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CALCULATION OF SOME TOPOLOGICAL INDICES OF SPLICES AND LINKS OF GRAPHS†

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Abstract. Explicit formulas are given for the first and second Zagreb index, degree-distance and Wiener-type invariants of splice and link of graphs. As a consequence, the first and second Zagreb coindex of these classes of composite graphs are also computed.

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1. Introduction

Throughout this paper graph means simple connected graphs. The distance $d_G(u,v)$ between the vertices u and v of a graph G is defined as the length of a shortest path connecting u and v. Let $d(G,k)$ be the number of pairs of vertices of G that are at distance k, λ a real number, and $W_\lambda(G) = \sum_{k\geq 1} d(G,k)k^\lambda$. $W_\lambda(G)$ is called the Wiener-type invariant of G associated to real number λ, see [7, 11] for details. Note that $d(G,0)$ and $d(G,1)$ represent the number of vertices and edges, respectively. The case of $\lambda = 1$ is called the classical Wiener index [25]. The quantities $WW = \frac{1}{2}[W_1 + W_2]$ and $TSZ = \frac{1}{6}W_3 + \frac{1}{2}W_2 + \frac{1}{3}W_1$ are the so-called hyper-Wiener index and Tratch−Stankevich−Zefirov index [7].

Suppose G and H are graphs with disjoint vertex sets. Following Došlić [5], for given vertices $y \in V(G)$ and $z \in V(H)$ a splice of G and H by vertices y and z, $(G \cdot H)(y,z)$, is defined by identifying the vertices y and z in the union of G and H. Similarly, a link of G and H by vertices y and z is defined as the graph $(G \sim H)(y,z)$ obtained by joining y and z by an edge in the union of these graphs.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestic, [8]. They are defined as $M_1(G) = \sum_{u\in V(G)} deg_G(u)^2$

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and $M_2(G) = \sum_{uv\in E(G)} deg_G(u)deg_G(v)$. As the sums involved run over the edges of the complement of G, such quantities were called Zagreb coindices. More formally, the first Zagreb coindex of a graph G is defined as $\bar{M}_1(G) = \sum_{uv\notin E(G)}[deg_G(u) + deg_G(v)]$, and the second Zagreb coindex of a graph G is given by $\bar{M}_2(G) = \sum_{uv\notin E(G)} deg_G(u)deg_G(v)$. The reader should note that Zagreb coindices of G are not Zagreb indices of $\bar{G}$; the defining sums run over $E(\bar{G})$, but the degrees are with respect to G. We encourage the reader to consult papers [6, 9, 14, 22, 27] and references therein for more information on Zagreb indices of graphs.

In some recent papers Dobrynin and Kochetova [4] and Gutman [10] introduced a new graph invariant defined as follows: the degree-distance of a vertex x, denoted by $D'(x)$, is defined as $D'(x) = D(x)deg_G(x)$, where $deg_G(x)$ is the degree of x, $D(x) = \sum_{y\in V(G)} d_G(x, y)$ and the degree-distance of G, denoted by $D'(G)$, is

$$\begin{aligned} D'(G) &= \sum_{x\in V(G)} D'(x) = \sum_{x\in V(G)} D(x)deg_G(x) \\ &= \frac{1}{2}\sum_{x,y\in V(G)} d(x, y)[deg_G(x) + deg_G(y)]. \end{aligned}$$

Throughout this paper our other notations are standard and taken mainly from [3, 24].

2. Main Results

In this section, formulas for the Zagreb index, Zagreb coindex, degree-distance and Wiener-type invariants of the splices and links of graphs are presented. The interested readers for more information on topological indices of graph operations can be referred to the papers $[1, 2, 12-21, 23, 26]$ and their references. The following simple lemma is crucial in this paper:

Lemma 2.1. Suppose G_1 and G_2 are connected graphs. Then the following are hold:

a) $|E((G_1 \cdot G_2)(y, z))| = |E(G_1)| + |E(G_2)|$,

b) $|V((G_1 \cdot G_2)(y, z))| = |V(G_1)| + |V(G_2)| - 1$,

c) $|E((G_1 \sim G_2)(y, z))| = |E(G_1)| + |E(G_2)| + 1$,

d) $|V((G_1 \sim G_2)(y, z))| = |V(G_1)| + |V(G_2)|$,

e) If $u, y \in V(G_1)$ and $u \neq y$ then $deg_{(G_1\cdot G_2)(y,z)}(u) = deg_{G_1}(u)$. If $u, z \in V(G_2)$ and $u \neq z$ then $deg_{(G_1\cdot G_2)(y,z)}(u) = deg_{G_2}(u)$. Moreover, $deg_{(G_1\cdot G_2)(y,z)}(y) = deg_{G_1}(y) + deg_{G_2}(z)$ and $deg_{(G_1\cdot G_2)(y,z)}(z) = deg_{G_1}(y) + deg_{G_2}(z)$.

f) If $u, y \in V(G_1)$ and $u \neq y$ then $deg_{(G_1 \sim G_2)(y,z)}(u) = deg_{G_1}(u)$ and if $u, z \in V(G_2)$ and $u \neq z$ then $deg_{(G_1 \sim G_2)(y,z)}(u) = deg_{G_2}(u)$.
Moreover, $deg_{(G_1 \sim G_2)(y,z)}(y) = deg_{G_1}(y)+1$ and $deg_{(G_1 \sim G_2)(y,z)}(z) = 1+deg_{G_2}(z)$.

g) If $V(G_1) = \{u_1, \dots, u_n\}$ and $V(G_2) = \{v_1, \dots, v_m\}$ then

$$\begin{aligned} d_{(G_1 \cdot G_2)(y,z)}(u_i, u_j) &= d_{G_1}(u_i, u_j), \\ d_{(G_1 \cdot G_2)(y,z)}(u_i, v_j) &= d_{G_1}(u_i, y) + d_{G_2}(v_j, z), \\ d_{(G_1 \cdot G_2)(y,z)}(v_i, v_j) &= d_{G_2}(v_i, v_j). \end{aligned}$$

and

$$\begin{aligned} d_{(G_1 \sim G_2)(y,z)}(u_i, u_j) &= d_{G_1}(u_i, u_j), \\ d_{(G_1 \sim G_2)(y,z)}(u_i, v_j) &= d_{G_1}(u_i, y) + d_{G_2}(v_j, z) + 1, \\ d_{(G_1 \sim G_2)(y,z)}(v_i, v_j) &= d_{G_2}(v_i, v_j). \end{aligned}$$

Proof. The proof is trivial and so omitted. □

In the following lemma, the set of all neighbors of z in G is denoted by $N_1(z)$.

Lemma 2.2. If G_1 and G_2 are connected graphs then $M_1((G_1 \cdot G_2)(y,z)) = M_1(G_1) + M_1(G_2) + 2deg_{G_1}(y)deg_{G_2}(z)$ and $M_2((G_1 \cdot G_2)(y,z)) = M_2(G_1) + M_2(G_2) + deg_{G_2}(z)(\sum_{u \in N_1(y)} deg_{G_1}(u)) + deg_{G_1}(y)(\sum_{v \in N_1(z)} deg_{G_2}(v))$.

Proof. By definition of the first Zagreb index, one can prove the first part of the lemma. To prove the second part, we can see that:

$$\begin{aligned} M_2((G_1 \cdot G_2)(y,z)) &= \sum_{uv \in E(G_1); u,v \neq y} deg_{G_1}(u)deg_{G_1}(v) \\ &+ \sum_{uv \in E(G_2); u,v \neq z} deg_{G_2}(u)deg_{G_2}(v) \\ &+ \sum_{\substack{uv \in E(G_1) \\ u=y \\ v \in V(G_1)}} (deg_{G_1}(y) + deg_{G_2}(z))deg_{G_1}(v) \\ &+ \sum_{\substack{uv \in E(G_2) \\ u=z \\ v \in V(G_2)}} (deg_{G_1}(y) + deg_{G_2}(z))deg_{G_2}(v) \\ &= M_2(G_1) + M_2(G_2) + deg_{G_2}(z)(\sum_{u \in N_1(y)} deg_{G_1}(u)) \\ &+ deg_{G_1}(y)(\sum_{v \in N_1(z)} deg_{G_2}(v)), \end{aligned}$$

which completes our argument. □

Corollary 2.3. With notation of Lemma 2.2, $\bar{M}_1((G_1 \cdot G_2)(y,z)) = \bar{M}_1(G_1) + \bar{M}_1(G_2) + 2\ (|V(G_2)||E(G_1)| + |E(G_2)||V(G_1)| - |E(G_1)| - deg_{G_1}(y)deg_{G_2}(z)$

$-\ |E(G_2)|)$ and $\bar{M}_2((G_1 \cdot G_2)(y,z)) = \bar{M}_2(G_1) + \bar{M}_2(G_2) + 4|E(G_1)||E(G_2)| - deg_{G_2}(z)\sum_{u\in N_1(y)} deg_{G_1}(u) - deg_{G_1}(y)\sum_{v\in N_1(z)} deg_{G_2}(v) - deg_{G_1}(y)deg_{G_2}(z)$.

Proof. The proof is straightforward and follows from $\bar{M}_1(G) = 2|E(G)|(|V(G)|-1) - M_1(G)$ and $\bar{M}_2(G) = 2|E(G)|^2 - M_2(G) - \frac{1}{2}M_1(G)$. □

Lemma 2.4. If G_1 and G_2 are connected graphs then $M_1((G_1 \sim G_2)(y,z)) = M_1(G_1) + M_1(G_2) + 2(deg_{G_1}(y) + deg_{G_2}(z) + 1)$ and $M_2((G_1 \sim G_2)(y,z)) = \sum_{u\in N_1(y)} deg_{G_1}(u) + \sum_{v\in N_1(z)} deg_{G_2}(v) + (deg_{G_1}(y) + 1)(1 + deg_{G_2}(z)) + M_2(G_1) + M_2(G_2)$.

Proof. The first part is an easy consequence of definition and for the second part, we have:

$$\begin{aligned}
M_2((G_1 \sim G_2)(y,z)) &= \sum_{uv\in E(G_1);u,v\neq y} deg_{G_1}(u)deg_{G_1}(v)\\
&+ \sum_{uv\in E(G_2);u,v\neq z} deg_{G_2}(u)deg_{G_2}(v)\\
&+ \sum_{\substack{uv\in E(G_1)\\ u=y\\ v\in V(G_1)}} (deg_{G_1}(y)+1)deg_{G_1}(v)\\
&+ \sum_{\substack{uv\in E(G_2)\\ u=z\\ v\in V(G_2)}} (1+deg_{G_2}(z))deg_{G_2}(v)\\
&+ (deg_{G_1}(y)+1)(1+deg_{G_2}(z))\\
&= M_2(G_1) + M_2(G_2) + \sum_{u\in N_1(y)} deg_{G_1}(u)\\
&+ \sum_{v\in N_1(z)} deg_{G_2}(v) + (deg_{G_1}(y)+1)(1+deg_{G_2}(z)),
\end{aligned}$$

proving the lemma. □

Corollary 2.5. By the notation of Lemma 2.4, $\bar{M}_1((G_1 \sim G_2)(y,z)) = \bar{M}_1(G_1) + \bar{M}_1(G_2) + 2|V(G_2)||E(G_1)| + 2|E(G_2)||V(G_1)| + 2|V(G_1)| + 2|V(G_2)| - 2(deg_{G_1}(y) + deg_{G_2}(z) + 1)$ and $\bar{M}_2((G_1 \sim G_2)(y,z)) = \bar{M}_2(G_1) + \bar{M}_2(G_2) + 4|E(G_2)| + 4|E(G_1)|(|E(G_2)| + 1) - \sum_{u\in N_1(y)} deg_{G_1}(u) - \sum_{v\in N_1(z)} deg_{G_2}(v) - deg_{G_1}(y)deg_{G_2}(z) - 2deg_{G_1}(y) - 2deg_{G_2}(z)$. □

In what follows, the degree distance of splices and links of graphs are computed.

Lemma 2.6. If G_1 and G_2 are connected graphs then the degree-distance of splices and links are computed as follows:

$$\begin{aligned}
D'((G_1 \cdot G_2)(y,z)) &= (|V(G_2)|-1)\sum_{y\neq u\in V(G_1)} deg_{G_1}(u)d_{G_1}(u,y)\\
&+ (|V(G_1)|-1)\sum_{z\neq v\in V(G_2)} deg_{G_2}(v)d_{G_2}(v,z)\\
&+ 2|E(G_1)|D(z)+D'(G_1)+D'(G_2)\\
&+ 2|E(G_2)|D(y),
\end{aligned}$$

$$\begin{aligned}
D'((G_1 \sim G_2)(y,z)) &= |V(G_2)|\sum_{y\neq u\in V(G_1)} deg_{G_1}(u)d_{G_1}(u,y)\\
&+ |V(G_1)|\sum_{z\neq v\in V(G_2)} deg_{G_2}(v)d_{G_2}(v,z)\\
&+ (2|E(G_1)|+2)D(z)+D'(G_1)\\
&+ (2|E(G_2)|+2)D(y)+D'(G_2)\\
&+ (|V(G_2)|-1)(2|E(G_1)|+1)+2|E(G_1)|\\
&+ (|V(G_1)|-1)(2|E(G_2)|+1)+2|E(G_2)|+2.
\end{aligned}$$

Proof. By Lemma 2.1 and definition, we have:

$$\begin{aligned}
D'((G_1 \cdot G_2)(y,z)) &= \frac{1}{2}\sum_{u,v}(deg_{(G_1\cdot G_2)(y,z)}(u)+deg_{(G_1\cdot G_2)(y,z)}(v))d_{(G_1\cdot G_2)(y,z)}(u,v)\\
&= \frac{1}{2}\sum_{\substack{u,v\neq y\\ u,v\in V(G_1)}}(deg_{G_1}(u)+deg_{G_1}(v))d_{G_1}(u,v)\\
&+ \frac{1}{2}\sum_{\substack{u=y\\ v\in V(G_1)}}(deg_{G_1}(y)+deg_{G_2}(z)+deg_{G_1}(v))d_{G_1}(v,y)\\
&+ \frac{1}{2}\sum_{\substack{u,v\neq z\\ u,v\in V(G_2)}}(deg_{G_2}(u)+deg_{G_2}(v))d_{G_2}(u,v)\\
&+ \frac{1}{2}\sum_{\substack{u=z\\ v\in V(G_2)}}(deg_{G_1}(y)+deg_{G_2}(z)+deg_{G_2}(v))d_{G_2}(v,z)\\
&+ \frac{1}{2}\sum_{\substack{y\neq u\in V(G_1)\\ z\neq v\in V(G_2)}}(deg_{G_1}(u)+deg_{G_2}(v))(d_{G_1}(u,y)+d_{G_2}(v,z))\\
&= D'(G_1)+D'(G_2)+(|V(G_2)|-1)\sum_{y\neq u\in V(G_1)} deg_{G_1}(u)d_{G_1}(u,y)\\
&+ (|V(G_1)|-1)\sum_{z\neq v\in V(G_2)} deg_{G_2}(v)d_{G_2}(v,z)+2|E(G_1)|D(z)\\
&+ 2|E(G_2)|D(y),
\end{aligned}$$

and,

$$
\begin{aligned}
D'((G_1 \sim G_2)(y,z)) &= \frac{1}{2}\sum_{u,v}(deg_{(G_1\sim G_2)(y,z)}(u))d_{(G_1\sim G_2)(y,z)}(u,v)\\
&+ \frac{1}{2}\sum_{u,v}(deg_{(G_1\sim G_2)(y,z)}(v))d_{(G_1\sim G_2)(y,z)}(u,v)\\
&= \frac{1}{2}\sum_{\substack{u,v\neq y\\ u,v\in V(G_1)}}(deg_{G_1}(u)+deg_{G_1}(v))d_{G_1}(u,v)\\
&+ \frac{1}{2}\sum_{\substack{u=y\\ v\in V(G_1)}}(deg_{G_1}(y)+1+deg_{G_1}(v))d_{G_1}(v,y)\\
&+ \frac{1}{2}\sum_{\substack{u,v\neq z\\ u,v\in V(G_2)}}(deg_{G_2}(u)+deg_{G_2}(v))d_{G_2}(u,v)\\
&+ \frac{1}{2}\sum_{\substack{u=z\\ v\in V(G_2)}}(1+deg_{G_2}(z)+deg_{G_2}(v))d_{G_2}(v,z)\\
&+ \sum_{\substack{v=z\\ u\neq y\in V(G_1)}}(1+deg_{G_2}(z)+deg_{G_1}(u))(1+d_{G_1}(u,y))\\
&+ \frac{1}{2}\sum_{\substack{y\neq u\in V(G_1)\\ z\neq v\in V(G_2)}}(deg_{G_1}(u)+deg_{G_2}(v))(d_{G_1}(u,y)+d_{G_2}(v,z)+1)\\
&+ deg_{G_1}(y)+deg_{G_2}(z)+2\\
&= |V(G_2)|\sum_{y\neq u\in V(G_1)}deg_{G_1}(u)d_{G_1}(u,y)\\
&+ |V(G_1)|\sum_{z\neq v\in V(G_2)}deg_{G_2}(v)d_{G_2}(v,z)\\
&+ (2|E(G_1)|+2)D(z)+D'(G_1)\\
&+ (2|E(G_2)|+2)D(y)+D'(G_2)\\
&+ (|V(G_2)|-1)(2|E(G_1)|+1)+2|E(G_1)|\\
&+ (|V(G_1)|-1)(2|E(G_2)|+1)+2|E(G_2)|+2.
\end{aligned}
$$

This completes our proof. □

Lemma 2.7. Suppose G_1 and G_2 are connected graphs and λ, j are positive integers. Define $D^j(z,G) = \sum_{v\in V(G)} d_G^j(v,z)$. Then

$$
\begin{aligned}
W_\lambda((G_1 \cdot G_2)(y,z)) &= W_\lambda(G_1)+W_\lambda(G_2)+[(|V(G_1)|-1)D^\lambda(z,G_2)\\
&+ \binom{\lambda}{1}D(y,G_1)D^{\lambda-1}(z,G_2)+\cdots\\
&+ \binom{\lambda}{\lambda-1}D^{\lambda-1}(y,G_1)D(z,G_2)\\
&+ (|V(G_2)|-1)D^\lambda(y,G_1)].
\end{aligned}
$$

Proof. By Lemma 2.1,

$$\begin{aligned}
W_\lambda((G_1\cdot G_2)(y,z)) &= \frac{1}{2}\sum_{u,v} d^{\lambda}_{G_1\cdot G_2}(u,v)\\
&= \frac{1}{2}\sum_{u,v\in V(G_1)} d^{\lambda}_{G_1}(u,v)+\frac{1}{2}\sum_{u,v\in V(G_2)} d^{\lambda}_{G_2}(u,v)\\
&+ \sum_{\substack{y\neq u\in V(G_1)\\ z\neq v\in V(G_2)}} d^{\lambda}_{G_1\cdot G_2}(u,v)\\
&= W_\lambda(G_1)+W_\lambda(G_2)+\sum_{\substack{y\neq u\in V(G_1)\\ z\neq v\in V(G_2)}} (d_{G_1}(u,y)+d_{G_2}(v,z))^{\lambda}\\
&= W_\lambda(G_1)+W_\lambda(G_2)+\sum_{\substack{y\neq u\in V(G_1)\\ z\neq v\in V(G_2)}} \left(\sum_{i=0}^{\lambda}\binom{\lambda}{i} d^{i}_{G_1}(u,y)d^{\lambda-i}_{G_2}(v,z)\right)\\
&= W_\lambda(G_1)+W_\lambda(G_2)+\Big[(|V(G_1)|-1)D^{\lambda}(z,G_2)\\
&+ \binom{\lambda}{1}D(y,G_1)D^{\lambda-1}(z,G_2)+\cdots+\binom{\lambda}{\lambda-1}D^{\lambda-1}(y,G_1)D(z,G_2)\\
&+ (|V(G_2)|-1)D^{\lambda}(y,G_1)\Big],
\end{aligned}$$

proving the lemma. □

Corollary 2.8. The Wiener, hyper-Wiener and TSZ indices of the splices of graphs are computed as follows:

$$\begin{aligned}
W((G_1\cdot G_2)(y,z)) &= W(G_1)+W(G_2)+(|V(G_1)|-1)D(z,G_2)\\
&+ (|V(G_2)|-1)D(y,G_1),
\end{aligned}$$

$$\begin{aligned}
WW((G_1\cdot G_2)(y,z)) &= WW(G_1)+WW(G_2)+\frac{1}{2}(|V(G_1)|-1)D(z,G_2)\\
&+ \frac{1}{2}(|V(G_2)|-1)D(y,G_1)+2D(y,G_1)D(z,G_2)\\
&+ \frac{1}{2}(|V(G_2)|-1)D^2(y,G_1)+\frac{1}{2}(|V(G_1)|-1)D^2(z,G_2),
\end{aligned}$$

$$\begin{aligned}
TSZ((G_1\cdot G_2)(y,z)) &= TSZ(G_1)+TSZ(G_2)+\frac{1}{3}(|V(G_1)|-1)D(z,G_2)\\
&+ D(y,G_1)D^2(z,G_2)+D^2(y,G_1)D(z,G_2)\\
&+ \frac{1}{6}(|V(G_1)|-1)D^3(z,G_2)+2D(y,G_1)D(z,G_2)\\
&+ \frac{1}{2}(|V(G_2)|-1)D^2(y,G_1)+\frac{1}{2}(|V(G_1)|-1)D^2(z,G_2)\\
&+ \frac{1}{3}(|V(G_2)|-1)D(y,G_1)+\frac{1}{6}(|V(G_2)|-1)D^3(y,G_1).
\end{aligned}$$

□

Lemma 2.9. Suppose G_1 and G_2 are connected graphs. The Wiener and hyper-Wiener indices of the links of graphs G_1 and G_2 are computed as follows:

$$\begin{aligned} W((G_1 \sim G_2)(y,z)) &= W(G_1) + W(G_2) + |V(G_1)|D(z,G_2) \\ &+ |V(G_2)|D(y,G_1) + |V(G_1)||V(G_2)|, \end{aligned}$$

$$\begin{aligned} WW((G_1 \sim G_2)(y,z)) &= WW(G_1) + WW(G_2) + \frac{3}{2}|V(G_1)|D(z,G_2) \\ &+ \frac{3}{2}|V(G_2)|D(y,G_1) + D(y,G_1)D(z,G_2) \\ &+ \frac{1}{2}|V(G_2)|D^2(y,G_1) + \frac{1}{2}|V(G_1)|D^2(z,G_2) \\ &+ |V(G_1)||V(G_2)|. \end{aligned}$$

□

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