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ANTI-PERIODIC SOLUTIONS FOR BAM NEURAL NETWORKS WITH MULTIPLE DELAYS ON TIME SCALES

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Abstract. In this paper, we consider anti-periodic solutions of the following BAM neural networks with multiple delays on time scales:

$$\begin{cases} x_i^{\Delta}(t) = -a_i(t)e_i(x_i(t)) + \sum\limits_{j=1}^{m} c_{ji}(t)f_j(y_j(t-\tau_{ji})) + I_i(t), \\ y_j^{\Delta}(t) = -b_j(t)h_j(y_j(t)) + \sum\limits_{i=1}^{n} d_{ij}(t)g_i(x_i(t-\delta_{ij})) + J_j(t), \end{cases}$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, m$. Using some analysis skills and Lyapunov method, some sufficient conditions on the existence and exponential stability of the anti-periodic solution to the above system are established.

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1. Introduction

Bi-directional associative memory (BAM) neural network model, proposed by Kosko [1-3], is a two-layer nonlinear feedback network model. It is often known as an extension of the unidirectional auto-associator of the Hopfield model, generalizing the single-layer auto-associative Hebbian correlation to a two-layer pattern-matched heteroassociative circuit. Since the BAM model presents a flexible nonlinear mapping from input space to output one, it has promising potential for applications in pattern recognition, artificial intelligence, diagnosing cancer and solving optimization problems. Recent years, dynamical behaviors, in particular, the existence and stability of the equilibrium points, periodic and almost periodic solutions of the continuous time delayed neural networks have been extensively studied by a large number of scholars (see i.e. [4-7]). Also, there are some papers to study the dynamics of the discrete time neural networks, such

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as Refs. [8-10]. However, most of the investigations focused on the continuous or discrete systems, respectively.

It is meaningful to research continuous and discrete systems under the same framework. Furthermore, dynamic equations on time scales can unify continuous and discrete time models very well. The theory of time scales was initiated by S.Hilger (1988), it has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, population dynamics, biotechnology, economics and so on. We refer the readers to the landmark paper of Hilger [11,12], books by Bohner and Peterson [13,14], which summarize much of time scales calculus.

During the past twenty years, anti-periodic problem of nonlinear differential equations were considered by many authors, see [15-21] and the references therein. However, few researches investigated the existence and stability of anti-periodic solutions of neural networks on time scales. In this paper, we consider the following BAM neural networks with multiple delays on time scales

$$\begin{cases} x_i^{\Delta}(t) = -a_i(t)e_i(x_i(t)) + \sum\limits_{j=1}^{m} c_{ji}(t)f_j(y_j(t-\tau_{ji})) + I_i(t), i = 1, 2, \dots, n, \\ y_j^{\Delta}(t) = -b_j(t)h_j(y_j(t)) + \sum\limits_{i=1}^{n} d_{ij}(t)g_i(x_i(t-\delta_{ij})) + J_j(t), j = 1, 2, \dots, m, \end{cases} \tag{1}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is a periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$; $x_i(t)$, $y_j(t)$ are the states of the ith neuron in neural field $\mathrm{F_X}$ and jth neuron in neural field $\mathrm{F_Y}$; $a_i(t)$, $b_j(t)$ represent the neuron charging times; $c_{ji}(t)$, $d_{ij}(t)$ tell the weights of the neuron interconnections; τ_{ji}, δ_{ij} show the axonal signal transmission delays; f_j, g_i denote the activation functions of the neurons; $I_i(t)$, $J_j(t)$ are the external inputs on the neurons. To the best of our knowledge, there are no papers published on the existence of anti-periodic solutions of (1). Our main aim of this paper is to establish some sufficient conditions for the existence and exponential stability of anti-periodic solutions of (1).

For the sake of simplicity, set $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$ and assume that $0 \in \mathbb{T}$, $\mathbb{T}$ is unbounded above, i.e. $\sup \mathbb{T} = \infty$. What's more, we will use $x = (x_1, \dots, x_k)^T \in \mathbb{R}^k$ to denote a column vector, in which the symbol $(\cdot)^T$ denotes the transpose of a vector. Let $|x|$ be the absolute-value vector given by $|x| = (|x_1|, \dots, |x_k|)$, and define $\| x \| = \sum\limits_{i=1}^{k} |x_i|$.

Let $u(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T \in C(\mathbb{T}, \mathbb{R}^{n+m})$, $u(t)$ is said to be ω-anti-periodic on $\mathbb{T}$, if $x_i(t + \omega) = -x_i(t), y_j(t + \omega) = -y_j(t)$ for all $t \in \mathbb{T}$, $t + \omega \in \mathbb{T}$, $i = 1, \dots, n$, $j = 1, \dots, m$. The initial conditions of (1) are of the form

$$\begin{cases} x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0]_{\mathbb{T}}, \quad \tau = \max\limits_{1\leq i\leq n, 1\leq j\leq m} \{\tau_{ji}\}, \quad i = 1, \dots, n, \\ y_j(s) = \psi_j(s), \quad s \in [-\delta, 0]_{\mathbb{T}}, \quad \delta = \max\limits_{1\leq i\leq n, 1\leq j\leq m} \{\delta_{ij}\}, \quad j = 1, \dots, m, \end{cases} \tag{2}$$

where $\varphi_i \in C([-\tau, 0]_{\mathbb{T}}, \mathbb{R})$, $\psi_j \in C([-\delta, 0]_{\mathbb{T}}, \mathbb{R})$.

For convenience, we introduce some notations

$$\overline{I}_i = \sup_{t\in\mathbb{T}} \big|I_i(t)\big|, \quad \overline{I} = \max_{1\le i\le n} \{\overline{I}_i\}, \quad \overline{J}_j = \sup_{t\in\mathbb{T}} \big|J_j(t)\big|, \quad \overline{J} = \max_{1\le j\le m} \{\overline{J}_j\}$$

$$\overline{c}_{ji} = \sup_{t\in\mathbb{T}} \big|c_{ji}(t)\big|, \quad \overline{d}_{ij} = \sup_{t\in\mathbb{T}} \big|d_{ij}(t)\big|, \quad \underline{a}_i = \inf_{t\in\mathbb{T}} \big|a_i(t)\big|, \quad \underline{b}_i = \inf_{t\in\mathbb{T}} \big|b_i(t)\big|.$$

Denote $\mathbb{R}^+ = (0, \infty)$, $\mathbb{T}^+ = (0, \infty)_{\mathbb{T}}$. Throughout this paper, for $i = 1, \dots, n$, $j = 1, .., m$, it will be assumed that

- (H_0) $a_i,\ b_j \in C(\mathbb{T}, \mathbb{R}^+)$; $a_i(t+\omega)e_i(r) = -a_i(t)e_i(-r), b_j(t+\omega)h_j(r) = -b_i(t)h_j(-r)$; $c_{ji}(t+\omega)f_j(r) = -c_{ji}(t)f_j(-r)$;
  $d_{ij}(t+\omega)g_i(r) = -d_{ij}(t)g_i(-r)$; $I_i(t+\omega) = -I_i(t)$,$J_j(t+\omega) = -J_j(t)$; for all $t \in \mathbb{T}$, $r \in \mathbb{R}$.
- (H_1) $e_i, d_j \in C(\mathbb{R}, \mathbb{R})$, there exist constants $\underline{e}_i > 0, \underline{h}_j > 0$ such that $\underline{e}_i|r_1 - r_2| \le \operatorname{sgn}(r_1 - r_2)[e_i(r_1) - e_i(r_2)], \underline{h}_j|r_1 - r_2| \le \operatorname{sgn}(r_1 - r_2)[h_j(r_1) - h_j(r_2)]$, for all $r_1, r_2 \in \mathbb{R}$, and $e_i(0) = 0, h_j(0) = 0$.
- (H_2) There exist nonnegative constants L_j^f, L_i^g such that $|f_j(r_1) - f_j(r_2) \le L_j^f|r_1 - r_2|, |g_i(r_1) - g_i(r_i)| \le L_i^g|r_1 - r_2|$, for all $r_1, r_2 \in \mathbb{R}$, and $f_j(0) = 0, g_i(0) = 0$.
- (H_3) There exists a constant $\eta > 0$ such that

$$-\underline{a}_i\underline{e}_i + \sum_{j=1}^{m} \overline{c}_{ji} L_j^f < -\eta < 0, \quad -\underline{b}_j\underline{h}_j + \sum_{i=1}^{n} \overline{d}_{ij} L_i^g < -\eta < 0.$$

The organization of the rest of this paper is as follow. In Section 2, we introduce some definitions and lemmas to make preparations for later sections. In Section 3, we establish our main results for the existence and exponential stability of anti-periodic solution of system (1).

2. Preliminaries

In this section, we first recall some basic definitions and lemmas on time scales used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. Throughout this paper we assume that the time scale $\mathbb{T}$ has uniformly bounded graininess $\mu(t)$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ and } \mu(\mathrm{t}) = \sigma(\mathrm{t}) - \mathrm{t}.$$

A point $t \in \mathbb{T}$ is called left-dense while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, also, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scatter if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$ otherwise $\mathbb{T}_k = \mathbb{T}$.

The notation $[a,b]_{\mathbb{T}}$ means that $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. The interval $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$, $(a,b)_{\mathbb{T}}$ are defined similarly.

Definition 1. *We say that a time scale $\mathbb{T}$ is periodic if there exist $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of time scale.*

Remark 1. *Let $\omega \in \mathbb{R}$, $\omega > 0$, $\mathbb{T}$ is a ω-periodic time scale if $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ such that $t + \omega \in \mathbb{T}$ whenever $t \in \mathbb{T}$. Clearly, we have $\mu(t) = \mu(t+\omega)$.*

Definition 2. *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with periodic p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is anti-periodic with period ω if there exist a natural number n such that $\omega = np$, $f(t+\omega) = -f(t)$ for all $t \in \mathbb{T}$ and ω is the smallest number such that $f(t+\omega) = -f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is anti-periodic with period $\omega > 0$, if ω is the smallest positive number such that $f(t+\omega) = -f(t)$ for all $t \in \mathbb{T}$.*

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point and its left side limits exist at left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$. We define $C(J, \mathbb{R}) = \{u(t) \text{ is continuous on } J\}$.

Definition 3. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$, we define $f^{\Delta}(t)$ to be the number if it exist with the property that for a given $\varepsilon > 0$, there exist a neighborhood U of t (i.e.$U = (t-\delta, t+\delta) \cap \mathbb{T}$,for some $\delta > 0$) such that*

$$\Big|\big[f(\sigma(t)) - f(s)\big] - f^{\Delta}(t)\big[\sigma(t) - s\big]\Big| < \varepsilon\Big|\sigma(t) - s\Big|,$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t. Moreover, we say that f is differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

Lemma 1. [13] *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

(*i*) *If f is differentiable at t, then f is continuous at t.*

(*ii*) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

(*iii*) *If t is not right scattered, then*

$$f^{\Delta} = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Let f be right-dense continuous. If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(s)\Delta s = F(b) - F(a).$$

Definition 4. *If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)_{\mathbb{T}}$, then we define the improper integral by*

$$\int_a^b f(s)\Delta s := \lim_{b\to\infty} \int_a^b f(s)\Delta s,$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}^k$.

If r is regressive and right-dense continuous function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\} \text{ for s, t} \in \mathbb{T}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\log\left(1+hz\right)}{h} & if\ h \neq 0, \\ z & if\ h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq,\ \ominus p := -\frac{p}{1 + \mu p},\ p \ominus q := p \oplus (\ominus q).$$

Then the generalized exponential function has the following properties.

Lemma 2. [13] *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*

(*i*) $e_0(t, s) \equiv 1$ and $\mathrm{e_p(t, t)} \equiv 1$;

(*ii*) $e_p(\sigma(t), t) = (1 + \mu(t)p(t))e_p(t, s)$;

(*iii*) $e_p(t, \sigma(s)) = \dfrac{e_p(t, s)}{1 + \mu(t)p(t)}$;

(*iv*) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$;

(*v*) $e_p(t, s) = \dfrac{1}{e_p(t, s)} = e_{\ominus p}(s, t)$;

(*vi*) $e_p(t, s)e_q(s, t) = e_{p\oplus q}(t, s)$;

(*vii*) $\dfrac{e_p(t, s)}{e_q(t, s} = e_{p\ominus q}(t, s)$.

Definition 5. (*Lakshmikantham and Vatsala* [22]) *For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, we define the generalized derivative (or Dini derivative), $D^+u^\Delta(t)$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N(\varepsilon) \subset N$ of t such that*

$$\frac{u(\sigma(t)) - u(s)}{\mu(t,s)} < D^+u^\Delta(t) + \varepsilon,$$

for each $s \in N(\varepsilon)$, $s > t$, where $\mu(t,s) = \sigma(t) - s$. In the case t is right-scattered and $u(t)$ is continuous at t, this reduce to

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t}.$$

Definition 6. *Let $u^*(t) = (x_1^*(t), \dots, x_n^*(t), y_1^*(t), \dots, y_m^*(t))^T$ be the solution of system (1) with initial value $\theta^* = (\varphi_1^*(t), \dots, \varphi_n^*(t), \psi_1^*(t), \dots, \psi_m^*(t))^T$ is said to be global exponential stable, if for all solution of system (1) $u(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$ with initial value $\theta = (\varphi_1(t), \dots, \varphi_n(t), \psi_1(t), \dots, \psi_m(t))^T$, there exist a positive constant ϵ and $N = N(\epsilon)$ such that*

$$\sum_{i=1}^{n} \big|x_i(t) - x_i^*(t)\big| + \sum_{j=1}^{m} \big|y_j(t) - y_j^*(t)\big| \le N(\epsilon) e_{\ominus\epsilon}(t,\alpha) \parallel \theta - \theta^* \parallel, t > 0,$$

where $\parallel \theta - \theta^* \parallel = \sum_{i=1}^{n} \sup_{s\in[-\delta,0]} \big|\varphi_i(s) - \varphi_i^*(s)\big| + \sum_{j=1}^{m} \sup_{s\in[-\tau,0]} \big|\psi_j(s) - \psi_j^*(s)\big|.$

Lemma 3. *Let $(H_0) - (H_3)$ hold. Suppose that $u(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$ is a solution of system (1) with the initial condition*

$$\begin{cases} x_i(s) = \varphi_i(s), \big|\varphi_i(s)\big| < \frac{\overline{I}}{\eta}, s \in [-\tau, 0]_{\mathbb{T}}, \\ y_j(s) = \psi_j(s), \ \big|\psi_j(s)\big| < \frac{\overline{J}}{\eta}, s \in [-\delta, 0]_{\mathbb{T}}. \end{cases} \tag{3}$$

Then

$$\big|x_i(t)\big| < \frac{\overline{I}}{\eta}, \quad \big|y_j(t)\big| < \frac{\overline{J}}{\eta}, \quad for\, all\, t \in [0, +\infty)_{\mathbb{T}}. \tag{4}$$

where $i = 1, \dots, n$, $j = 1, \dots, m$

Proof. By way of contradiction, assume that (4) does not hold. Then there exist $i \in \{1, 2, \dots, n\}$ or $j \in \{1, 2, \dots, m\}$ and the fist $t_0 > 0$, $t_0 \in \mathbb{T}$ such that

$$\big|x_i(t_0)\big| \ge \frac{\overline{I}}{\eta}, \quad \big|x_i(\rho(t_0))\big| \le \frac{\overline{I}}{\eta}, \quad \big|x_i(t)\big| < \frac{\overline{I}}{\eta}, \quad t \in [-\tau, t_0)_{\mathbb{T}}, \tag{5}$$

or

$$\big|y_j(t_0)\big| \ge \frac{\overline{J}}{\eta}, \quad \big|y_j(\rho(t_0))\big| \le \frac{\overline{J}}{\eta}, \quad \big|y_j(t)\big| < \frac{\overline{J}}{\eta}, \quad t \in [-\delta, t_0)_{\mathbb{T}}, \tag{6}$$

If (5) holds, calculating the Dini derivative of $\big|x_i(t_0)\big|$, together with $(H_0)-(H_4)$, we obtain

$$\begin{aligned} 0 &\le D^+(|x_i(t_0)|^\Delta) \\ &= \mathrm{sgn}x_i(t_0)\Big\{-a_i(t_0)e_i(t_0)+\sum_{j=1}^{m}c_{ji}(t_0)f_j(t_0-\tau_{ji})+I_i(t_0)\Big\} \\ &\le -\underline{a}_i\underline{e}_i\frac{\overline{I}}{\eta}+\sum_{j=1}^{m}\overline{c}_{ji}L_j^f|y_j(t_0-\tau_{ji})|+\overline{I}_i \\ &\le (-\underline{a}_i\underline{e}_i+\sum_{j=1}^{m}\overline{c}_{ji}L_j^f)\frac{\overline{I}}{\eta}+\overline{I}_i<-\eta\frac{\overline{I}}{\eta}+\overline{I}_i<0, \end{aligned}$$

which is a contradiction. Similarly, we can prove that (6) does not hold. The proof of Lemma 3 is now completed. □

Lemma 4. *Suppose that* $(H_0)-(H_3)$ *are satisfied. Suppose additionally that*

$(H4)$ *There exist some constants* $\epsilon>0$, $\xi_i>0$, $\xi_j^{'}>0$, *such that*

$$\begin{aligned} &\xi_i\big[\epsilon-\underline{a}_i\underline{e}_i(1+\mu(t)\epsilon)\big]+\sum_{j=1}^{m}\xi_j^{'}\overline{d}_{ij}L_i^g(1+\epsilon\mu(t+\delta_{ij}))e_\epsilon(t+\delta_{ij},t)<0, \\ &\xi_j^{'}\big[\epsilon-\underline{b}_j\underline{h}_j(1+\mu(t)\epsilon)\big]+\sum_{i=1}^{n}\xi_i\overline{c}_{ij}L_j^f(1+\epsilon\mu(t+\tau_{ji}))e_\epsilon(t+\tau_{ji},t)<0, \\ &\qquad\qquad i=1,2,\ldots,n,\quad j=1,2,\ldots,m. \end{aligned}$$

Then the solution of system (1) is globally exponentially stable.

Proof. Let $u^*(t)=(x_1^*(t),\ldots,x_n^*(t),y_1^*(t),\ldots,y_m^*(t))^T$ be the solution of system (1) with initial value $\theta^*=(\varphi_1^*(t),\ldots,\varphi_n^*(t),\psi_1^*(t),\ldots,\psi_m^*(t))^T$, and $u(t)=(x_1(t),\ldots,x_n(t),y_1(t),\ldots,y_m(t))^T$ be the solution of system (1) with initial value $\theta=(\varphi_1(t),\ldots,\varphi_n(t),\psi_1(t),\ldots,\psi_m(t))^T$. Then we have

$$\begin{aligned} (x_i(t)-x_i^*(t))^\Delta &= -a_i(t)\big[e_i(x_i(t))-e_i(x_i^*(t))\big] \\ &\quad+\sum_{j=1}^{m}c_{ji}(t)\big[f_j(y_j(t-\tau_{ji}))-f_j(y_j^*(t-\tau_{ji}))\big], t>0, \quad (7) \end{aligned}$$

and

$$\begin{aligned} (y_j(t)-y_j^*(t))^\Delta &= -b_j(t)\big[h_j(y_j(t))-h_j(y_j^*(t))\big] \\ &\quad+\sum_{i=1}^{n}d_{ij}(t)\big[g_i(x_i(t-\delta_{ij}))-g_i(x_i^*(t-\delta_{ij}))\big], t>0. \quad (8) \end{aligned}$$

In view of above, for $t > 0$, we have

$$D^+\big|x_i(t) - x_i^*(t)\big|^\Delta \leq -\underline{a}_i\underline{e}_i\big|x_i(t) - x_i^*(t)\big| + \sum_{j=1}^{m}\underline{c}_{ji}L_j^f\big|y_j(t-\tau_{ji}) - y_j^*(t-\tau_{ji})\big|, \tag{9}$$

$$D^+\big|y_j(t) - y_j^*(t)\big|^\Delta \leq -\underline{b}_j\underline{h}_j\big|y_j(t) - y_j^*(t)\big| + \sum_{i=1}^{n}\underline{d}_{ij}L_i^g\big|x_i(t-\delta_{ij}) - x_i^*(t-\delta_{ij})\big|. \tag{10}$$

For any $\alpha \in [-\max\{\tau, \delta\}, 0]$, we construct the Lyapunov functional

$$\begin{aligned}
V(t) &= V_1(t) + V_2(t) + V_3(t) + V_4(t),\\
V_1(t) &= \sum_{i=1}^{n}\xi_i e_\epsilon(t,\alpha)\big|x_i(t) - x_i^*(t)\big|,\\
V_2(t) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\overline{c}_{ji}L_j^f\int_{t-\tau_{ji}}^{t}(1+\epsilon\mu(s+\tau_{ji}))e_\epsilon(s+\tau_{ji},\alpha)\big|y_j(s) - y_j^*(s)\big|\Delta s,\\
V_3(t) &= \sum_{j=1}^{m}\xi_j^{'}e_\epsilon(t,\alpha)\big|y_j(t) - y_j^*(t)\big|,\\
V_4(t) &= \sum_{j=1}^{m}\sum_{i=1}^{n}\xi_j^{'}\overline{d}_{ij}L_i^g\int_{t-\delta_{ij}}^{t}(1+\epsilon\mu(s+\delta_{ij}))e_\epsilon(s+\delta_{ij},\alpha)\big|x_i(s) - x_i^*(s)\big|\Delta s.
\end{aligned}$$

Calculating the delta derivative $D^+V^\Delta(t)$ along the system (1), we can get

$$\begin{aligned}
D^+V_1^\Delta(t)\Big|_{(1)} &= \sum_{i=1}^{n}\xi_i\Big[\epsilon e_\epsilon(t,\alpha)\big|x_i(t) - x_i^*(t)\big| + e_\epsilon(\sigma(t),\alpha)D^+|x_i(t) - x_i^*(t)|^\Delta\Big]\\
&\leq \sum_{i=1}^{n}\xi_i\Big\{\epsilon e_\epsilon(t,\alpha)\big|x_i(t) - x_i^*(t)\big| + e_\epsilon(\sigma(t),\alpha)\Big[-\underline{a}_i\underline{e}_i\big|x_i(t) - x_i^*(t)\big|\\
&\quad + \sum_{j=1}^{m}\underline{c}_{ji}L_j^f\big|y_j(t-\tau_{ji}) - y_j^*(t-\tau_{ji})\big|\Big]\Big\}\\
&= \Big\{\sum_{i=1}^{n}\xi_i\Big[\epsilon - \underline{a}_i\underline{e}_i(1+\mu(t)\epsilon)\Big]e_\epsilon(t,\alpha)\big|x_i(t) - x_i^*(t)\big|\Big\}\\
&\quad + \Big\{(1+\mu(t)\epsilon)e_\epsilon(t,\alpha)\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\overline{c}_{ji}L_j^f\big|y_j(t-\tau_{ji}) - y_j^*(t-\tau_{ji})\big|\Big\},
\end{aligned}$$

$$D^+V_2^{\Delta}(t)\Big|_{(1)} \leq \sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\overline{c}_{ji}L_j^f\Big(1+\epsilon\mu(t+\tau_{ji})\Big)e_\epsilon(t+\tau_{ji},\alpha)\Big|y_j(t)-y_j^*(t)\Big| - \sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\overline{c}_{ji}L_j^f\Big(1+\epsilon\mu(t)\Big)e_\epsilon(t,\alpha)\Big|y_j(t-\tau_{ji})-y_j^*(t\tau_{ji})\Big|.$$

It concludes that

$$\begin{aligned} & D^+\Big(V_1(t)+V_2(t)\Big)^{\Delta}\Big|_{(1)} \\ \leq\ & \Big\{\sum_{i=1}^{n}\xi_i[\epsilon-\underline{a}_i\underline{e}_i(1+\mu(t)\epsilon)]e_\epsilon(t,\alpha)|x_i(t)-x_i^*(t)|\Big\} \\ & +\Big\{\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\overline{c}_{ji}L_j^f\Big(1+\epsilon\mu(t+\tau_{ji})\Big)e_\epsilon(t+\tau_{ji},\alpha)\Big|y_j(t)-y_j^*(t)\Big|\Big\}. \end{aligned} \tag{11}$$

Noting that

$$\begin{aligned} D^+V_3^{\Delta}(t)\Big|_{(1)} \leq\ & \Big\{\sum_{j=1}^{m}\xi_j'\Big[\epsilon-\underline{b}_j\underline{h}_j(1+\mu(t)\epsilon)\Big]e_\epsilon(t,\alpha)\Big|y_j(t)-y_j^*(t)\Big|\Big\} \\ & +\Big\{(1+\mu(t)\epsilon)e_\epsilon(t,\alpha)\sum_{j=1}^{m}\sum_{i=1}^{n}\xi_j'\overline{d}_{ij}L_i^g\Big|x_i(t-\delta_{ij}) \\ & -x_i^*(t-\delta_{ij})\Big|\Big\}, \\ D^+V_4^{\Delta}(t)\Big|_{(1)} \leq\ & \sum_{j=1}^{m}\sum_{i=1}^{n}\xi_i'\overline{d}_{ij}L_i^g(1+\epsilon\mu(t+\delta_{ij}))e_\epsilon(t+\delta_{ij},\alpha)\Big|x_i(t)-x_i^*(t)\Big| \\ & -\sum_{j=1}^{m}\sum_{i=1}^{n}\xi_i'\overline{d}_{ij}L_i^g(1+\epsilon\mu(t))e_\epsilon(t,\alpha)\Big|x_i(t-\delta_{ij})-x_i^*(t\delta_{ij})\Big|, \end{aligned}$$

which imply

$$\begin{aligned} & D^+\Big(V_3(t)+V_4(t)\Big)^{\Delta}\Big|_{(1)} \\ \leq\ & \Big\{\sum_{j=1}^{m}\xi_j'\Big[\epsilon-\underline{b}_j\underline{h}_j(1+\mu(t)\epsilon)\Big]e_\epsilon(t,\alpha)\Big|y_j(t)-y_j^*(t)\Big|\Big\} \\ & +\Big\{\sum_{j=1}^{m}\sum_{i=1}^{n}\xi_j'\overline{d}_{ij}L_i^g(1+\epsilon\mu(t+\delta_{ij}))e_\epsilon(t+\delta_{ij},\alpha)\Big|x_i(t)-x_i^*(t)\Big|\Big\}. \end{aligned} \tag{12}$$

From (11) and (12) we can get

$$\begin{aligned} D^+V^\Delta(t) \ \le\ & \sum_{i=1}^{n}\Big\{\xi_i\Big[\epsilon-\underline{a}_i\underline{e}_i(1+\mu(t)\epsilon)\Big] \\ & +\sum_{j=1}^{m}\xi_j^{'}\overline{d}_{ij}L_i^g(1+\epsilon\mu(t+\delta_{ij}))e_\epsilon(t+\delta_{ij},t)\Big\}e_\epsilon(t,\alpha)\Big|x_i(t)-x_i^*(t)\Big| \\ & +\sum_{j=1}^{m}\Big\{\xi_j^{'}[\epsilon-\underline{b}_j\underline{h}_j(1+\mu(t)\epsilon)] \\ & +\sum_{i=1}^{n}\xi_i\overline{c}_{ij}L_j^f(1+\epsilon\mu(t+\tau_{ji}))e_\epsilon(t+\tau_{ji},t)\Big\}e_\epsilon(t,\alpha)\Big|y_j(t)-y_j^*(t)\Big|. \end{aligned} \tag{13}$$

By assumption $(H4)$, it follows that $D^+V^\Delta(t)<0$, i.e. $V(t)<V(0)$, for $t>0$.

On the other hand, we have

$$\begin{aligned} V(0) \ =\ & \sum_{i=1}^{n}\xi_ie_\epsilon(0,\alpha)\Big|x_i(0)-x_i^*(0)\Big|+\sum_{j=1}^{m}\xi_j^{'}e_\epsilon(0,\alpha)\Big|y_j(0)-y_j^*(0)\Big| \\ & +\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_i\overline{c}_{ji}L_j^f\int_{-\tau_{ji}}^{0}(1+\epsilon\mu(s+\tau_{ji}))e_\epsilon(s+\tau_{ji},\alpha)\Big|y_j(s)-y_j^*(s)\Big|\Delta s \\ & +\sum_{j=1}^{m}\sum_{i=1}^{n}\xi_i\overline{d}_{ij}L_i^g\int_{-\delta_{ij}}^{0}(1+\epsilon\mu(s+\delta_{ij}))e_\epsilon(s+\delta_{ij},\alpha)\Big|x_i(s)-x_i^*(s)\Big|\Delta s \\ \le\ & \sum_{i=1}^{n}\Big\{\xi_ie_\epsilon(0,\alpha)+\sum_{j=1}^{m}\xi_j^{'}\overline{d}_{ij}L_i^g\int_{-\delta_{ij}}^{0}(1+\epsilon\mu(s+\delta_{ij}))e_\epsilon(s+\delta_{ij},\alpha)\Delta s\Big\} \\ & \times\sup_{s\in[-\delta,0]}\Big|x_i(s)-x_i^*(s)\Big| \\ & +\sum_{j=1}^{m}\Big\{\xi_j^{'}e_\epsilon(0,\alpha)+\sum_{i=1}^{n}\xi_i\overline{c}_{ji}L_j^f\int_{-\tau_{ji}}^{0}(1+\epsilon\mu(s+\tau_{ji}))e_\epsilon(s+\tau_{ji},\alpha)\Delta s\Big\} \\ & \times\sup_{s\in[-\tau,0]}\Big|y_j(s)-y_j^*(s)\Big| \\ \le\ & \overline{N}(\epsilon)\Big\{\sum_{i=1}^{n}\sup_{s\in[-\delta,0]}\Big|x_i(s)-x_i^*(s)\Big|+\sum_{j=1}^{m}\sup_{s\in[-\tau,0]}\Big|y_j(s)-y_j^*(s)\Big|\Big\}, \end{aligned}$$

where $\overline{N}(\epsilon)=\max\limits_{1\le i\le n,1\le j\le m}\{N_i,N_j^{'}\}$,

$$N_i=\xi_ie_\epsilon(0,\alpha)+\sum_{j=1}^{m}\xi_j^{'}\overline{d}_{ij}L_i^g\int_{-\delta_{ij}}^{0}(1+\epsilon\mu(s+\delta_{ij}))e_\epsilon(s+\delta_{ij},\alpha)\Delta s,$$

$$N_j^{'}=\xi_j^{'}e_\epsilon(0,\alpha)+\sum_{i=1}^{n}\xi_i\overline{c}_{ji}L_j^f\int_{-\tau_{ji}}^{0}(1+\epsilon\mu(s+\tau_{ji}))e_\epsilon(s+\tau_{ji},\alpha)\Delta s.$$

And

$$\sum_{i=1}^{n}\xi_i e_\epsilon(t,\alpha)\big|x_i(t)-x_i^*(t)\big|+\sum_{j=1}^{m}\xi_j^{'}e_\epsilon(t,\alpha)\big|y_j(t)-y_j^*(t)\big|\le V(t)\le V(0),$$

that is

$$\min_{1\le i\le n,1\le j\le m}\{\xi_i,\xi_j^{'}\}e_\epsilon(t,\alpha)\left\{\sum_{i=1}^{n}\big|x_i(t)-x_i^*(t)\big|+\sum_{j=1}^{m}\big|y_j(t)-y_j^*(t)\big|\right\}\le V(0).$$

Thus we can finally conclude that

$$\sum_{i=1}^{n}\big|x_i(t)-x_i^*(t)\big|+\sum_{j=1}^{m}\big|y_j(t)-y_j^*(t)\big|\le N(\epsilon)e_{\ominus\epsilon}(t,\alpha)\parallel\theta^*-\theta\parallel,$$

where $N(\epsilon)=\frac{\overline{N}(\epsilon)}{\min\limits_{1\le i\le n,1\le j\le m}\{\xi_i,\xi_j^{'}\}}$. Therefore, By Definition 5, the solution $u^*(t)$ of system (1) is globally exponentially stable. The proof is completed. □

3. Main result

In this section, we will state and prove our main result of this paper.

Theorem 1. *Assume that $(H_0)-(H_4)$ hold. Then system (1) has an ω-anti-periodic solution, which is globally exponentially stable.*

Proof. Let $u(t)=(x_1(t),\dots,x_1(t),y_1(t),\dots,y_1(t))^T$ is a solution of system (1) with the initial condition

$$\begin{cases}x_i(s)=\varphi_i(s),\big|\varphi_i(s)\big|<\frac{\overline{I}}{\eta},s\in[-\tau,0]_{\mathbb{T}},\\ y_j(s)=\psi_j(s),\ \big|\psi_j(s)\big|<\frac{\overline{J}}{\eta},s\in[-\delta,0]_{\mathbb{T}}.\end{cases}\tag{14}$$

Then by Lemma 3, the solution $u(t)$ is bounded and

$$\big|x_i(t)\big|<\frac{\overline{I}}{\eta},\quad\big|y_j(t)\big|<\frac{\overline{J}}{\eta},\text{ for all }t\in[0,+\infty)_T,\ i=1,\dots,n,\ j=1,\dots,m.\tag{15}$$

Then

$$\begin{aligned}&\big((-1)^{k+1}x_i(t+(k+1)\omega)\big)^{\Delta}\\ =\ &(-1)^{k+1}x_i^{\Delta}(t+(k+1)\omega)\\ =\ &(-1)^{k+1}\bigg\{-a_i(t+(k+1)\omega)e_i(x_i(t+(k+1)\omega))\\ &+\sum_{j=1}^{m}c_{ji}(t+(k+1)\omega)f_j(y_j(t+(k+1)\omega-\tau_{ji}))\\ &+I_i(t+(k+1)\omega)\bigg\}\\ =\ &-a_i(t)e_i((-1)^{k+1}x_i(t+(k+1)\omega))\\ &+\sum_{j=1}^{m}c_{ji}(t)f_j((-1)^{k+1}y_j(t+(k+1)\omega-\tau_{ji}))+I_i(t).\end{aligned}\tag{16}$$

Similarly,

$$\begin{aligned}
&\Big((-1)^{k+1}y_j(t+(k+1)\omega)\Big)^{\Delta}\\
=\;& -b_j(t)h_j((-1)^{k+1}y_j(t+(k+1)\omega))\\
&+\sum_{i=1}^{n}d_{ij}(t)g_i((-1)^{k+1}x_i(t+(k+1)\omega-\delta_{ij}))+J_j(t). \qquad (17)
\end{aligned}$$

Thus, for any natural number k, $(-1)^{k+1}u(t+(k+1)\omega)$ are the solution of system (1). Then by Lemma 3, there exists a constant $N(\epsilon)$ such that

$$\begin{aligned}
&\sum_{i=1}^{n}\Big|(-1)^{k+1}x_i(t+(k+1)\omega)-(-1)^{k}x_i(t+k\omega)\Big|\\
&\quad+\sum_{j=1}^{m}\Big|(-1)^{k+1}y_j(t+(k+1)\omega)-(-1)^{k}y_j(t+k\omega)\Big|\\
\le\;& N(\epsilon)e_{\ominus\epsilon}(t+k\omega,\alpha)\Bigg\{\sum_{i=1}^{n}\sup_{s\in[-\tau,0]}\Big|x_i(s+\omega)+x_i(s)\Big|\\
&\quad+\sum_{j=1}^{m}\sup_{s\in[-\delta,0]}\Big|y_j(s+\omega)+y_j(s)\Big|\Bigg\}\\
\le\;& 2N(\epsilon)e_{\ominus\epsilon}(t+k\omega,\alpha)\Bigg(\frac{n\overline{I}}{\eta}+\frac{m\overline{J}}{\eta}\Bigg). \qquad (18)
\end{aligned}$$

Then for a natural number l, we obtain

$$\begin{aligned}
&(-1)^{l+1}x_i(t+(l+1)\omega)\\
&=x_i(t)+\sum_{k=0}^{l}\Bigg[(-1)^{k+1}x_i(t+(k+1)\omega)-(-1)^{k}x_i(t+(k)\omega)\Bigg], \qquad (19)\\
&(-1)^{l+1}y_j(t+(l+1)\omega)\\
&=y_j(t)+\sum_{k=0}^{l}\Bigg[(-1)^{k+1}y_j(t+(k+1)\omega)-(-1)^{k}y_j(t+(k)\omega)\Bigg]. \qquad (20)
\end{aligned}$$

Then

$$\begin{aligned}
&\Big|(-1)^{l+1}x_i(t+(l+1)\omega)\Big|\\
&\le\Big|x_i(t)\Big|+\sum_{k=0}^{l}\Big|(-1)^{k+1}x_i(t+(k+1)\omega)-(-1)^{k}x_i(t+(k)\omega)\Big|, \qquad (21)\\
&\Big|(-1)^{l+1}y_j(t+(l+1)\omega)\Big|\\
&\le\Big|y_j(t)\Big|+\sum_{k=0}^{l}\Big|(-1)^{k+1}y_j(t+(k+1)\omega)-(-1)^{k}y_j(t+(k)\omega)\Big|. \qquad (22)
\end{aligned}$$

Noting that $\mu(t)$ is bounded and (18) holds, then there exist a sufficient large constant $K>0$ and a positive constant β such that

$$\Big|(-1)^{k+1}x_i(t+(k+1)\omega)-(-1)^{k}x_i(t+k\omega)\Big|\le\beta(e^{-c\omega})^{k},\ \text{for all}\ k>K, \qquad (23)$$

and

$$\left|(-1)^{k+1}y_j(t+(k+1)\omega)-(-1)^k y_j(t+k\omega)\right|\le\beta(e^{-c\omega})^k,\ \text{for all}\ k>K. \qquad (24)$$

where $i=1,\dots,n, j=1,\dots,m$. It follow from (19) − (24) that $(-1)^k u(t+k\omega)$ uniformly converges to a continuous function $v(t)=(x_1^*(t),\dots,x_n^*(t),y_1^*(t),\dots,$ $y_m^*(t))$ in time scales sense.

Now we will prove that $v(t)$ is an ω-anti-periodic of (1). First, we have

$$v(t+\omega)=\lim_{k\to\infty}(-1)^k u(t+k\omega+\omega)=-\lim_{k\to\infty}(-1)^{k+1}u(t+(k+1)\omega)=-v(t).$$

Next, we show that $v(t)$ is a solution of (1). In fact, together with the continuity of right hand of (1), (16) and (17) implies that $\{(-1)^{k+1}u(t+(k+1)\omega)\}$ uniformly converges to a continuous function in the sense time scales. Letting $k\to\infty$, we obtain

$$(x_i^*(t))^{\Delta}=-a_i(t)e_i(x_i^*(t))+\sum_{j=1}^{m}c_{ji}(t)f_j(y_j^*(t-\tau_{ji}))+I_i(t), i=1,\dots,n$$

and

$$(y_j^*(t))^{\Delta}=-b_j(t)h_j(y_j^*(t))+\sum_{i=1}^{n}d_{ij}(t)g_i(x_i^*(t-\delta_{ij}))+J_j(t), j=1,\dots,m.$$

Therefore, $v(t)$ is a solution of (1). Moreover, by Lemma 4 we can show that $v(t)$ is globally exponentially stable. This completes the proof. □

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