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NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR FUZZY LINEAR PROGRAMMING

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Abstract. This paper is concerned with deriving necessary and sufficient optimality conditions for a fuzzy linear programming problem. Toward this end, an equivalence between fuzzy and crisp linear programming problems is established by means of a specific ranking function. Under this setting, a main theorem gives optimality conditions which do not seem to be in conflict with the so-called Karush-Kuhn-Tucker conditions for a crisp linear programming problem.

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1. Introduction

Nowadays, the term "fuzzy optimization" has been much extended in hundreds practical fields such as artificial intelligence, manufacturing and management and many papers have been published in this area. As one of the first efforts in this direction, [2] was devoted to the study of fuzzy optimization problems in which introduced the aggregation operators combining the fuzzy goals and the fuzzy decision space. This study was followed up and extended by the others [3],[4],[13],[20] and [21]. Moreover, there are many fuzzy-model-based techniques have been developed as a useful tool for solving fuzzy optimization problems. For instance, in [8] a fuzzy linear programming problem was defuzzified by symmetric method of Bellman and Zadeh [2] and then the modified subgradient method used for solving the foregoing problem. A geometric approach was proposed in [12] for solving fuzzy linear programming problems. Furthermore, genetic algorithm was implemented to deal with optimization problems in [10] and [14]. The other method involving ranking function has been developed in [5],[7] and [9]. This method which is based on the concept of comparison of fuzzy numbers, transforms a fuzzy linear programming problem to a classical one.

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Recently, studies have also concerned on the fuzzy optimality conditions. In [15] the attempting is to develop the saddle point optimality conditions for a fuzzy optimization problem via its fuzzy dual. The duality theorems and saddle point optimality conditions in fuzzy nonlinear programming problems have been derived in [17] based on some solution concepts for primal and dual problems. The Karush-Kuhn-Tucker conditions in an optimization problem with interval-valued objective function have been elicited in [16]. In [11] the Karush-Kuhn-Tucker optimality conditions have been presented using the concepts of differentiability and convexity of a fuzzy mapping.

The purpose of this paper is to derive the Karush-Kuhn-Tucker optimality conditions for a fuzzy linear programming problem using a ranking function similar to that proposed in [7]. This present contribution is outlined as follows: Section 2 is devoted to give the definitions of fuzzy numbers and some related results of fuzzy arithmetic on LR fuzzy numbers. In Section 3 using a ranking function, the ordering of fuzzy numbers which is the key to fuzzy optimization problems is discussed. Formulation and solution concept of fuzzy linear programming problem are introduced in Section 4 and then it is proved the equivalence relationship between a fuzzy linear programming problem and a crisp one via the ranking function that maps the set of fuzzy numbers into the real line. Finally, Section 5 presents the necessary and sufficient optimality conditions those are well-known as the Karush-Kuhn-Tucker optimality conditions for a fuzzy linear programming problem.

2. Preliminaries

It is quoted some preliminary notions, definitions and results in fuzzy sets theory to be used throughout this article.

A *fuzzy set* $\widetilde{A}$ in X is characterized by a membership function $\mu_{\widetilde{A}} : X \to [0, 1]$, and denoted by

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) \mid x \in X\}.$$

An α-*cut* or α-*level* of the set $\widetilde{A}$, is the crisp set $[\widetilde{A}]_\alpha = \{x \in X \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$ and the *support* of a fuzzy set $\widetilde{A}$, is the crisp set $\mathtt{Supp}(\widetilde{A}) = cl\{x \in X \mid \mu_{\widetilde{A}}(x) > 0\}$. Generally, a fuzzy set $\widetilde{A}$ is called a *fuzzy number* if the following conditions are satisfied:

$\widetilde{A}$ is normal. It means that there exists an $x \in X$ such that $\mu_{\widetilde{A}}(x) = 1$;

$\mu_{\widetilde{A}}$ is quasi-concave. It means that for every $x, y \in X$

$$\mu_{\widetilde{A}}(\gamma x + (1-\gamma)y) \geq \min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)\}, \quad \gamma \in [0, 1];$$

$\mu_{\widetilde{A}}$ is upper semi-continuous. It means that $[\widetilde{A}]_\alpha$ are closed subsets of X for all $\alpha \in [0, 1]$;

$\mathtt{Supp}(\widetilde{A})$ is compact. In other words, it is closed and bounded in X.

Let $F(\mathbb{R})$ be the set of all fuzzy numbers on $\mathbb{R}$.

It is well known that the α-level set of a fuzzy number is a closed and bounded interval $[\underline{A}(\alpha), \overline{A}(\alpha)]$, where $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$ denote respectively the left- and right-hand endpoints of $[\widetilde{A}]_\alpha$. A fuzzy number $\widetilde{A} \in F(\mathbb{R})$ is said to be *positive (strict positive)* if $\underline{A}(\alpha) \geq 0$ ($\underline{A}(\alpha) > 0$) for all $0 \leq \alpha \leq 1$, *negative (strict negative)* if $\underline{A}(\alpha) \leq 0$ ($\underline{A}(\alpha) < 0$) for all $0 \leq \alpha \leq 1$. One shall remark that if $\widetilde{A} \in F(\mathbb{R})$ is positive (negative) then $-\widetilde{A} \in F(\mathbb{R})$ is negative (positive).

Let $L, R : [0, \infty) \to [0, 1]$ be two upper semi-continuous, non-increasing functions satisfying $L(0) = R(0) = 1$, $L(1) = R(1) = 0$, invertible on $[0, 1]$. Samples of $L(.)$ and $R(.)$ can be found in [19]. Furthermore, let $\underline{a}$ and $\overline{a}$ be positive numbers. The fuzzy number $\widetilde{a} \in F(\mathbb{R})$ is an *LR fuzzy number* if

$$\widetilde{a}(x) \equiv \mu_{\widetilde{a}}(x) = \begin{cases} L(\frac{\underline{\underline{a}}-x}{\underline{a}}), & x \leq \underline{\underline{a}}, \\ 1, & \underline{\underline{a}} \leq x \leq \overline{\overline{a}}, \\ R(\frac{x-\overline{\overline{a}}}{\overline{a}}), & x \geq \overline{\overline{a}}. \end{cases} \tag{1}$$

It is symbolically written $\widetilde{a} = (\underline{\underline{a}}, \overline{\overline{a}}, \underline{a}, \overline{a})_{LR}$, where $\underline{\underline{a}}$ and $\overline{\overline{a}}$ are called the mean values satisfying $\underline{\underline{a}} \leq \overline{\overline{a}}$ and $\underline{a}$, $\overline{a}$ are the left and right spreads, respectively.

An LR fuzzy number $\widetilde{a} \in F(\mathbb{R})$ is said to be a *trapezoidal fuzzy number* if the functions L and R are linear. Under the latter assumption, a real-numbered quadruple $(\underline{\underline{a}}, \overline{\overline{a}}, \underline{a}, \overline{a})_{LR}$ represents a trapezoidal fuzzy number.

One obtains the so-called *triangular fuzzy number* when the mean values of a trapezoidal fuzzy number fulfilling $\underline{\underline{a}} = \overline{\overline{a}} = a$. In this case, triple $(\underline{a}, a, \overline{a})_{LR}$ characterizes the triangular fuzzy number $\widetilde{a} \in F(\mathbb{R})$.
Remark that if $\widetilde{a} \in F(\mathbb{R})$ is a triangular fuzzy number then

$$[\widetilde{a}]_\alpha = [a - L^{-1}(\alpha)\underline{a}, a + R^{-1}(\alpha)\overline{a}], \quad 0 \leq \alpha \leq 1. \tag{2}$$

As particular case, one can get

$$[\widetilde{a}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)] = [a - (1-\alpha)\underline{a}, a + (1-\alpha)\overline{a}], \quad 0 \leq \alpha \leq 1, \tag{3}$$

when $L(x) = R(x) = 1 - x$.
In the sequel, it will be denoted the set of all triangular fuzzy numbers by $F_\tau(\mathbb{R})$ and if there is no confusion, $\widetilde{a} \in F_\tau(\mathbb{R})$ is represented by $(a, \underline{a}, \overline{a})_{LR}$ instead of $(\underline{a}, a, \overline{a})_{LR}$.

Let $\widetilde{a} = (a, \underline{a}, \overline{a})_{LR}$ and $\widetilde{b} = (b, \underline{b}, \overline{b})_{LR}$ be two triangular fuzzy numbers in $F_\tau(\mathbb{R})$, a binary operation $\odot$ between $\widetilde{a}$ and $\widetilde{b}$ is defined as the triangular fuzzy number characterizing by the membership function

$$(\widetilde{a} \odot \widetilde{b})\ (z) = \sup_{z = x \circ y} \min\{\widetilde{a}(x), \widetilde{b}(y)\},$$

where using *the extension principle* stated in [18], the operation $\odot$ can be taken as $\oplus$ or $\otimes$ corresponding to the conventional operations $+$ or $\times$, respectively, between two real numbers.
Fuzzy arithmetic operations on $\widetilde{a} = (a, \underline{a}, \overline{a})_{LR}$ and $\widetilde{b} = (b, \underline{b}, \overline{b})_{LR}$ in $F_\tau(\mathbb{R})$ are given by:

Fuzzy addition: $\widetilde{a} \oplus \widetilde{b} = (a, \underline{a}, \overline{a})_{LR} \oplus (b, \underline{b}, \overline{b})_{LR} = (a + b, \underline{a} + \underline{b}, \overline{a} + \overline{b})_{LR}$;

Fuzzy subtraction: $\widetilde{a} \ominus \widetilde{b} = \widetilde{a} \oplus (-\widetilde{b}) = (a, \underline{a}, \overline{a})_{LR} \oplus (-b, \underline{b}, \overline{b})_{LR} = (a-b, \underline{a} + \underline{b}, \overline{a} + \overline{b})_{LR}$;

Fuzzy scalar product: $k \otimes \widetilde{a} = (k, 0, 0)_{LR} \otimes (a, \underline{a}, \overline{a})_{LR}$

$$= \begin{cases} (ka, k\underline{a}, k\overline{a})_{LR}, & k > 0, \\ (ka, -k\overline{a}, -k\underline{a})_{LR}, & k < 0. \end{cases}$$

The details of proofs are omitted and the interested reader is referred to [6] and [19].

$\widetilde{\mathbf{A}} = [\widetilde{a}_{ij}]_{m\times n}$ is a *fuzzy matrix* if and only if each element of $\widetilde{\mathbf{A}}$ is a fuzzy number, particularly, $\widetilde{a}_{ij} = (a_{ij}, \underline{a_{ij}}, \overline{a_{ij}})_{LR}$. A fuzzy matrix with the LR fuzzy number representation of entries, can be characterized by three crisp matrices $\mathbf{A} = [a_{ij}]_{m\times n}$, $\underline{\mathbf{A}} = [\underline{a_{ij}}]_{m\times n}$ and $\overline{\mathbf{A}} = [\overline{a_{ij}}]_{m\times n}$. In this sense, it is proper to regard $\widetilde{\mathbf{A}} = [\widetilde{a}_{ij}]_{m\times n}$ as the *fuzzy LR matrix* and denote it by $\widetilde{\mathbf{A}} = [\mathbf{A}, \underline{\mathbf{A}}, \overline{\mathbf{A}}]_{LR}$.

Given two fuzzy matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$. Any binary operation $\odot$ between two LR fuzzy matrices, is defined as follows:

$\widetilde{\mathbf{A}}_{m\times n} \oplus \widetilde{\mathbf{B}}_{m\times n} = [\mathbf{A}, \underline{\mathbf{A}}, \overline{\mathbf{A}}]_{LR} \oplus [\mathbf{B}, \underline{\mathbf{B}}, \overline{\mathbf{B}}]_{LR} = [\mathbf{A}+\mathbf{B}, \underline{\mathbf{A}}+\underline{\mathbf{B}}, \overline{\mathbf{A}}+\overline{\mathbf{B}}]_{LR}$;

$\widetilde{\mathbf{A}}_{m\times n} \otimes \widetilde{\mathbf{B}}_{n\times s} = [\mathbf{A}, \underline{\mathbf{A}}, \overline{\mathbf{A}}]_{LR} \otimes [\mathbf{B}, \underline{\mathbf{B}}, \overline{\mathbf{B}}]_{LR} = [\sum_{k=1}^{n} \widetilde{a}_{ik} \otimes \widetilde{b}_{kj}]_{m\times s}$.

3. Ranking fuzzy numbers

Up to now, a lot of progress has been made in the study of the ordering of fuzzy numbers from different angles [5],[7] and [9] to map each fuzzy number into a number of the real line and thus realize a comparison and ordering of fuzzy numbers.

Ranking function is one of the most convenient approach to describe the order relation of fuzzy numbers.

Definition 1. A mapping $R : F_\tau(\mathbb{R}) \to \mathbb{R}$ is called a *ranking function* if for any $\widetilde{a}, \widetilde{b} \in F_\tau(\mathbb{R})$ it satisfies:

$\widetilde{a} \succeq_R \widetilde{b}$ if and only if $R(\widetilde{a}) \geq R(\widetilde{b})$;

$\widetilde{a} \succ_R \widetilde{b}$ if and only if $R(\widetilde{a}) > R(\widetilde{b})$;

$\widetilde{a} \approx_R \widetilde{b}$ if and only if $R(\widetilde{a}) = R(\widetilde{b})$.

Note that $\widetilde{a} \preceq_R \widetilde{b}$ if and only if $\widetilde{b} \succeq_R \widetilde{a}$.

The following proposition can be easily verified by using the above definition.

Proposition 1. *If R is a linear ranking function, that is*

$$R(k\widetilde{a} \oplus \widetilde{b}) = kR(\widetilde{a}) + R(\widetilde{b}), \quad \text{for all } \widetilde{a}, \widetilde{b} \in F_\tau(\mathbb{R}), \text{ and } k \in \mathbb{R}, \tag{4}$$

then, for any $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ and $\widetilde{d}$ in $F_\tau(\mathbb{R})$ the following results hold:

$\widetilde{a} \succeq_R \widetilde{b}$ if and only if $\widetilde{a} \ominus \widetilde{b} \succeq_R \widetilde{0}$ if and only if $-\widetilde{b} \succeq_R -\widetilde{a}$;

$\widetilde{a} \succeq_R \widetilde{b}$ and $\widetilde{c} \succeq_R \widetilde{d}$ deduces $\widetilde{a} \oplus \widetilde{c} \succeq_R \widetilde{b} \oplus \widetilde{d}$.

Throughout this paper, the considered ranking function R is that introduced in [7] given by

$$R(\widetilde{a}) = \frac{1}{2}\int_{[0,1]} (\underline{a}(\alpha) + \overline{a}(\alpha))\, d\alpha, \quad \textit{for any } \widetilde{a} \in F_\tau(\mathbb{R}), \tag{5}$$

where $\underline{a}(\alpha)$ and $\overline{a}(\alpha)$ are respectively the left- and right-hand endpoints of $[\widetilde{a}]_\alpha$.

In view of $\widetilde{a} \in F_\tau(\mathbb{R})$ with parameterizations given by $(a, \underline{a}, \overline{a})_{LR}$ and (3), the ranking function (5) is reduced to

$$R(\widetilde{a}) = a + \frac{1}{4}(\overline{a} - \underline{a}). \tag{6}$$

Furthermore, in accordance with the foregoing ranking function and Definition 1, for any $\widetilde{a},\ \widetilde{b} \in F_\tau(\mathbb{R})$

$$\widetilde{a} \succeq_R \widetilde{b} \quad \textit{if and only if} \quad a + \frac{1}{4}(\overline{a} - \underline{a}) \geq b + \frac{1}{4}(\overline{b} - \underline{b}). \tag{7}$$

Ordering fuzzy numbers by ranking function is the key to fuzzy linear programming problems and provides a proper foundation for solving them via classical approaches.

4. Formulation of fuzzy linear programming problem

Generally, a fuzzy linear programming problem is to find minimum or maximum of uncertain objective function under some nondeterministic constraints. Here, a class of such problems will be considered in which both objective function and constraints are with triangular fuzzy numbers. The general form of a fuzzy linear programming problem, here can be stated as:

Problem FLP: Minimize

$$\widetilde{\mathbf{c}} \otimes \mathbf{x},$$

subject to

$$\widetilde{\mathbf{A}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{b}},$$
$$\mathbf{x} \geq 0,$$

where $\mathbf{x} \in \mathbb{R}^n$, $\widetilde{\mathbf{c}} \in F_\tau(\mathbb{R}^n)$, $\widetilde{\mathbf{A}} \in F_\tau(\mathbb{R}^{m\times n})$ and $\widetilde{\mathbf{b}} \in F_\tau(\mathbb{R}^m)$.

Definition 2. Any $\mathbf{x} \in \mathbb{R}^n$ which satisfies the constraints and non-negativity restrictions of Problem FLP is called a *fuzzy feasible solution*, and $S = \{\mathbf{x} \in \mathbb{R}^n \mid \widetilde{\mathbf{A}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{b}},\ \mathbf{x} \geq 0\}$ denoted as the set of all fuzzy feasible solutions to Problem FLP.

Definition 3. Any $\mathbf{x}^* \in S$ is said to be a *fuzzy optimum solution* to Problem FLP if $\widetilde{\mathbf{c}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{c}} \otimes \mathbf{x}^*$ for all $\mathbf{x} \in S$.

Remark that $\widetilde{\mathbf{c}} \otimes \mathbf{x} \approx_R \widetilde{c}_1 \otimes x_1 \oplus ... \oplus \widetilde{c}_n \otimes x_n$. If no confusion may arise, $\sum$ is written for fuzzy summation $\widetilde{\sum}$. Hence, $\widetilde{\mathbf{c}} \otimes \mathbf{x}$ will be indicated by $\sum_{i=1}^n \widetilde{c}_i \otimes x_i$.

Definition 4. Given $\hat{\mathbf{x}} \in \mathbb{R}^n$ and some fuzzy constraints $\widetilde{\mathbf{p}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{q}}$, where $\widetilde{\mathbf{p}},\ \widetilde{\mathbf{q}} \in F_\tau(\mathbb{R}^n)$. Such constraints are referred to as *binding* or *active constraints* at $\hat{\mathbf{x}}$ if $\widetilde{\mathbf{p}} \otimes \hat{\mathbf{x}} \approx_R \widetilde{\mathbf{q}}$.

Now, consider following conventional linear programming problem which is defined in terms of ranking function R.

<u>Problem RLP:</u> Minimize

$$R(\widetilde{\mathbf{c}})\mathbf{x},$$

subject to

$$\begin{aligned} &R(\widetilde{\mathbf{A}})\mathbf{x} \ \geq R(\widetilde{\mathbf{b}}),\\ &\mathbf{x} \geq 0, \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$, $R(\widetilde{\mathbf{c}}) = (R(\widetilde{c}_1), ..., R(\widetilde{c}_n)) \in \mathbb{R}^n$, $R(\widetilde{\mathbf{A}}) = [R(\widetilde{a}_{ij})] \in \mathbb{R}^{m\times n}$ and $R(\widetilde{\mathbf{b}}) = (R(\widetilde{b}_1), ..., R(\widetilde{b}_m)) \in \mathbb{R}^m$.

Let $S_R = \{\mathbf{x} \in \mathbb{R}^n \mid R(\widetilde{\mathbf{A}})\mathbf{x} \ \geq R(\widetilde{\mathbf{b}}),\ \mathbf{x} \geq 0\}$ be the set of all crisp feasible solutions to Problem RLP.
The next theorem gives the relationship between Problem FLP and Problem RLP.

Theorem 1. *Problem FLP is equivalent to Problem RLP.*

Proof. Firstly, it should be shown that $S = S_R$, says, each fuzzy feasible solution to Problem FLP corresponds to a feasible solution to Problem RLP and vise versa. Then, the proof will be completed by verifying the correspondence between a fuzzy optimum solution to Problem FLP and an optimum solution to Problem RLP, provided they exist.

Suppose that $\mathbf{x} \in S$. Then, fuzzy feasibility requires that

$$\widetilde{\mathbf{A}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{b}}, \quad \mathbf{x} \geq 0.$$

Clearly, the foregoing inequalities hold if and only if

$$\sum_{j=1}^{n} \widetilde{a}_{ij} \otimes x_j \succeq_R \widetilde{b}_i,\ i = 1, ..., m, \quad x_j \geq 0,\ j = 1, ..., n,$$

or

$$\sum_{j=1}^{n} (a_{ij}, \underline{a}_{ij}, \overline{a}_{ij})_{LR} \otimes x_j \succeq_R (b_i, \underline{b}_i, \overline{b}_i)_{LR},\ i = 1, ..., m,\ x_j \geq 0,\ j = 1, ..., n,$$

$$\sum_{j=1}^{n} (a_{ij}x_j, \underline{a}_{ij}x_j, \overline{a}_{ij}x_j)_{LR} \succeq_R (b_i, \underline{b}_i, \overline{b}_i)_{LR},\ i = 1, ..., m,\ x_j \geq 0,\ j = 1, ..., n,$$

$$(\sum_{j=1}^{n} a_{ij}x_j, \sum_{j=1}^{n} \underline{a}_{ij}x_j, \sum_{j=1}^{n} \overline{a}_{ij}x_j)_{LR} \succeq_R (b_i, \underline{b}_i, \overline{b}_i)_{LR},\ i = 1, ..., m,$$

$$x_j \geq 0,\ j = 1, ..., n.$$

By virtue of Definition 1, the latter inequalities fulfill if and only if

$$R(\sum_{j=1}^{n} \widetilde{a}_{ij} x_j) \geq R(\widetilde{b}_i),\ i = 1, ..., m, \quad x_j \geq 0,\ j = 1, ..., n.$$

Linearity of R leads to

$$\sum_{j=1}^{n} R(\widetilde{a}_{ij}) x_j \geq R(\widetilde{b}_i),\ i = 1, ..., m, \quad x_j \geq 0,\ j = 1, ..., n,$$

that is

$$R(\widetilde{\mathbf{A}})\mathbf{x} \geq R(\widetilde{\mathbf{b}}), \quad \mathbf{x} \geq 0.$$

Consequently, $\mathbf{x} \in S_R$.

To complete the proof, assume that $\mathbf{x}^*$ is an fuzzy optimum solution to Problem FLP. Then, fuzzy optimality requires that for all $\mathbf{x} \in S$

$$\widetilde{\mathbf{c}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{c}} \otimes \mathbf{x}^*,$$

or

$$\sum_{j=1}^{n} \widetilde{c}_j \otimes x_j \succeq_R \sum_{j=1}^{n} \widetilde{c}_j \otimes {x^*}_j,$$
$$\sum_{j=1}^{n} (c_j, \underline{c}_j, \overline{c}_j)_{LR} \otimes x_j \succeq_R \sum_{j=1}^{n} (c_j, \underline{c}_j, \overline{c}_j)_{LR} \otimes {x^*}_j,$$
$$(\sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n} \underline{c}_j x_j, \sum_{j=1}^{n} \overline{c}_j x_j)_{LR} \succeq_R (\sum_{j=1}^{n} c_j {x^*}_j, \sum_{j=1}^{n} \underline{c}_j {x^*}_j, \sum_{j=1}^{n} \overline{c}_j {x^*}_j)_{LR}.$$

The foregoing inequalities hold if and only if

$$R(\sum_{j=1}^{n} \widetilde{c}_j x_j) \geq R(\sum_{j=1}^{n} \widetilde{c}_j {x^*}_j),$$

or

$$\sum_{j=1}^{n} R(\widetilde{c}_j) x_j \geq \sum_{j=1}^{n} R(\widetilde{c}_j) {x^*}_j,$$

that is

$$R(\widetilde{\mathbf{c}})\mathbf{x} \geq R(\widetilde{\mathbf{c}})\mathbf{x}^*.$$

Follows from the fact that $S = S_R$, the latter inequality holds for all $\mathbf{x} \in S_R$. Hence, $\mathbf{x}^*$ is an optimum solution to Problem RLP. $\square$

5. The Karush-Kuhn-Tucker conditions

Many authors made useful explorations in optimality conditions in fuzzy optimization problems from different angles. Saddle point optimality conditions in fuzzy nonlinear programming problem have been discussed in [17]. The Karush-Kuhn-Tucker conditions for constrained fuzzy minimization problem have been derived in [11] based on the concepts of differentiability, convexity and minimization of a fuzzy mapping. In [16] the Karush-Kuhn-Tucker conditions in an optimization problem with interval-valued objective function have been stated. In contrast to many attempts have been made for the Karush-Kuhn-Tucker optimality conditions in fuzzy optimization problems, these optimality conditions in fuzzy linear programming problems based on comparison of fuzzy numbers by means of ranking function had a high representation of minorities.

Proposition 2. *(Farkas' Lemma) One and only one of the following two crisp systems has a solution.*

$$\text{System } 1: \boldsymbol{A}\boldsymbol{y} \leq 0,\ \ \boldsymbol{y} \leq 0,\ \ \text{and}\ \ \boldsymbol{c}\boldsymbol{y} > 0; \tag{8}$$

$$\text{System } 2: \boldsymbol{w}\boldsymbol{A} \leq \boldsymbol{c},\ \ \text{and}\ \ \boldsymbol{w} \geq 0; \tag{9}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^n$ *and* $\boldsymbol{w} \in \mathbb{R}^m$.

Proof. See [1]. □

The aim here is to derive the necessary and sufficient optimality conditions for Problem FLP. The following theorem gives a characterization of optimality which does not seem to be in conflict with the crisp one.

Theorem 2. *(K.K.T conditions) Let* $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \widetilde{\boldsymbol{A}} \otimes \boldsymbol{x} \succeq_R \widetilde{\boldsymbol{b}},\ \boldsymbol{x} \geq 0\}$ *be non-empty. Then,* $\boldsymbol{x}^* \in S$ *is an optimum solution to Problem FLP if and only if* $(\boldsymbol{x}^*, \boldsymbol{w}, \boldsymbol{v}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ *is a solution to the following system:*

$$\widetilde{\boldsymbol{A}} \otimes \boldsymbol{x} \succeq_R \widetilde{\boldsymbol{b}}, \quad \boldsymbol{x} \geq 0; \tag{10}$$

$$\boldsymbol{w} \otimes \widetilde{\boldsymbol{A}}\ \oplus \boldsymbol{v} \approx_R \widetilde{\boldsymbol{c}}, \quad \boldsymbol{w} \geq 0,\ \ \boldsymbol{v} \geq 0; \tag{11}$$

$$\boldsymbol{w} \otimes (\widetilde{\boldsymbol{A}} \otimes \boldsymbol{x}\ \ominus \widetilde{\boldsymbol{b}}) \approx_R \widetilde{\boldsymbol{0}}, \quad ,\ \boldsymbol{v}\boldsymbol{x} = 0. \tag{12}$$

Proof. Let's turn to Problem FLP where $\widetilde{\mathbf{c}} \otimes \mathbf{x}$ is minimized on S. Suppose that $\mathbf{x}^* \in S$. Hence,

$$\widetilde{\mathbf{A}} \otimes \mathbf{x}^* \succeq_R \widetilde{\mathbf{b}}, \quad \mathbf{x}^* \geq 0. \tag{13}$$

In view of the ranking function R, the above inequalities hold if and only if

$$R(\widetilde{\mathbf{A}} \otimes \mathbf{x}^*) \geq R(\widetilde{\mathbf{b}}), \quad \mathbf{x}^* \geq 0,$$

or

$$R(\widetilde{\mathbf{A}})\mathbf{x}^* \geq R(\widetilde{\mathbf{b}}), \quad \mathbf{x}^* \geq 0. \tag{14}$$

Let the set of latter inequalities which are binding at $\mathbf{x}^*$, be denoting by

$$R(\widetilde{\mathbf{G}})\mathbf{x} \geq R(\widetilde{\mathbf{g}}), \tag{15}$$

equivalently,

$$R(\widetilde{\mathbf{G}} \otimes \mathbf{x}) \geq R(\widetilde{\mathbf{g}}),$$

or

$$\widetilde{\mathbf{G}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{g}}. \tag{16}$$

If $\mathbf{x}^*$ is an optimum solution to Problem FLP then, there cannot exists any improving feasible direction $\mathbf{d}$ at $\mathbf{x}^*$. That is, a direction $\mathbf{d}$ cannot be found fulfilling

$$\widetilde{\mathbf{G}} \otimes \mathbf{d} \succeq_R \widetilde{\mathbf{0}}, \quad \text{and} \qquad \widetilde{\mathbf{c}} \otimes \mathbf{d} \prec_R \widetilde{\mathbf{0}},$$

equivalently,

$$R(\widetilde{\mathbf{G}} \otimes \mathbf{d}) \geq \mathbf{0}, \quad \text{and} \qquad R(\widetilde{\mathbf{c}} \otimes \mathbf{d}) < \mathbf{0},$$

or

$$R(\widetilde{\mathbf{G}})\mathbf{d} \geq \mathbf{0}, \quad \text{and} \qquad R(\widetilde{\mathbf{c}})\mathbf{d} < \mathbf{0}. \tag{17}$$

If this is not the case, moving along the ray $\mathbf{x}^* + \lambda\mathbf{d}$ decreases the objective function value because of $\widetilde{\mathbf{c}} \otimes \mathbf{d} \prec_R \widetilde{\mathbf{0}}$ while the feasibility of $\mathbf{x}^* + \lambda\mathbf{d}$ for any $\lambda > 0$ is deduced by

$$\widetilde{\mathbf{G}} \otimes (\mathbf{x}^* + \lambda\mathbf{d}) \approx_R \widetilde{\mathbf{G}} \otimes \mathbf{x}^* \oplus \ \lambda\widetilde{\mathbf{G}} \otimes \mathbf{d} \approx_R \widetilde{\mathbf{g}} \oplus \ \lambda\widetilde{\mathbf{G}} \otimes \mathbf{d} \succeq_R \widetilde{\mathbf{g}}.$$

The preceding discussion contradicts optimality of $\mathbf{x}^*$ and hence, system (17) cannot have a solution. Furthermore, as follows from Proposition 2, it can be deduced there exists a $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{u}R(\widetilde{\mathbf{G}}) = R(\widetilde{\mathbf{c}}), \quad \text{and} \qquad \mathbf{u} \geq \mathbf{0}, \tag{18}$$

equivalently,

$$\mathbf{u} \otimes \widetilde{\mathbf{G}} \approx_R \widetilde{\mathbf{c}}, \quad \text{and} \qquad \mathbf{u} \geq \mathbf{0}. \tag{19}$$

The latter equality may be read in terms of (14) under the assumptions of binding inequalities at $\mathbf{x}^*$. To do this end, denote the i-th row of $\widetilde{\mathbf{A}}$, by $\widetilde{\mathbf{A}}_i$ for $i = 1, ..., m$ and a vector of zeros except for a 1 in the j-th position by $\mathbf{e}_j$. Indicate two sets of indices as follows:

$$I = \{i : R(\widetilde{\mathbf{A}}_i)\mathbf{x}^* = R(\widetilde{\mathbf{b}}_i)\} = \{i : \widetilde{\mathbf{A}}_i \otimes \mathbf{x}^* \approx_R \widetilde{\mathbf{b}}_i\}, \tag{20}$$

and

$$J = \{j : \mathbf{x}_j^* = 0\}. \tag{21}$$

Taking $\mathbf{u} = (w_i \text{ for } i \in I,\ v_j \text{ for } j \in J)$ and respecting to (18), the next result follows immediately

$$R(\sum_{i \in I} w_i \otimes \widetilde{\mathbf{A}}_i \ \oplus \sum_{j \in J} v_j \mathbf{e}_j) = R(\widetilde{\mathbf{c}}), \quad w_i \geq 0,\ i \in I, \quad v_j \geq 0,\ j \in J,$$

or

$$\sum_{i\in I} w_i R(\widetilde{\mathbf{A}}_i) \ + \sum_{j\in J} v_j \mathbf{e}_j = R(\widetilde{\mathbf{c}}), \quad w_i \geq 0,\ i\in I, \quad v_j \geq 0,\ j\in J, \tag{22}$$

equivalently,

$$\sum_{i\in I} w_i \otimes \widetilde{\mathbf{A}}_i \ \oplus \sum_{j\in J} v_j \mathbf{e}_j \approx_R \widetilde{\mathbf{c}}, \quad w_i \geq 0,\ i\in I, \quad v_j \geq 0,\ j\in J. \tag{23}$$

It is not hard to see that the obtained system of equations is another form of (19). Equations (23) together with (13) are called the Karush-Kuhn-Tucker optimality conditions for Problem FLP.

One direction of the assertion has been already proved so far. On the other words, the Karush-Kuhn-Tucker conditions must necessarily hold at an optimum solution $\mathbf{x}^*$ to Problem FLP.

Next, it is shown that if $\mathbf{x}^*$ is a feasible solution of Problem FLP fulfilling the Karush-Kuhn-Tucker conditions then it will be optimum solution as well. Toward this end, assume that equations (23) hold at $\mathbf{x}^*$ with the two sets of indices I and J given respectively by (20) and (21).

Now, for any $\mathbf{x} \in S$ one gets

$$R(\widetilde{\mathbf{c}} \otimes \mathbf{x} \ \ominus \widetilde{\mathbf{c}} \otimes \mathbf{x}^*) = R(\widetilde{\mathbf{c}} \otimes \mathbf{x}) - R(\widetilde{\mathbf{c}} \otimes \mathbf{x}^*) = R(\widetilde{\mathbf{c}})\mathbf{x} - R(\widetilde{\mathbf{c}})\mathbf{x}^*.$$

Substituting $R(\widetilde{\mathbf{c}})$ from the right-hand side of (22) into the foregoing relation, results in

$$\begin{aligned} R(\widetilde{\mathbf{c}} \otimes \mathbf{x} \ \ominus \widetilde{\mathbf{c}} \otimes \mathbf{x}^*) \quad &= \quad (\sum_{i\in I} w_i R(\widetilde{\mathbf{A}}_i)\mathbf{x} \ + \sum_{j\in J} v_j \mathbf{e}_j \mathbf{x}) \\ &- \quad (\sum_{i\in I} w_i R(\widetilde{\mathbf{A}}_i)\mathbf{x}^* \ + \sum_{j\in J} v_j \mathbf{e}_j \mathbf{x}^*). \end{aligned}$$

Since $R(\widetilde{\mathbf{A}}_i)\mathbf{x}^* = R(\mathbf{b}_i)$, for $i \in I$ and $v_j \mathbf{e}_j \mathbf{x}^* = 0$, for $j \in J$, it is reduced to

$$R(\widetilde{\mathbf{c}} \otimes \mathbf{x} \ \ominus \widetilde{\mathbf{c}} \otimes \mathbf{x}^*) = \sum_{i\in I} w_i (R(\widetilde{\mathbf{A}}_i)\mathbf{x} - R(\widetilde{\mathbf{b}}_i)) + \sum_{j\in J} v_j \mathbf{e}_j \mathbf{x}.$$

Furthermore, feasibility of $\mathbf{x}$ implies that

$$R(\widetilde{\mathbf{c}} \otimes \mathbf{x} \ \ominus \widetilde{\mathbf{c}} \otimes \mathbf{x}^*) \geq 0,$$

equivalently,

$$\widetilde{\mathbf{c}} \otimes \mathbf{x} \ \ominus \widetilde{\mathbf{c}} \otimes \mathbf{x}^* \succeq_R \widetilde{\mathbf{0}},$$

or

$$\widetilde{\mathbf{c}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{c}} \otimes \mathbf{x}^*. \tag{24}$$

The above inequality holds true for any $\mathbf{x} \in S$. Thus, $\mathbf{x}^*$ is an optimum solution to Problem FLP.

Consequently, the Karush-Kuhn-Tucker conditions are both necessary and sufficient for the optimality of $\mathbf{x}^*$.

To deal conveniently with conditions (23), let's put $\mathbf{w} = (w_1, ..., w_m) \geq 0$ and $\mathbf{v} = (v_1, ..., v_n) \geq 0$ where those w_i's and v_j's are corresponding with the non-binding constraints must be zeros. This allows conditions (23) to be written as

$$\mathbf{w} \otimes \widetilde{\mathbf{A}} \ \oplus \mathbf{vI} \approx_R \widetilde{\mathbf{c}}, \quad \mathbf{w} \geq 0, \quad \mathbf{v} \geq 0,$$

or simply,

$$\mathbf{w} \otimes \widetilde{\mathbf{A}} \ \oplus \mathbf{v} \approx_R \widetilde{\mathbf{c}}, \quad \mathbf{w} \geq 0, \quad \mathbf{v} \geq 0.$$

In summary, the above arguments show that the Karush-Kuhn-Tucker conditions can be presented by three conditions which are usually referred to as *primal feasibility*, *dual feasibility* and *complementary slackness* and they are mathematically stated as requiring a solution $(\mathbf{x}^*, \mathbf{w}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ to the following system:

$\widetilde{\mathbf{A}} \otimes \mathbf{x} \succeq_R \widetilde{\mathbf{b}},\ \mathbf{x} \geq 0;\ (Primal\ feasibility\ condition)$

$\mathbf{w} \otimes \widetilde{\mathbf{A}} \ \oplus \mathbf{v} \approx_R \widetilde{\mathbf{c}},\ \mathbf{w} \geq 0,\ \mathbf{v} \geq 0;\ (Dual\ feasibility\ condition)$

$\mathbf{w} \otimes (\widetilde{\mathbf{A}} \otimes \mathbf{x} \ \ominus \widetilde{\mathbf{b}}) \approx_R \widetilde{\mathbf{0}},\ \mathbf{vx} = 0.\ (Complementary\ slackness\ condition)$ □

6. Numerical example

Consider Problem FLP as follows:

$$\begin{array}{ll} \textit{Minimize} & (-1,1,1)_{LR} \otimes x_1 \oplus (-3,1,1)_{LR} \otimes x_2, \\ \textit{subject to} & (1,1,2)_{LR} \otimes x_1 \oplus (-2,2,1)_{LR} \otimes x_2 \succeq_R (-4,2,1)_{LR}, \\ & (-1,2,1)_{LR} \otimes x_1 \oplus (-1,1,2)_{LR} \otimes x_2 \succeq_R (-4,1,3)_{LR}, \\ & x_1, x_2 \geq 0. \end{array}$$

Let the point $(x_1, x_2) = (0, 0)$ be supposed to be an optimum one. To verify this assertion, it will be examined whether or not $(x_1, x_2) = (0, 0)$ satisfies conditions (10)-(12).

Obviously, primal feasibility condition (10) holds at $(x_1, x_2) = (0, 0)$ so that

$$(1,1,2)_{LR} \otimes 0 \oplus (-2,2,1)_{LR} \otimes 0 \succ_R (-4,2,1)_{LR},$$
$$(-1,2,1)_{LR} \otimes 0 \oplus (-1,1,2)_{LR} \otimes 0 \succ_R (-4,1,3)_{LR},$$

where none of the above constraints is binding. This leads to $(w_1, w_2) = (0, 0)$, in order to satisfy complementary slackness condition (12). Dual feasibility condition (11) with together $\mathbf{w} = \mathbf{0}$ imply that

$$\widetilde{\mathbf{0}} \oplus (v_1, v_2) \approx_R ((-1,1,1)_{LR}, (-3,1,1)_{LR}),$$

or

$$(v_1, 0, 0)_{LR} \approx_R (-1,1,1)_{LR}, \quad (v_2, 0, 0)_{LR} \approx_R (-3,1,1)_{LR}.$$

By Definition 1, the latter equalities turn out equivalent to

$$v_1 = -1 + \frac{1}{4}(1-1), \quad v_2 = -3 + \frac{1}{4}(1-1).$$

The negative vector $(v_1, v_2) = (-1, -3)$ violates the non-negativity restriction of $\mathbf{v}$ given in (11). Hence, the point $(x_1, x_2) = (0, 0)$ could not be an optimum solution to the considered Problem FLP.

Once again the above argument is illustrated with the point $(x_1, x_2) = (\frac{75}{12}, \frac{31}{12})$. Follows from the complementary slackness condition and $x_1, x_2 > 0$, one gets immediately $(v_1, v_2) = (0, 0)$. Then, considering dual feasibility condition, $\mathbf{w} = (w_1, w_2)$ can be obtained by solving $\mathbf{w} \otimes \widetilde{\mathbf{A}} \approx_R \widetilde{\mathbf{c}}$, that is

$$\begin{cases} w_1(1,1,2)_{LR} \oplus w_2(-1,2,1)_{LR} \approx_R (-1,1,1)_{LR}, \\ w_1(-2,2,1)_{LR} \oplus w_2(-1,1,2)_{LR} \approx_R (-3,1,1)_{LR}, \end{cases}$$

or

$$\begin{cases} (w_1 - w_2, w_1 + 2w_2, 2w_1 + w_2)_{LR} \approx_R (-1,1,1)_{LR}, \\ (-2w_1 - w_2, 2w_1 + w_2, w_1 + 2w_2)_{LR} \approx_R (-3,1,1)_{LR}, \end{cases}$$

equivalently,

$$\begin{cases} \frac{5}{4}w_1 - \frac{5}{4}w_2 = -1, \\ -\frac{9}{4}w_1 - \frac{3}{4}w_2 = -3, \end{cases}$$

hence, $(w_1, w_2) = (\frac{1}{20}, \frac{17}{20}) > 0$. On the other hand, one can verify easily that

$$(1,1,2)_{LR}\frac{75}{12} \oplus (-2,2,1)_{LR}\frac{31}{12} \approx_R (-4,2,1)_{LR},$$
$$(-1,2,1)_{LR}\frac{75}{12} \oplus (-1,1,2)_{LR}\frac{31}{12} \approx_R (-4,1,3)_{LR}.$$

This means that $\widetilde{\mathbf{A}} \otimes \mathbf{x} \approx_R \widetilde{\mathbf{b}}$ or $\widetilde{\mathbf{A}} \otimes \mathbf{x} \ominus \widetilde{\mathbf{b}} \approx_R \widetilde{\mathbf{0}}$. Therefore, the complementary slackness condition $\mathbf{w} \otimes (\widetilde{\mathbf{A}} \otimes \mathbf{x} \ominus \widetilde{\mathbf{b}}) \approx_R \widetilde{\mathbf{0}}$ holds.

Consequently, the Karush-Kuhn-Tucker conditions (10)-(12) hold true for $(x_1, x_2) = (\frac{75}{12}, \frac{31}{12})$. Thus, this point is really an optimum solution to the considered Problem FLP with optimal objective value

$$\widetilde{\mathbf{Z}} \approx_R (-1,1,1)_{LR}\frac{75}{12} \oplus (-3,1,1)_{LR}\frac{31}{12} \approx_R (-14, \frac{53}{6}, \frac{53}{6})_{LR}.$$

7. Conclusion

The argument of this paper explained that the necessary and sufficient optimality conditions for Problem FLP can be given as a characterization of optimality which does not seem to be in conflict with the so-called Karush-Kuhn-Tucker conditions for a crisp linear programming problem. Ranking function played the main role in this scenario. The implication of this result is that it is theoretically possible to investigate the fuzzy optimality conditions as same as the crisp ones. Although here a specific ranking function was just considered to explore optimality conditions, in fact, one still can derive the same results using any linear ranking function as well.

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