J. Appl. Math. & Informatics Vol. **29***(2011), No. 1- 2, pp. 351 - 367*
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THE $(\frac{G'}{G})$- EXPANSION METHOD COMBINED WITH THE RICCATI EQUATION FOR FINDING EXACT SOLUTIONS OF NONLINEAR PDES

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Abstract. In this article, we construct exact traveling wave solutions for nonlinear PDEs in mathematical physics via the (1+1)- dimensional combined Korteweg- de Vries and modified Korteweg- de Vries (KdV-mKdV) equation, the (1+1)- dimensional compouned Korteweg- de Vries Burgers (KdVB) equation,, the (2+1)- dimensional cubic Klien- Gordon (cKG) equation , the Generalized Zakharov- Kuznetsov- Bonjanmin- Bona Mahony (GZK-BBM) equation and the modified Korteweg- de Vries - Zakharov- Kuznetsov (mKdV-ZK) equation, by using the $(\frac{G'}{G})$ -expansion method combined with the Riccati equation, where $G = G(\xi)$ satisfies the Riccati equation $G'(\xi) = A + BG^2$ and A, B are arbitrary constants.

AMS Mathematics Subject Classification : 35K99, 35P05, 35P99.
Key words and phrases : The $(\frac{G'}{G})$- expansion method, The Riccati equation, Traveling wave solutions, The KdV-mKdV equation, The KdVB equation, The cKG equation , The GZK-BBM equation , The mKdV-ZK equation.

1. Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors(see for example [1-46]) who are interested in nonlinear physical phenomena. Many powerful different methods have been presented by those authors. For integrable nonlinear differential equations, the inverse scattering transform method [2], the Hirota method [9], the truncated Painleve expansion method [27,38], the Backlund transform method [15,16] and the exp-function method [8,31,39,40] are used in looking for the exact solutions. Among non-integrable nonlinear differential equations there is a wide class of the equations that referred to as the partially integrable, because these equations become integrable for some values of their parameters. There are many different methods

Received January 24, 2010. Accepted August 31, 2010.

to look for the exact solutions of these equations. The most famous algorithms are the truncated Painleve expansion method [12], the Weierstrass elliptic function method [11], the tanh- function method [1,7,34,41] and the Jacobi elliptic function expansion method [6,13,14,26,28,29,32].

Wang et.al [23] have introduced a simple method which is called the $(\frac{G'}{G})$-expansion method to look for traveling wave solutions of nonlinear evolution equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G^{''}(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are arbitrary constants. For further references see the articles [4,35,36,44,45]. Recently, Zayed [37] has introduced an alternative method where $G = G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$, and e_2, e_1, e_0, V are arbitrary constants . He has determined new families of exact solutions for some nonlinear evolution equation in mathematical physics.

In the present article, we shall use a different alternative approach. The main idea of this approach is that the traveling wave solutions of nonlinear partial differential equations can be expressed by a polynomial in $(\frac{G'}{G})$, where $G = G(\xi)$ satisfies the Riccati equation $G'(\xi) = A + BG^2$, where $\xi = x - Vt$ and A , B, V are arbitrary constants, while $' = \frac{d}{d\,\xi}$. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. This approach will play an important role in constructing exact traveling wave solutions for the nonlinear PDEs in mathematical physics via the KdV-mKdV equation, the KdVB equation, the cKG equation , the GZK-BBM equation and the mKdV-ZK equation. These equations have been paid attention by many researchers in engineering and physics.

2. Description of the $(\frac{G'}{G})$ -expansion method combined with the Riccati equation

Suppose we have the following nonlinear partial differential equation

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the proposed method:

Step 1 . The traveling wave variable

$$u(x,t) = u(\xi), \xi = x - Vt, \tag{2}$$

where V is a constant, permits us reducing Eq. (2.1) to an ODE for $u = u(\xi)$ in the form

$$P(u, u^{'}, u^{''}, u^{'''}, ...) = 0. \tag{3}$$

Step 2. Suppose the solution of Eq(2.3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows

$$u(\xi) = \sum_{i=0}^{n} \alpha_i (\frac{G'}{G})^i, \tag{4}$$

where $G = G(\xi)$ satisfies the following Riccati equation

$$G'(\xi) = A + BG^2, \tag{5}$$

where α_i ,A, B and V are arbitrary constants to be determined provided $\alpha_n \neq 0$. The positive integer "n" can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq (2.1) or (2.3) . More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = n$

which gives rise to the degree of other expressions as follows

$$D[\frac{d^q u}{d\xi^q}] = n + q, \; D[u^p (\frac{d^q u}{d\xi^q})^s] = np + s(q + n). \tag{6}$$

Therefore, we can get the value of n in (2.4), using (2.6).

Step 3. Substituting (2.4) into (2.3) and using Eq (2.5), we obtain polynomials in $G^j(\xi)$ $(j = 0, \pm 1, \pm 2, ...)$.Equating each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for α_i ,A, B and V.

Step 4 . Since the general solutions of Eq (2.5) have been well known for us (see Appendix A), then substituting α_i, V and the general solution of Eq (2.5) into (2.4) we have exact traveling wave solutions of the nonlinear partial differential equation (2.1).

3. Some applications

In this section, we apply the $(\frac{G'}{G})$ - expansion method combined with the Riccati equation to construct exact traveling wave solutions for the following nonlinear PDEs in mathematical physics:

Example 1. We start with the (1+1)- dimensional combined KdV-mKdV equation [30] in the form

$$u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx} = 0, \tag{1}$$

where α and β are nonzero constants. This equation may describe the wave propagation of the bound particle, sound wave and thermal pulse. This equation is the most popular soliton equation and often exists in practical problems such as fluid physics and quantum field theory. An extended Fan's sub-equation method is used in [30] for constructing exact traveling wave solutions of Eq.

(3.1). Let us now solve Eq. (3.1) by the proposed method. To this end, we see that the traveling wave variable (2.2) permits us converting Eq. (3.1) into the following ODE:

$$C - Vu + \frac{1}{2}\alpha u^2 + \frac{1}{3}\beta u^3 + u^{''} = 0, \tag{2}$$

where C is an integration constant. Suppose that the solution of Eq. (3.2) can

be expressed by a polynomial in $(\frac{G'}{G})$ as follows

$$u(\xi) \ = \sum_{i=0}^{n} \alpha_i (\frac{G'}{G})^i, \tag{3}$$

where α_i are arbitrary constants provided $\alpha_n \neq 0$, while $G(\xi)$ satisfies the Riccati equation (2.5).

Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.2), we deduce from (2.6) that $D(u^{''}) = D\ (u^3)$. Therefore $n + 2 = 3n$ and hence $n = 1$. Thus, we get

$$u(\xi) \ = \alpha_1 (\frac{G'}{G}) + \alpha_0. \tag{4}$$

From (2.5) and (3.4) we derive the following formulae:

$$u = \alpha_0 + \alpha_1 A G^{-1} + \alpha_1 B G,$$

$$u^{'} \ = \alpha_1 \left[B^2 G^2 - A^2 G^{-2} \right],$$

$$u^{''} \ = 2\alpha_1 \left[AB^2 G + BA^2 G^{-1} + B^3 G^3 + A^3 G^{-3} \right], \tag{5}$$

and so on.

Substituting (3.5) into (3.2) we get the following polynomial

$$\begin{aligned}
&G^{-1}[\alpha\alpha_0\alpha_1 A + A^2 B\beta\alpha_1^3 + \alpha_0^2\alpha_1 A\beta + 2\alpha_1 A^2 B - VA\alpha_1] \\
&+G[-V\alpha_1 B + \alpha\alpha_0\alpha_1 B + B\beta\alpha_0^2\alpha_1 + \beta\alpha_1^3 AB^2 + 2\alpha_1 AB^2] \\
&+G^{-2}[\frac{1}{2}\alpha\alpha_1^2 A^2 + A^2\beta\alpha_1^2\alpha_0] + G^2[\frac{1}{2}\alpha\alpha_1^2 B^2 + B^2\beta\alpha_1^2\alpha_0] \\
&+G^{-3}[\frac{1}{3}\beta\alpha_1^3 A^3 + 2\alpha_1 A^3] + G^3[\frac{1}{3}\beta\alpha_1^3 B^3 + 2\alpha_1 B^3]
\end{aligned}$$

$$+C - V\alpha_0 + \frac{1}{2}\alpha\alpha_0^2 + \alpha\alpha_1^2 AB + \frac{1}{3}\beta\alpha_0^3 = 0. \tag{6}$$

Consequently, we have the following system of algebraic equations

$$\begin{aligned}
\alpha\alpha_0\alpha_1 A + A^2B\beta\alpha_1^3 + \alpha_0^2\alpha_1 A\beta + 2\alpha_1 A^2 B - VA\alpha_1 &= 0,\\
-V\alpha_1 B + \alpha\alpha_0\alpha_1 B + B\beta\alpha_0^2\alpha_1 + \beta\alpha_1^3 AB^2 + 2\alpha_1 AB^2 &= 0,\\
\frac{1}{2}\alpha\alpha_1^2 A^2 + A^2\beta\alpha_1^2\alpha_0 &= 0,\\
\frac{1}{2}\alpha\alpha_1^2 B^2 + B^2\beta\alpha_1^2\alpha_0 &= 0,\\
\frac{1}{3}\beta\alpha_1^3 A^3 + 2\alpha_1 A^3 &= 0,\\
\frac{1}{3}\beta\alpha_1^3 B^3 + 2\alpha_1 B^3 &= 0,\\
C - V\alpha_0 + \frac{1}{2}\alpha\alpha_0^2 + \alpha\alpha_1^2 AB + \frac{1}{3}\beta\alpha_0^3 &= 0, \qquad (7)
\end{aligned}$$

which can be solved to get

$$\alpha_1 = \pm\sqrt{\frac{-6}{\beta}}, \quad \alpha_0 = -\frac{\alpha}{2\beta}, \quad V = -\frac{\alpha^2}{4\beta} - 4AB, \quad C = \frac{8\alpha AB}{\beta} + \frac{\alpha^3}{24\beta^2} \qquad (8)$$

Substituting (3.8) into (3.4) yields

$$u(\xi) = \pm\sqrt{\frac{-6}{\beta}}(\frac{G'}{G}) - \frac{\alpha}{2\beta}, \qquad (9)$$

where

$$\xi = x + t\ (\frac{\alpha^2}{4\beta} + 4AB). \qquad (10)$$

According to the appendix A, we have the following families of exact solutions

Family 1. If $A = \frac{1}{2}, B = -\frac{1}{2}$, then we get

$$u(\xi) = -\frac{\alpha}{2\beta} - \sqrt{\frac{-6}{\beta}}i \operatorname{sech}\xi, \qquad (11)$$

or

$$u(\xi) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{-6}{\beta}}\operatorname{csch}\xi, \qquad (12)$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} - 1)$ and $i = \sqrt{-1}$.

Family 2. If $A = B = \pm\frac{1}{2}$, then we get

$$u(\xi) = -\frac{\alpha}{2\beta} + \sqrt{\frac{-6}{\beta}} \sec\xi, \qquad (13)$$

or

$$u(\xi) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{-6}{\beta}}\csc\xi, \qquad (14)$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} + 1)$.

Family 3. If $A = 1, B = -1$, then we get

$$u(\xi)\ = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{-6}{\beta}}\ (\coth\xi - \tanh\xi), \tag{15}$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} - 4)$.

Family 4. If $A = B = 1$, then we get

$$u(\xi)\ = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{-6}{\beta}}\ (\cot\xi + \tan\xi), \tag{16}$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} + 4)$.

Family 5. If $A = 0, B \neq 0$, then we get

$$u(\xi)\ = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{-6}{\beta}}\ \left(\frac{B}{B\xi + c_1}\right) \tag{17}$$

where $\xi = x + \frac{\alpha^2 t}{4\beta}$ and c_1 is an arbitrary constant.

Example 2. We consider the (1+1)- dimensional compound KdVB equation [46] in the form:

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xx} - \delta u_{xxx} = 0, \tag{18}$$

where α, β, γ and δ are constants. This equation can be thought of as a generalization of KdV-mKdV and Burgers equations involving nonlinear dispersion and dissipation effects. The traveling wave solutions of Eq. (3.18) have been found in [46] using an improved sine-cosine method. Let us now solve Eq. (3.18) by the proposed method. To this end, we see that the traveling wave variable (2.2) permits us converting Eq. (3.18) into the following ODE:

$$C - Vu + \frac{1}{2}\alpha u^2 + \frac{1}{3}\beta u^3 + \gamma u^{'} - \delta u^{''} = 0, \tag{19}$$

where C is an integration constant. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.19), we get $n = 1$. Thus, we have the solution of the Eq. (3.19) in the same form (3.4). Substituting (3.5) into (3.19) we get the following polynomail

$$\begin{aligned}
&G^{-1}[-VA\alpha_1 + \alpha\alpha_0\alpha_1 A + A^2 B\beta\alpha_1^3 + \alpha_0^2\alpha_1 A\beta - 2\alpha_1 A^2 B\ \delta] \\
&+G[-V\alpha_1 B + \alpha\alpha_0\alpha_1 B + B\beta\alpha_0^2\alpha_1 + \beta\alpha_1^3 AB^2 - 2\alpha_1 AB^2\delta] \\
&+G^2[\frac{1}{2}\alpha\alpha_1^2 B^2 + B^2\beta\alpha_1^2\alpha_0 + \gamma\alpha_1 B^2] + G^{-2}[\frac{1}{2}\alpha\alpha_1^2 A^2 + A^2\beta\alpha_1^2\alpha_0 - \gamma\alpha_1 A^2] \\
&+G^{-3}[\frac{1}{3}\beta\alpha_1^3 A^3 - 2\delta\alpha_1 A^3] + G^3[\frac{1}{3}\beta\alpha_1^3 B^3 - 2\delta\alpha_1 B^3]
\end{aligned}$$

$$+C - V\alpha_0 + \frac{1}{2}\alpha\alpha_0^2 + \alpha\alpha_1^2 AB + \frac{1}{3}\beta\alpha_0^3 = 0. \tag{20}$$

Consequently, we have the following system of algebraic equations

$$\begin{aligned}
-VA\alpha_1 + \alpha\alpha_0\alpha_1 A + A^2B\beta\alpha_1^3 + \alpha_0^2\alpha_1 A\beta - 2\alpha_1 A^2B\ \delta &= 0,\\
-V\alpha_1 B + \alpha\alpha_0\alpha_1 B + B\beta\alpha_0^2\alpha_1 + \beta\alpha_1^3 AB^2 - 2\alpha_1 AB^2\delta &= 0,\\
\frac{1}{2}\alpha\alpha_1^2 A^2 + A^2\beta\alpha_1^2\alpha_0 - \gamma\alpha_1 A^2 &= 0,\\
\frac{1}{2}\alpha\alpha_1^2 B^2 + B^2\beta\alpha_1^2\alpha_0 + \gamma\alpha_1 B^2 &= 0,\\
\frac{1}{3}\beta\alpha_1^3 A^3 - 2\delta\alpha_1 A^3 &= 0,\\
\frac{1}{3}\beta\alpha_1^3 B^3 - 2\delta\alpha_1 B^3 &= 0,\\
C - V\alpha_0 + \frac{1}{2}\alpha\alpha_0^2 + \alpha\alpha_1^2 AB + \frac{1}{3}\beta\alpha_0^3 &= 0, \qquad (21)
\end{aligned}$$

which can be solved to get

$$\alpha_1 = \pm\sqrt{\frac{6\delta}{\beta}}, \quad \alpha_0 = -\frac{\alpha}{2\beta}, \quad V = -\frac{\alpha^2}{4\beta} + 4\delta AB, \quad C = \frac{\alpha^3}{24\beta^2} - \frac{8\alpha\delta AB}{\beta}, \quad \gamma = 0 \tag{22}$$

Substituting (3.22) into (3.4) yields

$$u(\xi) = \pm\sqrt{\frac{6\delta}{\beta}}(\frac{G'}{G}) - \frac{\alpha}{2\beta}, \tag{23}$$

where

$$\xi = x + t\ (\frac{\alpha^2}{4\beta} - 4\delta AB). \tag{24}$$

According to the appendix A, we have the following families of exact solutions

Family 1. If $A = \frac{1}{2}, B = -\frac{1}{2}$, then we get

$$u(\xi) = -\frac{\alpha}{2\beta} - \sqrt{\frac{6\delta}{\beta}} i\ \mathrm{sech}\xi, \tag{25}$$

or

$$u(\xi) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6\delta}{\beta}}\mathrm{csch}\ \xi, \tag{26}$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} + \delta)$ and $i = \sqrt{-1}$.

Family 2. If $A = B = \pm\frac{1}{2}$, then we get

$$u(\xi) \ = -\frac{\alpha}{2\beta} + \sqrt{\frac{6\delta}{\beta}}\ \sec\ \xi, \tag{27}$$

or

$$u(\xi) \ = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6\delta}{\beta}} \csc\ \xi, \tag{28}$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} - \delta)$.

Family 3. If $A = 1, B = -1$, then we get

$$u(\xi) \ = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6\delta}{\beta}}\ (\coth\xi - \tanh\xi), \tag{29}$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} + 4\delta)$.

Family 4. If $A = B = 1$, then we get

$$u(\xi) \ = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6\delta}{\beta}}\ (\cot\xi + \tan\xi), \tag{30}$$

where $\xi = x + t\ (\frac{\alpha^2}{4\beta} - 4\delta)$.

Family 5. If $A = 0, B \neq 0$, then we get

$$u(\xi) \ = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{6\delta}{\beta}}\ \left(\frac{B}{B\xi + c_1}\right) \tag{31}$$

where $\xi = x + \frac{\alpha^2 t}{4\beta}$ and c_1 is an arbitrary constant.

Note that the results of example 2 are in agreement with the results of example 1 when $\delta = -1$ and $\gamma = 0$.

Example 3. We consider the following (2+1)- dimensional cKG equation [24]:

$$u_{xx} + u_{yy} - u_{tt} + \alpha u + \beta u^3 = 0, \tag{32}$$

where α and β are nonzero constants. This equation is used to model many different nonlinear phenomena, including the propagation of dislocation in crystals and the behavior of elementary particles and the propagation of fluxons in Josephson junctions. Many types of complexiton solutions and soliton solutions for Eq. (3.32) have been found in [24] using the multi- function expansion method. Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable $u(x,t) = u(\xi),\ \xi = x + y - Vt$, permits us converting Eq. (3.32) into the following ODE:

$$(2 - V^2)u^{''} + \alpha u + \beta u^3 = 0, \tag{33}$$

where $V^2 \neq 2$. Considering the homogeneous balance between the highest order

derivative and the nonlinear term in (3.33), we get $n = 1$. Thus, the solution of Eq. (3.33) has the same form (3.4). Substituting (3.5) into (3.33) we get the following polynomial:

$$\begin{aligned}
&G^{-1}[2\alpha_1 A^2 B(2-V^2) + \alpha\alpha_1 A + 3\beta\alpha_1^3 A^2 B + 3\beta\alpha_0^2\alpha_1 A] \\
&+G[2\alpha_1 AB^2(2-V^2) + \alpha\alpha_1 B + 3B\beta\alpha_0^2\alpha_1 + 3\beta\alpha_1^3 AB^2] \\
&+G^{-2}[3A^2\beta\alpha_1^2\alpha_0] + G^2[3B^2\beta\alpha_1^2\alpha_0] \\
&+G^3[\beta\alpha_1^3 B^3 + 2\alpha_1 B^3(2-V^2)] + G^{-3}[\beta\alpha_1^3 A^3 + 2\alpha_1 A^3(2-V^2)]
\end{aligned}$$

$$\alpha\alpha_0 + \beta\alpha_0^3 = 0. \tag{34}$$

Consequently, we have the following system of algebraic equations

$$\begin{aligned}
2\alpha_1 A^2 B(2-V^2) + \alpha\alpha_1 A + 3\beta\alpha_1^3 A^2 B + 3\beta\alpha_0^2\alpha_1 A &= 0, \\
2\alpha_1 AB^2(2-V^2) + \alpha\alpha_1 B + 3B\beta\alpha_0^2\alpha_1 + 3\beta\alpha_1^3 AB^2 &= 0, \\
3A^2\beta\alpha_1^2\alpha_0 &= 0, \\
3B^2\beta\alpha_1^2\alpha_0 &= 0, \\
\beta\alpha_1^3 A^3 + 2\alpha_1 A^3(2-V^2) &= 0, \\
\beta\alpha_1^3 B^3 + 2\alpha_1 B^3(2-V^2) &= 0, \\
\alpha\alpha_0 + \beta\alpha_0^3 &= 0,
\end{aligned} \tag{35}$$

which can be solved to get

$$\alpha_1 = \pm\sqrt{\frac{-\alpha}{2\beta AB}}, \quad \alpha_0 = 0, \quad V = \pm\sqrt{2 - \frac{\alpha}{4AB}}. \tag{36}$$

Substituting (3.36) into (3.4) yields

$$u(\xi) \; = \pm\sqrt{\frac{-\alpha}{2\beta AB}}(\frac{G'}{G}), \tag{37}$$

where

$$\xi = x + y \mp t\sqrt{2 - \frac{\alpha}{4AB}} \; . \tag{38}$$

According to the appendix A, we have the following families of exact solutions:
Family 1. If $A = \frac{1}{2}, B = -\frac{1}{2}$, then we get

$$u(\xi) \; = -\sqrt{\frac{2\alpha}{\beta}} i \text{ sech } \xi, \tag{39}$$

or

$$u(\xi) \ = \pm\sqrt{\frac{2\alpha}{\beta}}\text{csch } \xi, \tag{40}$$

where $\xi = x + y \mp t\sqrt{2+\alpha}$.

Family 2. If $A = B = \pm\frac{1}{2}$, then we get

$$u(\xi) \ = \sqrt{\frac{-2\alpha}{\beta}} \sec \xi, \tag{41}$$

or

$$u(\xi) \ = \pm\sqrt{\frac{-2\alpha}{\beta}}\csc \ \xi, \tag{42}$$

where $\xi = x + y \mp t\sqrt{2-\alpha}$.

Family 3. If $A = 1, B = -1$, then we get

$$u(\xi) \ = \pm\sqrt{\frac{\alpha}{2\beta}} \ (\coth\xi - \tanh\xi), \tag{43}$$

where $\xi = x + y \mp t\sqrt{2+\frac{\alpha}{4}}$.

Family 4. If $A = B = 1$, then we get

$$u(\xi) \ = \pm\sqrt{\frac{-\alpha}{2\beta}} \ (\cot\xi + \tan\xi), \tag{44}$$

where $\xi = x + y \mp t\sqrt{2-\frac{\alpha}{4}}$

Example 4. We consider the following (2+1)- dimensional GZK-BBM equation [25]:

$$u_t + u_x + \alpha(u^3)_x + \beta(u_{xt} + u_{yy})_x = 0, \tag{45}$$

where α and β are nonzero constants. Wazwaz [25] has studied this equation and derived solutions of distinct physical structures: compactons, solitons, solitary patterns and periodic solutions using the tanh-method and the sine-cosine method. Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable $u(x,t) = u(\xi),\ \xi = x + y - Vt$, permits us converting Eq. (3.45) into the following ODE:

$$C + (1-V)u + \alpha u^3 + \beta(1-V)u^{''} = 0, \tag{46}$$

where $V \neq 1$ and C is the integration constant. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.46), we get $n = 1$. Thus, the solution of Eq. (3.46) has the same form (3.4). Substituting (3.5) into (3.46), we get the following polynomail:

$$
\begin{aligned}
&G^{-1}[\alpha_1 A(1-V)+3\alpha\alpha_1\alpha_0^2 A+3\alpha\alpha_1^3 A^2 B+2\beta B A^2\alpha_1(1-V)]\\
&+G[\alpha_1 B(1-V)+3\alpha\alpha_1\alpha_0^2 B+3\alpha\alpha_1^3 A B^2+2\beta\alpha_1(1-V)AB^2]\\
&+G^{-2}[3\alpha A^2\alpha_1^2\alpha_0]+G^2[3B^2\alpha\alpha_1^2\alpha_0]\\
&+G^{-3}[\alpha\alpha_1^3 A^3+2\alpha_1 A^3\beta(1-V)]+G^3[\alpha\alpha_1^3 B^3+2\alpha_1\beta B^3(1-V)]
\end{aligned}
$$

$$
+C+\alpha_0(1-V)+\alpha\alpha_0^3=0. \tag{47}
$$

Consequently, we have the following system of algebraic equations:

$$
\begin{aligned}
2\alpha_1 A(1-V)+3\alpha\alpha_1\alpha_0^2 A+3\alpha\alpha_1^3 A^2 B+2\beta B A^2\alpha_1(1-V) &= 0,\\
\alpha_1 B(1-V)+3\alpha\alpha_1\alpha_0^2 B+3\alpha\alpha_1^3 A B^2+2\beta\alpha_1(1-V)AB^2 &= 0,\\
3\alpha A^2\alpha_1^2\alpha_0 &= 0,\\
3B^2\alpha\alpha_1^2\alpha_0 &= 0,\\
\alpha\alpha_1^3 A^3+2\alpha_1 A^3\beta(1-V) &= 0,\\
\alpha\alpha_1^3 B^3+2\alpha_1\beta B^3(1-V) &= 0,\\
C+\alpha_0(1-V)+\alpha\alpha_0^3 &= 0,
\end{aligned} \tag{48}
$$

which can be solved to get

$$
\alpha_1=\pm\sqrt{\frac{V-1}{2\alpha AB}},\ \alpha_0=0,\ C=0. \tag{49}
$$

Substituting (3.49) into (3.4) yields

$$
u(\xi)\ =\pm\sqrt{\frac{V-1}{2\alpha AB}}(\frac{G'}{G}), \tag{50}
$$

where

$$
\xi=x+y-Vt\ . \tag{51}
$$

According to the appendix A, we have the following families of exact solutions:

Family 1. If $A=\frac{1}{2}, B=-\frac{1}{2}$, then we get

$$
u(\xi)\ =-\sqrt{\frac{2(1-V)}{\alpha}}\ i\ \text{sech}\ \xi, \tag{52}
$$

or

$$
u(\xi)\ =\pm\sqrt{\frac{2(1-V)}{\alpha}}\text{csch}\ \xi, \tag{53}
$$

Family 2. If $A=B=\pm\frac{1}{2}$, then we get

$$u(\xi) \ = \sqrt{\frac{2(V-1)}{\alpha}} \ \sec\xi, \tag{54}$$

or

$$u(\xi) \ = \pm\sqrt{\frac{2(V-1)}{\alpha}}\csc \ \ \xi, \tag{55}$$

Family 3. If $A = 1, B = -1$, then we get

$$u(\xi) \ = \pm\sqrt{\frac{1-V}{2\alpha}} \ (\coth\xi - \tanh\xi), \tag{56}$$

Family 4. If $A = B = 1$, then we get

$$u(\xi) \ = \pm\sqrt{\frac{V-1}{2\alpha}} \ (\cot\xi + \tan\xi), \tag{57}$$

Example 5. We consider the following (3+1)- dimensional mKdV-ZK equation [28]:

$$u_t + \alpha u^2 u_x + (u_{xx} + u_{yy} + u_{zz})_x = 0, \tag{58}$$

where α is a nonzero constants. Xu [28] has discussed this equation using an elliptic equation method and found many types of elliptic function solutions. Let us now solve this equation by the proposed method. To this end, we see that the traveling wave variable $u(x,t) = u(\xi),\ \xi = x + y + z - Vt$, permits us converting Eq. (3.58) into the following ODE:

$$C - Vu + \frac{1}{3}\alpha u^3 + 3u^{''} = 0, \tag{59}$$

where C is the integration constant Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.59), we get $n = 1$. Thus, the solution of Eq. (3.59) has the same form (3.4). Substituting (3.5) into (3.59), we get the following polynomial:

$$\begin{aligned}
&G^{-1}[-\alpha_1 AV + \alpha\alpha_1\alpha_0^2 A + \alpha\alpha_1^3 A^2 B + 6\alpha_1 BA^2] \\
&+G[-\alpha_1 BV + \alpha\alpha_1\alpha_0^2 B + \alpha\alpha_1^3 AB^2 + 6\alpha_1 AB^2] \\
&+G^{-2}[\alpha\alpha_0\alpha_1^2 A^2] + G^2[\alpha\alpha_0\alpha_1^2 B^2] \\
&+G^{-3}[\frac{1}{3}\alpha\alpha_1^3 A^3 + 6\alpha_1 A^3] + G^3[\frac{1}{3}\alpha\alpha_1^3 B^3 + 6\alpha_1 B^3] \\
&\qquad +C - V\alpha_0 + \frac{1}{3}\alpha\alpha_0^3 = 0.
\end{aligned} \tag{60}$$

Consequently, we have the following system of algebraic equations

$$
\begin{aligned}
-\alpha_1 AV + \alpha\alpha_1\alpha_0^2 A + \alpha\alpha_1^3 A^2 B + 6\alpha_1 BA^2 &= 0, \\
-\alpha_1 BV + \alpha\alpha_1\alpha_0^2 B + \alpha\alpha_1^3 AB^2 + 6\alpha_1 AB^2 &= 0, \\
\alpha\alpha_0\alpha_1^2 A^2 &= 0, \\
\alpha\alpha_0\alpha_1^2 B^2 &= 0, \\
\frac{1}{3}\alpha\alpha_1^3 A^3 + 6\alpha_1 A^3 &= 0, \\
\frac{1}{3}\alpha\alpha_1^3 B^3 + 6\alpha_1 B^3 &= 0, \\
C - V\alpha_0 + \frac{1}{3}\alpha\alpha_0^3 &= 0,
\end{aligned} \tag{61}
$$

which can be solved to get

$$\alpha_1 = \pm 3\sqrt{\frac{-2}{\alpha}}, \quad \alpha_0 = 0, \quad C = 0, \ V = -12AB, \tag{62}$$

Substituting (3.62) into (3.4) yields

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}\left(\frac{G'}{G}\right), \tag{63}$$

where

$$\xi = x + y + z + 12ABt. \tag{64}$$

According to the appendix A, we have the following families of exact solutions:
Family 1. If $A = \frac{1}{2}, B = -\frac{1}{2}$, then we get

$$u(\xi) = -3\sqrt{\frac{-2}{\alpha}}\ i \ \mathrm{sech}\ \xi, \tag{65}$$

or

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}\ \mathrm{csch}\ \xi, \tag{66}$$

where $\xi = x + y + z - 3t$.

Family 2. If $A = B = \pm\frac{1}{2}$, then we get

$$u(\xi) = 3\sqrt{\frac{-2}{\alpha}}\ \sec\ \xi, \tag{67}$$

or

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}} \csc\ \xi, \tag{68}$$

where $\xi = x + y + z + 3t$.

Family 3. If $A = 1, B = -1$, then we get

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}\,(\coth\xi - \tanh\xi), \tag{69}$$

where $\xi = x + y + z - 12t$.

Family 4. If $A = B = 1$, then we get

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}\,(\cot\xi + \tan\xi), \tag{70}$$

where $\xi = x + y + z + 12t$.

Family 5. If $A = 0, B \neq 0$, then we get

$$u(\xi) = \pm 3\sqrt{\frac{-2}{\alpha}}\,\left(\frac{B}{B\xi + c_1}\right), \tag{71}$$

where $\xi = x + y + z$.

Appendix A

The general solutions to the Riccati equation (2.5) are well known [5,24] which are listed in the following table:

A	B	The solution $G(\xi)$
$\frac{1}{2}$	$-\frac{1}{2}$	$\tanh\xi \pm i\ \mathrm{sech}\xi$, $\coth\xi \pm \mathrm{csch}\xi$, $\tanh\frac{\xi}{2}$, $\coth\frac{\xi}{2}$
$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\sec\xi \pm \tan\xi$, $\pm\tan\frac{\xi}{2}$, $\mp\cot\frac{\xi}{2}$, $\pm(\csc\xi - \cot\xi)$
1	-1	$\tanh\xi$, $\coth\xi$
1	1	$\tan\xi$, $-\cot\xi$
0	$\neq 0$	$\frac{-1}{B\xi + c_1}$, where c_1 is an arbitrary constant.

Other values for the solution $G(\xi)$ of the Eq(2.5) can be found for arbitrary values of A and B.

4. Conclusions

The main idea of the $(\frac{G'}{G})$- expansion method (see [4,23,35,36,44,45]) is that the traveling wave solutions of nonlinear partial differential equations can be expressed as a polynomial in $(\frac{G'}{G})$, where $G(\xi)$ satisfies a second order linear ordinary differential equation. In the present article, we have developed this method, where we have assumed that $G(\xi)$ satisfies the Riccati equation (2.5) instead of the standard technique used by Wang et.al.[23]. We have applied this alternative method to some nonlinear PDEs in mathematical physics via the KdV-mKdV equation, the KdVB equation, the cKG equation, the GZKBBM equation and the mKdV-ZK equation. We have obtained families of exact solutions of these equations in terms of hyperbolic, trigonometeric and rational functions.

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