,\ \theta_0\}.$$

The operator L is considered from a space $E = E_1 \times E_2$ into the space $E^* = E_1^* \times E_2^*$, where E is a Banach space consisting of all functions $(w,\ \theta) \in L_2(\Omega) \times L_2(\Omega)$ satisfying conditions (3)-(4) and having the finite norm

$$\| U \|_E^2 = \sup_{0\le\tau\le T}\left[\int_0^1 \left[w^2 + \theta^2 + \theta_x^2 + w_x^2\right](\tau,x)dx\right] + \int_\Omega \left[w_t^2 + w_x^2 + \theta_t^2 + \theta_x^2\right] dtdx + \int_\Omega \left[w_{xx}^2 + \theta_{xx}^2\right] dtdx, \tag{6}$$

and $E^* = E_1^* \times E_2^*$ is the completion of the Hilbert space $\left\{L_2(\Omega) \times H^{1,2}(0,\ 1) \times L_2(\Omega)\right\} \times \left\{L_2(\Omega) \times H^{1,2}(0,\ 1)\right\}$ with respect to the norm

$$\| F \|_{E^*}^2 = \| f \|_{L_2(\Omega)}^2 + \| g \|_{L_2(\Omega)}^2 + \| w_0 \|_{H^1(0,\ 1)}^2 + \| \theta_0 \|_{H^1(0,\ 1)}^2 .$$

3. A priori estimate

Here we establish an a priori estimate which ensures the uniqueness of the solution of the boundary value problem of the coupled system (2)-(4). For that, we need the following.

Lemma 1. *Let $(w(t,x),\ \theta(t,x))$ be a solution of problem (2)-(4, then*

$$\int_0^1 w^2(\tau,x)dx \leq \eta_1 \left(\int_{\Omega^\tau} w_t^2(t,x)dtdx + \int_0^1 w_0^2(x)dx \right), \tag{7}$$

$$\int_0^1 \theta^2(\tau,x)dx \leq \eta_2 \left(\int_{\Omega^\tau} \theta_t^2(t,x)dtdx + \int_0^1 \theta_0^2(x)dx \right), \tag{8}$$

$$\int_{\Omega^\tau} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dtdx \leq \eta_3 \int_{\Omega^\tau} \left[w_t^2(t,x) + \theta_t^2(t,x) + f^2(t,x) \right] dtdx, \tag{9}$$

$$\int_{\Omega^\tau} \left(\frac{\partial^2 \theta}{\partial x^2} \right)^2 dtdx \leq \eta_4 \int_{\Omega^\tau} \left[\theta_t^2(t,x) + w_{xx}^2(t,x) + g^2(t,x) \right] dtdx, \tag{10}$$

where $\Omega^\tau = (0,\ \tau) \times (0,\ 1),\ \eta_i > 0,\ i = 1, ..., 4$ and $\eta_3 = 3\max\left(\frac{1}{\alpha^2},\ \frac{\beta^2}{\alpha^2}\right)$ and $\eta_4 = 3\max\left(\frac{1}{\gamma^2},\ \frac{\delta^2}{\gamma^2}\right)$.

Proof. To prove (7), it is easy to observe that

$$\int_{\Omega^\tau} (w^2)_t dt = 2\int_{\Omega^\tau} w_t w dt, \tag{11}$$

it follows after applying the $\overline{\gamma_1} -$ inequality : $2 \mid ab \mid \leq \overline{\gamma_1} a^2 + \frac{1}{\overline{\gamma_1}} b^2,\ \overline{\gamma_1} > 0$ to the right hand-side of this equality

$$\int_0^1 w^2(\tau,\ x)dx - \int_0^1 w^2(0,\ x)dx \leq \frac{1}{\overline{\gamma_1}} \int_{\Omega^\tau} w^2(t,\ x)dtdx + \overline{\gamma_1} \int_{\Omega^\tau} w_t^2(t,\ x)dtdx, \tag{12}$$

and using Gronwall's inequality to this inequality we obtain (7).
By the same method we prove (8).
Now from the second equation of (2), we see that

$$\alpha^2 \left[\frac{\partial w}{\partial x} \right]^2 (t,x) = \left[w_t(t,x) - \beta\theta_t(t,x) + f(t,x) \right]^2, \tag{13}$$

by integration over Ω^τ and using the fact that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we obtain the desired inequality (9). □

By the same method we prove (10). The following theorem shows that the solution of (2)-(4) is unique.

Theorem 1. *For any any solution $U = (w(t,x),\ \theta(t,x)) \in E$ of (2)-(4) there exists a positive constant C independent on U such that*

$$\parallel U \parallel_E \leq C \parallel LU \parallel_{E^*}. \tag{14}$$

Proof. Firstly, consider the scalar product $(\ell_1(w,\ \theta),\ w_t)_{L_2(\Omega^\tau)}$. Employing integration by parts, and taking into account initial and boundary conditions (3)-(4), we obtain

$$2\left(\ell_1(w,\ \theta),\ w_t\right)_{L_2(\Omega^\tau} = 2\int_{\Omega^\tau} w_t^2 dtdx + 2\int_0^1 \alpha_x w_t w_x(\tau,x)dx + \int_0^1 \alpha w_x^2(\tau,x)dx - \int_0^1 \alpha(0,x)w_0(x)^2 dx - \int_{\Omega^\tau} \alpha_t w_x^2 dtdx - 2\left(\beta\theta_t, w_t\right)_{L_2(\Omega^\tau)}. \quad (15)$$

Similarly, for the scalar product $(\ell_2(w,\ \theta),\ \theta_t)_{L_2(\Omega^\tau)}$, we obtain

$$2\left(\ell_2(w,\ \theta),\ \theta_t\right)_{L_2(\Omega^\tau} = 2\int_{\Omega^\tau} \theta_t^2 dtdx + 2\int_0^1 \gamma_x \theta_t \theta_x(\tau,x)dx + \int_0^1 \gamma\theta_x^2(\tau,x)dx - \int_0^1 \gamma(0,x)\theta_0'(x)^2 dx - \int_{\Omega^\tau} \gamma_t \theta_x^2 dtdx - 2\left(\gamma w_{xx}, \theta_t\right)_{L_2(\Omega^\tau)}. \quad (16)$$

Adding side to side (15) and (16) we obtain

$$2\int_{\Omega^\tau}\left[w_t^2+\theta_t^2\right]dtdx + \int_0^1\left[\alpha w_x^2+\gamma\theta_x^2\right](\tau,x)dx - \int_{\Omega^\tau}\alpha_t w_x^2 dtdx - \int_{\Omega^\tau}\gamma_t\theta_x^2 dtdx = 2\left(\ell_1, w_t\right)_{L_2(\Omega^\tau)} + 2\left(\ell_2,\theta_t\right)_{L_2(\Omega^\tau)} + \int_0^1 \alpha(0,x)w_0'(x)^2 dx + \int_0^1 \gamma(0,x)\theta_0'(x)^2 dx + 2\left(\beta\theta_t, w_t\right)_{L_2(\Omega^\tau)} + 2\left(\gamma w_{xx},\theta_t\right)_{L_2(\Omega^\tau)}. \quad (17)$$

Using $\varepsilon-$ inequality to estimate the following terms which arise in the right-hand side and the left-hand side of (17) as follows:

$$2\left(\ell_1(w,\theta),\ w_t\right)_{L_2(\Omega^\tau)} \le \frac{1}{\varepsilon_1}\int_{\Omega^\tau} f^2 dtdx + \varepsilon_1\int_{\Omega^\tau} w_t^2 dtdx, \quad (18)$$

$$2\left(\ell_2(w,\theta),\ \theta_t\right)_{L_2(\Omega^\tau)} \le \frac{1}{\varepsilon_2}\int_{\Omega^\tau} g^2 dtdx + \varepsilon_2\int_{\Omega^\tau} \theta_t^2 dtdx, \quad (19)$$

$$2\left(\beta\theta_t, w_t\right)_{L_2(\Omega^\tau)} \le \frac{1}{\varepsilon_3}\int_{\Omega^\tau} \beta\theta_t^2 dtdx + \varepsilon_3\int_{\Omega^\tau} \beta w_t^2 dtdx, \quad (20)$$

$$2\left(\gamma w_{xx},\theta_t\right)_{L_2(\Omega^\tau)} \le \frac{1}{\varepsilon_4}\int_{\Omega^\tau} \gamma w_{xx}^2 dtdx + \varepsilon_4\int_{\Omega^\tau} \gamma\theta_t^2 dtdx. \quad (21)$$

where $\varepsilon_i > 0,\ i = 1, ..., 4.$
From (17), (18)-(21) and (7)-(10),and using the above conditions (5), we get

$$\begin{aligned}
2\int_{\Omega^\tau}\left[w_t^2+\theta_t^2\right]dtdx &+ \alpha_1\int_0^1 w_x^2(\tau,x)dx+\gamma_1\int_0^1\theta_x^2(\tau,x)dx\\
&+\alpha_3\int_{\Omega^\tau} w_x^2(\tau,x)dx+\gamma_3\int_{\Omega^\tau}\theta_x^2dtdx+\int_0^1 w^2(\tau,x)dx+\int_0^1\theta^2(\tau,x)dx\\
&+\int_{\Omega^\tau} w_{xx}^2dtdx+\int_{\Omega^\tau}\theta_{xx}^2dtdx\\
\le \frac{1}{\varepsilon_1}\int_{\Omega^\tau} f^2dtdx &+ \varepsilon_1\int_{\Omega^\tau} w_t^2dtdx+\frac{1}{\varepsilon_2}\int_{\Omega^\tau} g^2dtdx\\
&+\varepsilon_2\int_{\Omega^\tau}\theta_t^2dtdx+\alpha_2\int_0^1\theta_0'(x)^2dx+\gamma_2\int_0^1 w_0'(x)^2dx\\
&+\frac{\beta_2}{\varepsilon_3}\int_{\Omega^\tau}\theta_t^2dtdx+\beta_2\varepsilon_3\int_{\Omega^\tau} w_t^2dtdx+\frac{\delta_2}{\varepsilon_4}\int_{\Omega^\tau} w_{xx}^2dtdx\\
&+\varepsilon_4\delta_2\int_{\Omega^\tau}\theta_t^2dtdx+\eta_1\int_{\Omega^\tau} w_t^2dtdx\\
&+\eta_1\int_0^1 w_0^2(x)dx+\eta_2\int_{\Omega^\tau}\theta_t^2dtdx+\eta_2\int_0^1\theta_0^2(x)dx\\
&+\eta_3\int_{\Omega^\tau} f^2dtdx+\eta_3\int_{\Omega^\tau}\theta_t^2dtdx+\eta_3\int_{\Omega^\tau} w_t^2dtdx\\
&+\eta_4\int_{\Omega^\tau} g^2dtdx+\eta_4\int_{\Omega^\tau}\theta_t^2dtdx+\eta_4\int_{\Omega^\tau} w_{xx}^2dtdx. \qquad (22)
\end{aligned}$$

Thus

$$\begin{aligned}
k_1\int_{\Omega^\tau} w_t^2dtdx &+ k_2\int_{\Omega^\tau}\theta_t^2dtdx+\alpha_3\int_{\Omega^\tau} w_x^2(\tau,x)dx+\gamma_3\int_{\Omega^\tau}\theta_x^2dtdx\\
&+\alpha_1\int_0^1 w_x^2(\tau,x)dx+\gamma_1\int_0^1\theta_x^2(\tau,x)dx\\
&+\int_0^1 w^2(\tau,x)dx+\int_0^1\theta^2(\tau,x)dx+k_3\int_{\Omega^\tau} w_{xx}^2dtdx+\int_{\Omega^\tau}\theta_t^2dtdx\\
&\le k_4\int_{\Omega^\tau} f^2dtdx+k_5\int_{\Omega^\tau} g^2dtdx+\eta_1\int_0^1 w_0^2(x)dx\\
&+\gamma_2\int_0^1 w_0'(x)^2dx+\eta_2\int_0^1\theta_0^2(x)dx+\alpha_2\int_0^1\theta_0'(x)^2dx. \qquad (23)
\end{aligned}$$

where $k_1 = 2-\varepsilon_1-\beta_2\varepsilon_3-\eta_1-\eta_3,\ k_2 = 2-\varepsilon_2-\frac{\beta_2}{\varepsilon_3}-\varepsilon_4\delta_2-\eta_2-\eta_3-\eta_4,$ $k_3 = 1-\frac{\delta_2}{\varepsilon_2}-\eta_4,\ k_4 = \frac{1}{\varepsilon_1}+\eta_3$ and $k_5 = \frac{1}{\varepsilon_2}+\eta_4.$

Therefore,

$$\int_0^1 \left[w^2+\theta^2+\theta_x^2+w_x^2\right](\tau,x)dx+\int_{\Omega^\tau}\left[w_t^2+w_x^2+\theta_t^2+\theta_x^2\right]dtdx$$
$$+\int_{\Omega^\tau}\left[w_{xx}^2+\theta_{xx}^2\right]dtdx$$
$$\leq C\left[\int_{\Omega^\tau}(f^2+g^2)dtdx+\int_0^1\left(w_0^2(x)+w_0'(x)^2\right)dx+\int_0^1\left(\theta_0^2(x)+\theta_0'(x)^2\right)dx\right] \tag{24}$$

where $C=\frac{\min(1,\ k_1,\ k_2,\ k_3,\ \alpha_3,\ \gamma_3,\ \delta_1)}{\max(1,k_4,\ k_5,\ \alpha_2,\ \eta_1,\ \eta_2)}$.
Now, replacing the right-hand side of (24) by its upper bound with respect τ in the interval $(0,\ T)$, we obtain the desired inequality.
This completes the proof. □

Corollary 1. *There exists a unique solution $U=(w,\ \theta)\ \in E=E_1\times E_2$ to (2)-(4).*

4. Traveling wave solutions

4.1.Solvability of system with constant coefficients. Consider the coupled nonhomogeneous system

$$\begin{cases} \frac{\partial w}{\partial t}-\alpha\frac{\partial^2 w}{\partial x^2}-\beta\frac{\partial\theta}{\partial t} &= f(t,x),\\ \\ \frac{\partial\theta}{\partial t}-\gamma\frac{\partial^2\theta}{\partial x^2}-\delta\frac{\partial^2 w}{\partial x^2} &= g(t,x).\end{cases} \tag{25}$$

Introduce the similarity variable $\xi=x-ct$, where the frequency c is required to be non-zero. Then system (25) can be transformed into the linear system of ordinary differential equations

$$\begin{cases} -cw'-\alpha w''+c\beta\theta' &= f(\xi),\\ -c\theta'-\gamma\theta''-\delta w'' &= g(\xi).\end{cases} \tag{26}$$

From the first equation of system (26), we get

$$\theta'=\frac{1}{c\beta}f(\xi)+\frac{1}{\beta}w'+\frac{\alpha}{c\beta}w''. \tag{27}$$

It follows that,

$$\theta=\frac{1}{c\beta}\int f(\xi)d\xi+\frac{1}{\beta}w+\frac{\alpha}{c\beta}w'+c_1. \tag{28}$$

Integrating the second equation of system (26), we get

$$-c\theta-\gamma\theta'-\delta w'=\int g(\xi)d\xi+c_2 \tag{29}$$

Substituting (27) and (28) into (29), we get

$$w''+r_1w'+r_2w=F(\xi), \tag{30}$$

where

$$F(\xi) = \frac{-c}{\alpha\gamma}\left[\beta \int g(\xi)d\xi + \int f(\xi)d\xi + \frac{\gamma}{c} f(\xi) + \beta\left(cc_1 + c_2\right)\right]$$

and

$$r_1 = \frac{c}{\alpha\gamma}\left[\alpha + \gamma + \beta\delta\right], \; r_2 = \frac{c^2}{\alpha\gamma}.$$

If $r_1^2 - 4r_2 \geq 0$, then the general solution of (30) is

$$\begin{aligned} w = b_1 e^{\lambda_1\xi} + b_2 e^{\lambda_2\xi} + \left[\int \frac{-e^{\lambda_2\xi}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)\xi}} F(\xi)d\xi\right] e^{\lambda_1\xi} \\ + \left[\int \frac{e^{\lambda_1\xi}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)\xi}} F(\xi)d\xi\right] e^{\lambda_2\xi}, \end{aligned} \tag{31}$$

where $\lambda_i = \frac{-r_1 \pm \sqrt{r_1^2 - 4r_2}}{2}$ and $b_i, \; i = 1,2$ are arbitrary constants.
Thus, from Eq.(28), we obtain

$$\begin{aligned} \theta = \frac{1}{c\beta}\int f(\xi)d\xi + \frac{1}{\beta}\left(1+\frac{\alpha\lambda_1}{c}\right)e^{\lambda_1\xi}\int \frac{-e^{\lambda_2\xi}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)\xi}}F(\xi)d\xi \\ +\frac{1}{\beta}\left(1+\frac{\alpha\lambda_2}{c}\right)e^{\lambda_2\xi}\int \frac{e^{\lambda_1\xi}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)\xi}}F(\xi)d\xi \\ +\frac{b_1}{\beta}\left(1+\frac{\alpha\lambda_1}{c}\right)e^{\lambda_1\xi} + \frac{b_2}{\beta}\left(1+\frac{\alpha\lambda_2}{c}\right)e^{\lambda_2\xi} + c_1. \end{aligned} \tag{32}$$

By back transformation, we get $w(t,x)$ and $\theta(t,x)$.

4.2.Solvability of system with variable coefficients. Consider the coupled system

$$\begin{cases} \frac{\partial w}{\partial t} - \alpha(t,x)\frac{\partial^2 w}{\partial x^2} - \beta(t,x)\frac{\partial\theta}{\partial t} &= 0, \\ \frac{\partial\theta}{\partial t} - \gamma(t,x)\frac{\partial^2\theta}{\partial x^2} - \delta(t,x)\frac{\partial^2 w}{\partial x^2} &= 0 \end{cases} \tag{33}$$

and introduce the similarity variable $\xi = x - ct, \; u(t,x) = u(\xi), \; \theta(t,x) = \theta(\xi), \; \beta(t,x) = \beta(\xi), \; \gamma(t,x) = \gamma(\xi)$ and $\delta(t,x) = \delta(\xi)$. Then system (33) can be transformed into the linear system of ordinary differential equations with variable coefficients

$$\begin{cases} cw' + \alpha(\xi)w'' - c\beta(\xi)\theta' &= 0, \\ c\theta' + \gamma(\xi)\theta'' + \delta(\xi)w'' &= 0. \end{cases} \tag{34}$$

We shall prove a result and discuss the conditions which govern the separation of system of two coupled equations (34).
Consider now the following transformation

$$\theta' = \Phi(\xi)w', \tag{35}$$

where $\Phi(\xi)$ is unknown function.
The substitution of Eq.(35) into system (34) gives a system which we write as

$$\begin{cases} \alpha(\xi)w'' + c(1-\beta(\xi)\Phi(\xi))w' &= 0, \\ [\gamma(\xi)\Phi(\xi)+\delta(\xi)]\,w'' + [c\Phi(\xi)+\gamma(\xi)\Phi'(\xi)]\,w' &= 0. \end{cases} \tag{36}$$

Now, if we multiply the second equation of system (36) by $\lambda \in \Re^*$ and equate the coefficients of w' and w'' on the first equation of system (36) and the resulting equation, we get

$$\Phi(\xi) = \frac{\alpha(\xi) - \lambda\delta(\xi)}{\lambda\gamma(\xi)} \tag{37}$$

and

$$\lambda\gamma(\xi)\Phi'(\xi) + c\,[(\lambda+\beta(\xi))\,\Phi(\xi) - 1] = 0. \tag{38}$$

Therefore, the coupled system is also separated and reduced to the following equation:

$$\alpha(\xi)w'' + c\,[1-\beta(\xi)\Phi(\xi)]\,w' = 0, \tag{39}$$

which has infinite solutions depend on λ.
Substituting $w' = z$ into Eq.(39) we obtain.

$$\alpha(\xi)z' + c\,(1-\beta(\xi)\Phi(\xi))\,z = 0. \tag{40}$$

The general solution of Eq.(40) is

$$z = c_1 \exp\left[\int \frac{c(1-\beta(\xi)\Phi(\xi))}{\alpha(\xi)} d\xi\right]. \tag{41}$$

Then,

$$w = c_1 \int \exp\left[\int \frac{c(1-\beta(\xi)\Phi(\xi))}{\alpha(\xi)} d\xi\right] d\xi + c_2. \tag{42}$$

Thus, in view of (35), we get

$$\theta = c_1 \int \Phi(\xi) \exp\left[\int \frac{c(1-\beta(\xi)\Phi(\xi))}{\alpha(\xi)} d\xi\right] d\xi + c_3, \tag{43}$$

where $c_i,\ i = 1, ..., 3$ are arbitrary constants.
Thus we have proved the following result which governs the separation of system (33).

Lemma 2. *The system (33) can always be decoupled and solved without increase of the differential equations by the transformation $\theta' = \Phi(\xi)w'$ if the conditions (37)-(38) are satisfied.*
Furthermore, the general solutions of (33) can be expressed by (42) and (43).

Remark 1. *In order to find the particular solutions to systems (25) and (33) we use the initial and boundary conditions (3)-(4) to find the constants of integration $c_i,\ i = 1, ..., 3$.*

We will take an example on system (33) as an illustration of this method.

Example 1. *If we choose in system (33)* $\alpha(t,x) = c\left[1 - e^{x-ct}\right]$, $\beta(t,x) = e^{-(x-ct)}$, $\delta = c$ *and* $\gamma(t,x) = -c$.
Then, the coupled system (33) can be obtained by the following similarity variable $\xi = x - ct$.
A straightforward computation, with $\lambda = 1$, *yields* $\Phi(\xi) = \frac{c\left(1-e^{\xi}\right)-c}{-c} = e^{\xi}$ *and we see that the conditions of Lemma 2 are fulfilled. Then, we get the following solutions*

$$w(\xi) = c_1\xi + c_2 \tag{44}$$

and

$$\theta(\xi) = c_1 e^{\xi} + c_3. \tag{45}$$

By back transformation, we get

$$w(t,x) = c_1(x - ct) + c_2 \tag{46}$$

and

$$\theta(t,x) = c_1 e^{x-ct} + c_3. \tag{47}$$

5. Separation method

We shall find the solutions to system (33) by separation method. For this we rewrite system (33) as

$$\begin{cases} \frac{\partial w}{\partial t} - a(t,x)\frac{\partial^2 w}{\partial x^2} - b(t,x)\frac{\partial^2 \theta}{\partial x^2} &= 0, \\ \frac{\partial \theta}{\partial t} - \gamma(t,x)\frac{\partial^2 \theta}{\partial x^2} - \delta(t,x)\frac{\partial^2 w}{\partial x^2} &= 0, \end{cases} \tag{48}$$

where $a(t,x) = \alpha(t,x) + \beta(t,x)\delta(t,x)$ and $b(t,x) = \beta(t,x)\gamma(t,x)$.
We first begin with the following result on the separation of this system in which the two equations of (48) are decoupled.

Lemma 3. *The system (48) can always be decoupled if the functions* $a(t,x) - \gamma(t,x)$ *and* $b(t,x)$ *are proportional to the function* $\delta(t,x)$.

Proof. Multiplying second equation of (48) by κ and adding side to side the resulting equation and the first equation of (48) we get

$$w_t + \kappa\theta_t = (a + \kappa\delta)\left(w_{xx} + \frac{b + \kappa\gamma}{a + \kappa\delta}\theta_{xx}\right). \tag{49}$$

Now, if we choose $\kappa = \frac{b+\kappa\gamma}{a+\kappa\delta}$, then

$$\kappa^2 + \frac{a-\gamma}{\delta}\kappa - \frac{b}{\delta} = 0. \tag{50}$$

This condition means that κ is independent on t and x such that

$$\kappa_i = \frac{\frac{\gamma-a}{\delta} \pm \sqrt{(\frac{a-\gamma}{\delta})^2 + 4\frac{b}{\delta}}}{2}, \; i = 1, 2. \tag{51}$$

It may be shown that system (48) is separated into the following equations

$$\frac{\partial Z_i}{\partial t} = \Phi_i(t,x)\frac{\partial^2 Z_i}{\partial x^2}, \ i = 1,2. \tag{52}$$

where $Z_i = w + \kappa_i \theta$ and $\Phi_i(t,x) = a + \kappa_i \delta, \ i = 1,2.$ □

In the following we use separation method to obtain explicit solutions for each equations (52).
Consider the functions $Z_i(t,x) = X_i(t)Y_i(x)$ and $\Phi_i(t,x) = \chi_i(t)\psi_i(x), \ i = 1,2.$
Therefore, the equations of (52) can be written in the following form

$$X_i'(t)Y_i(x) = \chi_i(t)\psi_i(x)X_i(t)Y_i''(x), \ i = 1,2. \tag{53}$$

So that

$$\frac{1}{\chi_i(t)}\frac{X_i'(t)}{X_i(t)} = \psi_i(x)\frac{Y_i''(x)}{Y_i(x)} = -\lambda^2, \ i = 1,2, \tag{54}$$

where λ^2 is an arbitrary constant.
The direct computation yields

$$X_{i,\ n}(t) = \exp\left(-\lambda_n^2 \int \chi_i(t)dt\right), \tag{55}$$

where $\lambda_n^2, \ n = 1,2,...$ are the eigenvalues corresponding to the eigenfunctions $Y_{i,\ n}(x)$ satisfying

$$Y_{i,\ n}''(x) + \lambda_n^2 \frac{1}{\psi_i(x)} Y_{i,\ n}(x) = 0, \ i = 1,2 \tag{56}$$

and

$$Y_{i,\ n}'(0) = Y_{i,\ n}'(1) = 0, \ i = 1,2. \tag{57}$$

By principle of superposition, the solution can be expressed by the series

$$Z_i(t,x) = \sum_{n=1}^{\infty} c_n Y_{i,\ n}(x) \exp\left(-\lambda_n^2 \int \chi_i(t)dt\right), \ i = 1,2. \tag{58}$$

Therefore, by using the initial condition $Z_i(0,x) = w(0,x) + \kappa_i\theta(0,x) = w_0(x) + \kappa_i\theta_0(x) = g_i(x), \ i = 1,2$ and from (58) we get

$$c_n = \frac{\int_0^1 g_i(x)Y_{i,\ n}(x)\frac{1}{\psi_i(x)}dx}{r_{i,\ n}\int_0^1 Y_{i,\ n}^2(x)\frac{1}{\psi_i(x)}dx}, \ i = 1,2, \tag{59}$$

where $r_{i,\ n} = \exp\left(-\lambda_n^2 \int \chi_i(t)dt\right)|_{t=0}$.
Thus we have proved

Theorem 2. *The solution (w,θ) of system (48) can be represented by*

$$w(t,x) = \frac{\kappa_2 Z_1(t,x) - \kappa_1 Z_2(t,x)}{\kappa_2 - \kappa_1} \tag{60}$$

and

$$\theta(t,x) = \frac{Z_2(t,x) - Z_1(t,x)}{\kappa_2 - \kappa_1}, \tag{61}$$

where $Z_i(t,x),\ i = 1,2$ *are given by the series*

$$Z_i(t,x) = \sum_{n=1}^{\infty} c_n Y_{i,\ n}(x) \exp\left(-\lambda_n^2 \int \chi_i(t)dt\right),\ i = 1,2,$$

$$c_n = \frac{\int_0^1 g_i(x) Y_{i,\ n}(x) \frac{1}{\psi_i(x)} dx}{r_{i,\ n} \int_0^1 Y_{i,\ n}^2(x) \frac{1}{\psi_i(x)} dx},\ i = 1,2$$

and $r_{i,\ n} = \exp\left(-\lambda_n^2 \int \chi_i(t)dt\right)|_{t=0}$.

Reforences

1. A. V. Luikov, *Heat and Mass transfer in capillary-porous bodies*, Chap. 6. Pergamon Press, Oxford , 1966.
2. M. D. Mikhailov and M. N. Özisik, *Unified analysis and solutions of heat and mass diffusion,* Chap. 10. Wiley, New York , 1994.
3. A. V. Luikov and Y. A. Mikhailov and M. N. Özisik, *Thery of Energy and mass transfer,* Chaps. 1 and 2. Pergamon Press, Oxford , 1965.
4. Jen Y. liu and Shun Cheng, *Solutions of Luikov equations of heat and mass transfer in capillary-porous bodies*, Int. J. Heat Mass Transfer **34**(1991), 1747-1754.
5. H. R. Thomas, R. W. lewis and K. Morgan, *An application of the finite element method to the drying of timber*, Wood Fiber **11**(1980), 237-243.
6. R. Keylwerth, *The variation of the temperature of wood during the drying of veneers and sawn wood,* Holz roh-Wekstoff **10**(1952), 87-91.

Lazhar Bougoffa is a professor at Al-Imam University.

Department of Mathematics, Faculty of Science, Al-Imam University, P.O.Box 90950, Riyadh 11623, Kingdom Of Saudi Arabia.
e-mail: bougoffa@hotmail.com

Hind K. Al-Jeaid is a professor at Umm Al-Qura University.

Department of Mathematics, Umm Al-Qura University, Makkah, Kingdom Of Saudi Arabia.
e-mail: hkjeaid@uqu.edu.sa