J. Appl. Math. & Informatics Vol. **29***(2011), No. 1 - 2, pp. 227-245*
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A VISCOSITY APPROXIMATIVE METHOD TO CESÀRO MEANS FOR SOLVING A COMMON ELEMENT OF MIXED EQUILIBRIUM, VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS†

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Abstract. In this paper, we introduce a new iterative method for finding a common element of the set of solutions for mixed equilibrium problem, the set of solutions of the variational inequality for a β-inverse-strongly monotone mapping and the set of fixed points of a family of finitely nonexpansive mappings in a real Hilbert space by using the viscosity and Cesàro mean approximation method. We prove that the sequence converges strongly to a common element of the above three sets under some mind conditions. Our results improve and extend the corresponding results of Kumam and Katchang [A viscosity of extragradient approximation method for finding equilibrium problems, variational inequalities and fixed point problems for nonexpansive mapping, *Nonlinear Analysis: Hybrid Systems*, 3(2009), 475–486], Peng and Yao [Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems, *Mathematical and Computer Modelling*, 49(2009), 1816–1828], Shimizu and Takahashi [Strong convergence to common fixed points of families of nonexpansive mappings, *Journal of Mathematical Analysis and Applications*, 211(1) (1997), 71–83] and some authors.

AMS Mathematics Subject Classification : 46C05, 47D03, 47H09, 47H10.
Key words and phrases : Nonexpansive mapping, Fixed point, Variational inequality, Mixed equilibrium problem, Viscosity approximation method, Cesàro mean approximation method.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively and let C

Received January 10, 2010. Revised May 17, 2010. Accepted May 20, 2010. *Corresponding author. †This Research was Financially Supported by Faculty of Science, King Mongkut's University of Technology Thonburi.

be a nonempty closed convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C$. We use $F(T)$ to denote the set of *fixed points* of T, that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in [0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$.

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and ϕ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [4] considered the following *mixed equilibrium problem* for finding $x \in C$ such that

$$\phi(x, y) + \varphi(y) \geq \varphi(x), \quad \text{for all } y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $MEP(\phi, \varphi)$. We see that x is a solution of problem (1) implies that $x \in \text{dom}\varphi = \{x \in C | \varphi(x) < +\infty\}$. If $\varphi = 0$, then the mixed equilibrium problem (1) becomes the following *equilibrium problem* is to find $x \in C$ such that

$$\phi(x, y) \geq 0, \quad \text{for all } y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $EP(\phi)$. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems and the equilibrium problem as special cases. Numerous problems in physics, optimization and economics reduce to find a solution of (2). Some methods have been proposed to solve the equilibrium problem (see [2, 3, 5, 6, 7, 8, 9, 10, 12, 21, 22, 24, 25]).

Let $B : C \to H$ be a mapping. The *variational inequality problem*, denoted by $VI(C, B)$, is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \tag{3}$$

for all $y \in C$. The variational inequality problem has been extensively studied in the literature. See, e.g. [28, 29] and the references therein. A mapping B of C into H is called *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \tag{4}$$

for all $x, y \in C$. B is called β-*inverse-strongly monotone* if there exists a positive real number $\beta > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y\rangle \geq \beta\|Bx - By\|^2. \tag{5}$$

Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \text{for all } \ x \in H. \tag{6}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{7}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S on H, and b is a given point in H. Moreover, it is shown in [11] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A)Sx_n, \tag{8}$$

converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \;\; x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Recently, Kumam and Katchang [10] introduced a viscosity extragradient approximation method which solved the problem for finding the set of solutions for equilibrium problems, the set of solutions of the variational inequalites for k-Lipschitz continuous mapping and the set of fixed point problems for nonexpansive mapping in a real Hilbert space. The sequences were generated by $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \;\; \forall y \in C, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)SP_C(x_n - \lambda_n B y_n), \end{cases} \tag{9}$$

for all $n \geq 1$, where B is a monotone k-Lipschitz continuous mapping, S is a nonexpansive mapping and A is a strongly bounded linear operator. They proved the strong convergence theorems under suitable conditions.

In 2008, Peng and Yao [15] introduced an iterative algorithm based on extragradient method which solves the problem of finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a real Hilbert space. The sequences generated by $v \in C$,

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \;\forall y \in C, \\ y_n = P_C(u_n - \gamma_n B u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \lambda_n B y_n), \end{cases} \tag{10}$$

for all $n \geq 0$, where W_n is W-mapping. They proved the strong convergence theorems under some mind conditions.

On the other hand, Shimizu and Takahashi [19] originally studied the convergence of an iteration process $\{x_n\}$ for a family of nonexpansive mappings in the framework of a real Hilbert space. They restate the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n, \;\; for \;\; n = 0, 1, 2, ... \tag{11}$$

where x_0 and x are all elements of C and α_n is an appropriate in $[0,1]$. They proved that $\{x_n\}$ converges strongly to an element of fixed point of T which is the nearest to x.

In this paper, motivated by the above results and the iterative schemes considered in [10], [15] and [19], we introduce a new iterative process (17) below base on viscosity and Cesàro means approximation method for finding a common element of the set of fixed points of a family of finitely nonexpansive mappings, the set of solutions of the variational inequality problem for a β-inverse-strongly monotone mapping and the set of solutions of a mixed equilibrium problem in a real Hilbert space. Then, we prove strong convergence theorems which are connected with [6, 18, 20, 26, 27] and extend and improve the corresponding results of Kumam and Katchang [10], Peng and Yao [15] and Shimizu and Takahashi [19].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{12}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{13}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. For every point $x \in H$, there exists a unique *nearest point* in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x-y\|, \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{14}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{15}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{16}$$

for all $x \in H, y \in C$. Let B be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (15) implies the following:

$$u \in VI(C,B) \Leftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0.$$

It is also known that H satisfies the Opial condition [13], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $U : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Ux$ and $g \in Uy$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $U : H \to 2^H$ is *maximal* if the graph of $G(U)$ of U is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping U is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(U)$ implies $f \in Ux$. Let B be a monotone mapping of C into H and let $N_C\bar{y}$ be the *normal cone* to C at $\bar{y} \in C$, i.e., $N_C\bar{y} = \{w \in H : \langle u - \bar{y}, w\rangle \leq 0, \forall u \in C\}$ and define

$$U\bar{y} = \begin{cases} B\bar{y} + N_C\bar{y}, & \bar{y} \in C; \\ \emptyset, & \bar{y} \notin C. \end{cases}$$

Then U is the *maximal monotone* and $0 \in U\bar{y}$ if and only if $\bar{y} \in VI(C, B)$; see [16].

For solving the mixed equilibrium problem, let us give the following assumptions for a bifunction $\phi : C \times C \to \mathbb{R}$ and a proper extended real-valued function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;
(A2) ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\to 0} \phi(tz + (1 - t)x, y) \leq \phi(x, y)$;
(A4) for each $x \in C, y \mapsto \phi(x, y)$ is convex and lower semicontinuous;
(A5) for each $y \in C, x \mapsto \phi(x, y)$ is weakly upper semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exist abounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\phi(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

We need the following lemmas for proving our main results.

Lemma 1. (Peng and Yao [15]) *Let* C *be a nonempty closed convex subset of* H. *Let $\phi : C\times C \to \mathbb{R}$ be a bifunction satisfies (A1)-(A5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \Big\{z \in C : \phi(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C\Big\},$$

for all $z \in H$. Then, the following hold:

(1) *For each $x \in H, T_r(x) \neq \emptyset$;*
(2) *T_r is single-valued;*
(3) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_rx - T_ry\|^2 \leq \langle T_rx - T_ry, x - y\rangle$;*
(4) *$F(T_r) = MEP(\phi, \varphi)$;*
(5) *$MEP(\phi, \varphi)$ is closed and convex.*

Lemma 2. (Xu [23]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 3. (Osilike and Igbokwe [14]) *Let $(C, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in C$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 4. (Suzuki [17]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 5. (Marino and Xu [11]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 6. (Bruck [1]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a nonexpansive mapping. For each $x \in C$ and the Cesàro means $T_n x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x$, then $\limsup_{n\to\infty} \|T_n x - T(T_n x)\| = 0$.*

3. Main results

In this section, we show a strong convergence theorem for finding a common element of the set of fixed points of a family of finitely nonexpansive mappings, the set of solutions of mixed equilibrium problem and the set of solutions of a variational inequality problem for a β-inverse strongly monotone mapping in a real Hilbert space by using the viscosity of Cesàro mean approximation method.

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let ϕ be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1)-(A5)$ and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T^i : H \to H$ be a nonexpansive mappings for all $i = 1,2,3,...,n$, such that $\Theta := F(T) \cap VI(C,B) \cap MEP(\phi,\varphi) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$ and let B be a β-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Assume that either B_1 or B_2 holds. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$, $u_n \in C$ and*

$$\begin{cases} \phi(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \forall y \in C, \\ y_n = \delta_n u_n + (1-\delta_n)P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{n+1}\sum_{i=0}^{n} T^i y_n, \forall n \geq 0, \end{cases} \quad (17)$$

where $\{\alpha_n\}, \{\beta_n\}$ *and* $\delta_n \subset (0,1), \{\lambda_n\} \subset (0, 2\beta)$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy the following conditions:*
(i). $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
(ii). $\lim_{n\to\infty} \delta_n = 0$,
(iii). $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
(iv). $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(v). $\{\lambda_n\} \subset [a,b], \exists a,b \in (0,2\beta)$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,
(vi). $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$.
Then, $\{x_n\}$ *converges strongly to* $z \in \Theta$ *where* $z = P_\Theta(I - A + \gamma f)(z)$, *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)z, x - z\rangle \ge 0, \quad \forall x \in \Theta,$$

which is the optimality condition for the minimization problem

$$\min_{x\in\Theta} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h *is a potential function for* γf.

Proof. Now, we have $\|(1-\beta_n)I - \alpha_n A\| \le 1 - \beta_n - \alpha_n\bar{\gamma}$ (see [10] p.479). Since $\lambda_n \in (0, 2\beta)$ and B is a β-inverse-strongly monotone mapping. For any $x, y \in C$ we have

$$\begin{aligned}
\|(I-\lambda_n B)x - (I-\lambda_n B)y\|^2 &= \|(x-y) - \lambda_n(Bx - By)\|^2 \\
&= \|x-y\|^2 - 2\lambda_n\langle x-y, Bx - By\rangle + \lambda_n^2\|Bx - By\|^2 \\
&\le \|x-y\|^2 - 2\lambda_n\beta\|Bx-By\|^2 + \lambda_n^2\|Bx-By\|^2 \\
&= \|x-y\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bx-By\|^2 \\
&\le \|x-y\|^2.
\end{aligned} \tag{18}$$

It follow that $\|(I-\lambda_n B)x - (I-\lambda_n B)y\| \le \|x-y\|$, hence $I - \lambda_n B$ is nonexpansive.

Let $x^* \in \Theta$, T_{r_n} be a sequence of mapping defined as in Lemma 1 and $u_n = T_{r_n}x_n$, for all $n \ge 0$, we have

$$\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \le \|x_n - x^*\|.$$

By the fact that P_C and $I - \lambda_n B$ are nonexpansive and $x^* = P_C(x^* - \lambda_n Bx^*)$, we get

$$\begin{aligned}
\|y_n - x^*\| &= \|\delta_n u_n + (1-\delta_n)P_C(u_n - \lambda_n Bu_n) - x^*\| \\
&\le \delta_n\|u_n - x^*\| + (1-\delta_n)\|P_C(u_n - \lambda_n Bu_n) - P_C(x^* - \lambda_n Bx^*)\| \\
&\le \delta_n\|u_n - x^*\| + (1-\delta_n)\|(I-\lambda_n B)u_n - (I-\lambda_n B)x^*\| \\
&\le \delta_n\|u_n - x^*\| + (1-\delta_n)\|u_n - x^*\| \\
&= \|u_n - x^*\| \\
&\le \|x_n - x^*\|.
\end{aligned} \tag{19}$$

Let $T_n = \frac{1}{n+1}\sum_{i=0}^{n} T^i$, it follows that

$$
\begin{aligned}
\|T_n x - T_n y\| &= \Big\| \frac{1}{n+1} \sum_{i=0}^{n} T^i x - \frac{1}{n+1} \sum_{i=0}^{n} T^i y \Big\| \\
&\le \frac{1}{n+1} \sum_{i=0}^{n} \|T^i x - T^i y\| \\
&\le \frac{1}{n+1} \sum_{i=0}^{n} \|x - y\| \\
&= \frac{n+1}{n+1} \|x - y\| = \|x - y\|,
\end{aligned} \tag{20}
$$

which implies that T_n is nonexpansive. Since $x^* \in \Theta$, we have

$$
T_n x^* = \frac{1}{n+1} \sum_{i=0}^{n} T^i x^* = \frac{1}{n+1} \sum_{i=0}^{n} x^* = x^*
$$

for all $x, y \in C$. By (19) and (20), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(T_n y_n - x^*)\| \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|y_n - x^*\| \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - x^*\| \\
&\le \alpha_n \gamma \|f(x_n) - f(x^*)\| + \alpha_n \|\gamma f(x^*) - Ax^*\| + (1 - \alpha_n \bar{\gamma})\|x_n - x^*\| \\
&\le \alpha_n \gamma \alpha \|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Ax^*\| + (1 - \alpha_n \bar{\gamma})\|x_n - x^*\| \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - x^*\| + \alpha_n(\bar{\gamma} - \gamma\alpha) \frac{\|\gamma f(x^*) - Ax^*\|}{(\bar{\gamma} - \gamma\alpha)} \\
&\le \max\Big\{ \|x_0 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{(\bar{\gamma} - \gamma\alpha)} \Big\}.
\end{aligned}
$$

Hence $\{x_n\}$ is bounded and also $\{u_n\}, \{y_n\}$ and $\{T_n y_n\}$ are bounded.

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Observing that $u_n = T_{r_n} x_n \in$ dom φ and $u_{n+1} = T_{r_{n+1}} x_{n+1} \in$ dom φ, we get

$$
\phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, \forall y \in C \tag{21}
$$

and

$$
\phi(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0, \forall y \in C. \tag{22}
$$

Take $y = u_{n+1}$ in (21) and $y = u_n$ in (22), by using condition (A2), it follows that

$$
\Big\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \Big\rangle \quad \ge \quad 0.
$$

Thus $\Big\langle u_{n+1} - u_n, u_n - u_{n+1} + x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \Big\rangle \ge 0$. Without loss of generality, let us assume that there exists a nonnegative real number c such that $r_n > c, \forall n \ge 1$. Then, we have

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\| \Big\{ \|x_{n+1} - x_n\| + \Big|1 - \frac{r_n}{r_{n+1}}\Big| \|u_{n+1} - x_{n+1}\| \Big\}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| M_1, \end{aligned} \tag{23}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. On the other hand, setting $v_n = P_C(u_n - \lambda_n B u_n)$, it follows from the definition of $\{y_n\}$ that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|\{\delta_{n+1} u_{n+1} + (1 - \delta_{n+1}) P_C(u_{n+1} - \lambda_{n+1} B u_{n+1})\} \\ &\quad - \{\delta_n u_n + (1 - \delta_n) P_C(u_n - \lambda_n B u_n)\}\| \\ &= \|\delta_{n+1}(u_{n+1} - u_n) + (\delta_{n+1} - \delta_n) u_n \\ &\quad + (1 - \delta_{n+1}) P_C(u_{n+1} - \lambda_{n+1} B u_{n+1}) \\ &\quad - (1 - \delta_{n+1}) P_C(u_n - \lambda_n B u_n) + (1 - \delta_{n+1}) P_C(u_n - \lambda_n B u_n) \\ &\quad - (1 - \delta_n) P_C(u_n - \lambda_n B u_n)\| \\ &= \|\delta_{n+1}(u_{n+1} - u_n) + (\delta_{n+1} - \delta_n) u_n \\ &\quad + (1 - \delta_{n+1})\{P_C(u_{n+1} - \lambda_{n+1} B u_{n+1}) - P_C(u_n - \lambda_n B u_n)\} \\ &\quad + (\delta_n - \delta_{n+1}) P_C(u_n - \lambda_n B u_n)\| \\ &\leq \delta_{n+1} \|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n| (\|u_n\| + \|v_n\|) \\ &\quad + (1 - \delta_{n+1}) \|(u_{n+1} - \lambda_{n+1} B u_{n+1}) - (u_n - \lambda_n B u_n)\| \\ &= \delta_{n+1} \|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n| (\|u_n\| + \|v_n\|) \\ &\quad + (1 - \delta_{n+1}) \|(u_{n+1} - \lambda_{n+1} B u_{n+1}) - (u_n - \lambda_{n+1} B u_n) \\ &\quad + (u_n - \lambda_{n+1} B u_n) - (u_n - \lambda_n B u_n)\| \\ &\leq \delta_{n+1} \|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n| (\|u_n\| + \|v_n\|) \\ &\quad + (1 - \delta_{n+1})\{\|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n| \|B u_n\|\} \\ &= |\delta_{n+1} - \delta_n| (\|u_n\| + \|v_n\|) + \|u_{n+1} - u_n\| \\ &\quad + (1 - \delta_{n+1}) |\lambda_{n+1} - \lambda_n| \|B u_n\| \\ &\leq |\delta_{n+1} - \delta_n| (\|u_n\| + \|v_n\|) + \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n| \|B u_n\| \\ &\leq |\delta_{n+1} - \delta_n| (\|u_n\| + \|v_n\|) + \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| M_1 \\ &\quad + |\lambda_{n+1} - \lambda_n| \|B u_n\|. \end{aligned} \tag{24}$$

We compute that

$$
\begin{aligned}
&\|T_{n+1}y_{n+1} - T_n y_n\| \\
&\le \|T_{n+1}y_{n+1} - T_{n+1}y_n\| + \|T_{n+1}y_n - T_n y_n\| \\
&\le \|y_{n+1} - y_n\| + \left\| \frac{1}{n+2}\sum_{i=0}^{n+1} T^i y_n - \frac{1}{n+1}\sum_{i=0}^{n} T^i y_n \right\| \\
&= \|y_{n+1} - y_n\| + \left\| \frac{1}{n+2}\sum_{i=0}^{n} T^i y_n + \frac{1}{n+2}T^{n+1}y_n - \frac{1}{n+1}\sum_{i=0}^{n} T^i y_n \right\| \\
&= \|y_{n+1} - y_n\| + \left\| -\frac{1}{(n+1)(n+2)}\sum_{i=0}^{n} T^i y_n + \frac{1}{n+2}T^{n+1}y_n \right\| \\
&\le \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)}\sum_{i=0}^{n} \|T^i y_n\| + \frac{1}{n+2}\|T^{n+1}y_n\| \\
&\le \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)}\sum_{i=0}^{n} (\|T^i y_n - T^i x^*\| + \|x^*\|) \\
&\quad + \frac{1}{n+2}(\|T^{n+1}y_n - T^{n+1}x^*\| + \|x^*\|) \\
&\le \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)}\sum_{i=0}^{n} (\|y_n - x^*\| + \|x^*\|) + \frac{1}{n+2}(\|y_n - x^*\| + \|x^*\|) \\
&\le \|y_{n+1} - y_n\| + \frac{n+1}{(n+1)(n+2)}(\|y_n - x^*\| + \|x^*\|) + \frac{1}{n+2}\|y_n - x^*\| + \frac{1}{n+2}\|x^*\| \\
&= \|y_{n+1} - y_n\| + \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\| \\
&\le |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) + \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M_1 \\
&\quad + |\lambda_{n+1} - \lambda_n|\|Bu_n\| + \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\|.
\end{aligned}
$$

Let $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, then

$$
z_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)T_n y_n}{1-\beta_n},
$$

and hence

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)T_{n+1}y_{n+1}}{1-\beta_{n+1}} \right. \\
&\quad \left. - \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)T_n y_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})T_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}AT_{n+1}y_{n+1}}{1-\beta_{n+1}} \right. \\
&\quad \left. - \frac{\alpha_n \gamma f(x_n)}{1-\beta_n} - \frac{(1-\beta_n)T_n y_n}{1-\beta_n} + \frac{\alpha_n AT_n y_n}{1-\beta_n} \right\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}) + \frac{\alpha_n}{1-\beta_n}(AT_n y_n - \gamma f(x_n)) + T_{n+1}y_{n+1} - T_n y_n \right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|AT_n y_n - \gamma f(x_n)\| + \|T_{n+1}y_{n+1} - T_n y_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|AT_n y_n - \gamma f(x_n)\| \\
&\quad + |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) + \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M_1 + |\lambda_{n+1} - \lambda_n|\|Bu_n\| \\
&\quad + \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
\le\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|AT_n y_n - \gamma f(x_n)\| \\
&+ |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) + \frac{1}{c}\|r_{n+1} - r_n\|M_1 + |\lambda_{n+1} - \lambda_n|\|Bu_n\| \\
&+ \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\|.
\end{aligned}
$$

It follows from $n \to \infty$ and the conditions (i)-(v), that

$$
\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{25}
$$

From Lemma 4 and (25), we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and also

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.
$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$. For $x^* \in \Theta$, we obtain

$$
\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^* \rangle \\
&= \langle u_n - x^*, x_n - x^* \rangle \\
&= \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x^* - x_n + x^*\|^2) \\
&= \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n\|^2),
\end{aligned}
$$

and hence

$$
\|u_n - x^*\|^2 \quad \le \quad \|x_n - x^*\|^2 - \|u_n - x_n\|^2. \tag{26}
$$

Since $\|y_n - x^*\| \le \|u_n - x^*\|$ and from Lemma 3 and (26), we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&=\|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)T_n y_n-x^*\|^2\\
&=\|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n A)(T_n y_n-x^*)\|^2\\
&\le \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\|y_n-x^*\|^2\\
&\le \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-x^*\|^2\\
&\le \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})(\|x_n-x^*\|^2-\|x_n-u_n\|^2)\\
&\le \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2-(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-u_n\|^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-u_n\|^2\\
&\le \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&=\alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|).
\end{aligned}$$

By $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, (i) and (iv), imply that

$$\lim_{n\to\infty}\|u_n-x_n\|=0.$$

Since $\liminf_{n\to\infty} r_n>0$, we obtain

$$\lim_{n\to\infty}\left\|\frac{x_n-u_n}{r_n}\right\|=\lim_{n\to\infty}\frac{1}{r_n}\|x_n-u_n\|=0.$$

Next, we show that $\lim_{n\to\infty}\|T_n y_n-x_n\|=0$. Indeed, observe that

$$\begin{aligned}
\|x_n-T_n y_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T_n y_n\|\\
&=\|x_n-x_{n+1}\|+\|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)T_n y_n-T_n y_n\|\\
&=\|x_n-x_{n+1}\|+\|\alpha_n\gamma f(x_n)-\alpha_n AT_n y_n+\alpha_n AT_n y_n+\beta_n x_n\\
&\quad -\beta_n T_n y_n+\beta_n T_n y_n+((1-\beta_n)I-\alpha_n A)T_n y_n-T_n y_n\|\\
&\le \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-AT_n y_n\|+\beta_n\|x_n-T_n y_n\|
\end{aligned}$$

and then

$$\|x_n-T_n y_n\|\le \frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-AT_n y_n\|. \tag{27}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, (i) and (iv), we get $\lim_{n\to\infty}\|x_n-T_n y_n\|=0$.

Next, we show that $\lim_{n\to\infty}\|y_n-v_n\|=0$, where $v_n=P_C(u_n-\lambda_n Bu_n)$. From Lemma 3, (13) and (18), we obtain

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n \|x_n - x^*\|^2 + ((1-\beta_n)I - \alpha_n A)\|T_n y_n - x^*\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n \|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\bar{\gamma})\|y_n - x^*\|^2 \\
&= \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n \|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\bar{\gamma})\|\delta_n(u_n - x^*) \\
&\quad + (1-\delta_n)\{P_C(u_n - \lambda_n B u_n) - P_C(x^* - \lambda_n B x^*)\}\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n \|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\bar{\gamma})\delta_n\|u_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)\|(u_n - \lambda_n B u_n) - (x^* - \lambda_n B x^*)\|^2 \\
&= \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n \|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\bar{\gamma})\delta_n\|u_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)\|(u_n - x^*) - \lambda_n(B u_n - B x^*)\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n \|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\bar{\gamma})\delta_n\|u_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)\{\|u_n - x^*\|^2 + \lambda_n(\lambda_n - 2\beta)\|B u_n - B x^*\|^2\} \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + (1 - \alpha_n\bar{\gamma})\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)\lambda_n(\lambda_n - 2\beta)\|B u_n - B x^*\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \|x_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)a(b - 2\beta)\|B u_n - B x^*\|^2,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
0 &\le (1-\beta_n - \alpha_n\bar{\gamma})(1-\delta_n)a(2\beta - b)\|B u_n - B x^*\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \|x_{n+1} - x_n\|((\|x_n - x^*\| + \|x_{n+1} - x^*\|).
\end{aligned}
$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$, we obtain $\|B u_n - B x^*\| \to 0$ as $n \to \infty$. Using (12), we have

$$
\begin{aligned}
\|v_n - x^*\|^2 &= \|P_C(u_n - \lambda_n B u_n) - P_C(x^* - \lambda_n B x^*)\|^2 \\
&\le \langle (u_n - \lambda_n B u_n) - (x^* - \lambda_n B x^*), v_n - x^* \rangle \\
&= \frac{1}{2}\Big\{\|(u_n - \lambda_n B u_n) - (x^* - \lambda_n B x^*)\|^2 + \|v_n - x^*\|^2\Big\} \\
&\quad - \frac{1}{2}\Big\{\|(u_n - \lambda_n B u_n) - (x^* - \lambda_n B x^*) - (v_n - x^*)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|v_n - x^*\|^2 - \|(u_n - v_n) - \lambda_n(B u_n - B x^*)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|v_n - x^*\|^2\Big\} - \frac{1}{2}\Big\{\|u_n - v_n\|^2 - \lambda_n^2\|B u_n - B x^*\|^2 \\
&\quad - 2\langle u_n - v_n - \lambda_n(B u_n - B x^*), \lambda_n(B u_n - B x^*)\rangle\Big\} \\
&\le \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|v_n - x^*\|^2\Big\} - \frac{1}{2}\Big\{\|u_n - v_n\|^2 - \lambda_n^2\|B u_n - B x^*\|^2 \\
&\quad - 2\lambda_n\langle u_n - v_n, B u_n - B x^*\rangle + 2\lambda_n^2\|B u_n - B x^*\|^2\Big\},
\end{aligned}
$$

so, we obtain

$$\|v_n-x^*\|^2 \leq \|u_n-x^*\|^2-\|u_n-v_n\|^2-\lambda_n^2\|Bu_n-Bx^*\|^2+2\lambda_n\langle u_n-v_n, Bu_n-Bx^*\rangle,$$

and hence

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\|y_n-x^*\|^2 \\
&= \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\|\delta_n u_n+(1-\delta_n)v_n-x^*\|^2 \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\delta_n\|u_n-x^*\|^2 \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})(1-\delta_n)\|v_n-x^*\|^2 \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\delta_n\|u_n-x^*\|^2 \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})\|v_n-x^*\|^2 \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\delta_n\|u_n-x^*\|^2 \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})\Big\{\|u_n-x^*\|^2-\|u_n-v_n\|^2-\lambda_n^2\|Bu_n-Bx^*\|^2 \\
&\quad +2\lambda_n\langle u_n-v_n, Bu_n-Bx^*\rangle\Big\} \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\delta_n\|u_n-x^*\|^2 \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-x^*\|^2-(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-v_n\|^2 \\
&\quad -(1-\beta_n-\alpha_n\bar{\gamma})\lambda_n^2\|Bu_n-Bx^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})2\lambda_n\langle u_n-v_n, Bu_n-Bx^*\rangle \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\delta_n\|u_n-x^*\|^2 \\
&\quad -(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-v_n\|^2-(1-\beta_n-\alpha_n\bar{\gamma})\lambda_n^2\|Bu_n-Bx^*\|^2 \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})2\lambda_n\|u_n-v_n\|\|Bu_n-Bx^*\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-v_n\|^2 \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})\delta_n\|u_n-x^*\|^2 \\
&\quad -(1-\beta_n-\alpha_n\bar{\gamma})\lambda_n^2\|Bu_n-Bx^*\|^2+(1-\beta_n-\alpha_n\bar{\gamma})2\lambda_n\|u_n-v_n\|\|Bu_n-Bx^*\| \\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|) \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})\delta_n\|u_n-x^*\|^2-(1-\beta_n-\alpha_n\bar{\gamma})\lambda_n^2\|Bu_n-Bx^*\|^2 \\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})2\lambda_n\|u_n-v_n\|\|Bu_n-Bx^*\|.
\end{aligned}$$

Since $\|Bu_n-Bx^*\| \to 0$, $\|x_{n+1}-x_n\| \to 0$ as $n \to 0$ and the condition (i), (ii), (iv), we have $\|u_n-v_n\| \to 0$ as $n \to 0$. At the same time, we note that

$$\|y_n-v_n\| = \|\delta_n(u_n-v_n)+(1-\delta_n)(v_n-v_n)\| = \delta_n\|u_n-v_n\|,$$

since $\delta_n \to 0$, we have $\|y_n-v_n\| \to 0$ as $n \to \infty$. Consequently, we observe that

$$\|T_ny_n-y_n\| \leq \|T_ny_n-x_n\|+\|x_n-u_n\|+\|u_n-v_n\|+\|v_n-y_n\| \to 0, \text{ as } n \to \infty.$$

It is easy to see that $P_\Theta(I-A+\gamma f)(z)$ is a contradiction of H into itself. Hence H is complete, there exists a unique fixed point $z \in H$, such that $z = P_\Theta(I-A+\gamma f)(z)$.

Next, we show that

$$\limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle \le 0. \tag{28}$$

Indeed, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$, such that

$$\lim_{i\to\infty}\langle (A-\gamma f)z, z-y_{n_i}\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)z, z-y_n\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converge weakly to $\bar{y}\in C$. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup \bar{y}$. From $\|T_n y_n - y_n\|\to 0$, we obtain $T_n y_{n_i}\rightharpoonup \bar{y}$.

Let us show $\bar{y}\in MEP(\phi,\varphi)$. Since $u_n = T_{r_n}x_n \in \mathrm{dom}\varphi$, we obtain

$$\phi(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \forall y\in C.$$

From (A2), we also have

$$\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge F(y,u_n), \forall y\in C$$

and hence

$$\varphi(y)-\varphi(u_n)+\langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \ge F(y,u_{n_i}), \forall y\in C.$$

From $\|u_n-x_n\|\to 0, \|x_n-T_n y_n\|\to 0$ and $\|T_n y_n - y_n\|\to 0$, we get $u_{n_i}\to\bar{y}$. Since $\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\to 0$, thus that from (A4) and the weakly lower semicontinuity of φ that $\phi(y,\bar{y})+\varphi(\bar{y})-\varphi(y)\le 0, \forall y\in C$. For t with $0<t\le 1$ and $x\in C$, let $x_t = tx+(1-t)\bar{y}$. Since $x\in C$ and $\bar{y}\in C$, we have $x_t\in C$ and hence $\phi(x_t,\bar{y})+\varphi(\bar{y})-\varphi(x_t)\le 0$. So, from (A1), (A4) and the convexity of φ, we have

$$\begin{aligned}0 &= \phi(x_t,x_t)+\varphi(x_t)-\varphi(x_t)\\ &\le t\phi(x_t,x)+(1-t)\phi(x_t,\bar{y})+t\varphi(x)+(1-t)\varphi(\bar{y})-\varphi(x_t)\\ &\le t(\phi(x_t,x)+\varphi(x)-\varphi(x_t)).\end{aligned} \tag{29}$$

Dividing by t, we get $\phi(x_t,x)+\varphi(x)-\varphi(x_t)\ge 0$. From (A3) and the weakly lower semicontinuity of φ, we have $\phi(\bar{y},y)+\varphi(y)-\varphi(\bar{y})\ge 0$, for all $y\in C$ and hence $\bar{y}\in MEP(\phi,\varphi)$.

Next, we show that $\bar{y}\in F(T_n)=\frac{1}{n+1}\sum_{i=0}^{n}F(T^i)$. Assume that $\bar{y}\notin \frac{1}{n+1}\sum_{i=0}^{n}F(T^i)$, since $y_{n_i}\rightharpoonup\bar{y}$ and $T_n\bar{y}\ne\bar{y}$. From Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i}-\bar{y}\| &< \liminf_{i\to\infty}\|y_{n_i}-T_n\bar{y}\|\\ &\le \liminf_{i\to\infty}(\|y_{n_i}-T_n y_{n_i}\|+\|T_n y_{n_i}-T_n\bar{y}\|)\\ &\le \liminf_{i\to\infty}\|y_{n_i}-\bar{y}\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $\bar{y} \in F(T_n) = \frac{1}{n+1}\sum_{i=0}^{n} F(T^i)$.

Now, let us show that $\bar{y} \in VI(C,B)$. Let $U : H \to 2^H$ be a set-valued mapping is defined by

$$U\bar{y} = \begin{cases} B\bar{y} + N_C\bar{y}, & \bar{y} \in C, \\ \emptyset, & \bar{y} \notin C, \end{cases}$$

where $N_C\bar{y}$ is the normal cone to C at $\bar{y} \in C$. We have U is maximal monotone and $0 \in U\bar{y}$ if and only if $\bar{y} \in VI(C,B)$. Let $(\bar{y}, w) \in G(U)$, hence $w - B\bar{y} \in N_C\bar{y}$ and $v_n \in C$, we have $\langle \bar{y} - v_n, w - B\bar{y}\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n B u_n)$, we have

$$\langle \bar{y} - v_n, v_n - (u_n - \lambda_n B u_n)\rangle \geq 0,$$

that is

$$\left\langle \bar{y} - v_n, \frac{v_n - u_n}{\lambda_n} + Bu_n \right\rangle \geq 0.$$

Therefor, we have

$$\begin{aligned}
\langle \bar{y} - v_{n_i}, w\rangle &\geq \langle \bar{y} - v_{n_i}, B\bar{y}\rangle \\
&\geq \langle \bar{y} - v_{n_i}, B\bar{y}\rangle - \left\langle \bar{y} - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i} \right\rangle \\
&= \left\langle \bar{y} - v_{n_i}, B\bar{y} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} - Bu_{n_i} \right\rangle \\
&= \langle \bar{y} - v_{n_i}, B\bar{y} - Bv_{n_i}\rangle + \langle \bar{y} - v_{n_i}, Bv_{n_i} - Bu_{n_i}\rangle \\
&\quad - \left\langle \bar{y} - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \qquad (30)\\
&\geq \langle \bar{y} - v_{n_i}, Bv_{n_i} - Bu_{n_i}\rangle - \left\langle \bar{y} - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \|\bar{y} - v_{n_i}\|\|Bv_{n_i} - Bu_{n_i}\| - \|\bar{y} - v_{n_i}\|\left\|\frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\|.
\end{aligned}$$

Noting that $\|v_{n_i} - u_{n_i}\| \to 0$ as $i \to \infty$ and B is β-inverse-strongly monotone, hence from (30), we obtain $\langle \bar{y} - z, w\rangle \geq 0$ as $i \to \infty$. Since U is maximal monotone, we have $\bar{y} \in U^{-1}0$, and hence $\bar{y} \in VI(C,B)$. Therefore $\bar{y} \in \Theta$.

Since $z = P_\Theta(I - A + \gamma f)(z)$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle((\gamma f - A)z, x_n - z\rangle &= \limsup_{n\to\infty}\langle(\gamma f - A)z, T_n y_n - z\rangle \\
&= \limsup_{i\to\infty}\langle(\gamma f - A)z, T_n y_{n_i} - z\rangle \qquad (31)\\
&= \langle(\gamma f - A)z, \bar{y} - z\rangle \leq 0.
\end{aligned}$$

Finally we show that $\{x_n\}$ converge strongly to z, we obtain that

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&=\|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)T_n y_n-z\|^2\\
&=\|\alpha_n(\gamma f(x_n)-Az)+\beta_n(x_n-z)+((1-\beta_n)I-\alpha_n A)(T_n y_n-z)\|^2\\
&=\alpha_n^2\|\gamma f(x_n)-Az\|^2+\|\beta_n(x_n-z)+((1-\beta_n)I-\alpha_n A)(T_n y_n-z)\|^2\\
&\quad+2\langle\beta_n(x_n-z)+((1-\beta_n)I-\alpha_n A)(T_n y_n-z),\alpha_n(\gamma f(x_n)-Az)\rangle\\
&\le\alpha_n^2\|\gamma f(x_n)-Az\|^2+\{\beta_n\|x_n-z\|+(1-\beta_n-\alpha_n\bar\gamma)\|y_n-z\|\}^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(x_n)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(x_n)-Az\rangle\\
&\le\alpha_n^2\|\gamma f(x_n)-Az\|^2+\{\beta_n\|x_n-z\|+(1-\beta_n-\alpha_n\bar\gamma)\|x_n-z\|\}^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(x_n)-\gamma f(z)\rangle+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(x_n)-\gamma f(z)\rangle\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(z)-Az\rangle\\
&\le\alpha_n^2\|\gamma f(x_n)-Az\|^2+(1-\alpha_n\bar\gamma)^2\|x_n-z\|^2+2\alpha_n\beta_n\gamma\|x_n-z\|\|f(x_n)-f(z)\| \qquad (32)\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\gamma\|T_n y_n-z\|\|f(x_n)-f(z)\|\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(z)-Az\rangle\\
&\le\alpha_n^2\|\gamma f(x_n)-Az\|^2+(1-\alpha_n\bar\gamma)^2\|x_n-z\|^2+2\alpha_n\beta_n\gamma\alpha\|x_n-z\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\gamma\alpha\|x_n-z\|^2\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(z)-Az\rangle\\
&=\alpha_n^2\|\gamma f(x_n)-Az\|^2+(1-2\alpha_n\bar\gamma+\alpha_n^2\bar\gamma^2+2\alpha_n\gamma\alpha-2\alpha_n^2\bar\gamma\gamma\alpha)\|x_n-z\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(z)-Az\rangle\\
&\le\{1-\alpha_n(2\bar\gamma-\alpha_n\bar\gamma^2-2\gamma\alpha+2\alpha_n\bar\gamma\gamma\alpha)\}\|x_n-z\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(z)-Az\rangle\\
&\le\{1-\alpha_n(2\bar\gamma-\alpha_n\bar\gamma^2-2\gamma\alpha+2\alpha_n\bar\gamma\gamma\alpha)\}\|x_n-z\|^2+\alpha_n\sigma_n,
\end{aligned}$$

where $\sigma_n=\alpha_n\|\gamma f(x_n)-Az\|^2+2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n-\alpha_n\bar\gamma)\langle T_n y_n-z,\gamma f(z)-Az\rangle$. By (31) and (i), we get $\limsup_{n\to\infty}\sigma_n\le 0$. Applying Lemma 2 to (32) we conclude that $x_n\to z$. This complete the proof. □

Using Theorem 1, we obtain the following corollary.

Corollary 1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let ϕ be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying $(A1)-(A5)$ and let $T: H\to H$ be a nonexpansive mapping such that $\Omega := F(T)\cap VI(C,B)\cap EP(\phi)\neq\emptyset$. Let B be a β-inverse-strongly monotone mapping of C into H. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0\in H$, $u_n\in C$ and*

$$\begin{cases}\phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\ \forall y\in C,\\ y_n=P_C(u_n-\lambda_n Bu_n),\\ x_{n+1}=\alpha_n v+\beta_n x_n+(1-\beta_n-\alpha_n)Ty_n,\forall n\ge 0,\end{cases}\qquad (33)$$

where $\{\alpha_n\},\{\beta_n\}\subset(0,1),\{\lambda_n\}\subset(0,2\beta)$, for all $n\ge 0$ satisfy the condition $(i),(iii)-(v)$. Then, $\{x_n\}$ converges strongly to $z\in\Omega$ where $z=P_\Omega z$.

Proof. Taking $T^i = T$ for $i = 0, 1, \ldots, n$, $\gamma = 1$, $\delta_n = 0$, $f(x_n) = v$, $A = I$ and $\varphi \equiv 0$ in Theorem (17), we can conclude the desired conclusion easily. This completes the proof. □

Acknowledgments

The authors would like to thanks the "National Research University (NRU)", KMUTT, the Commission on Higher Education, Ministry of Education, Thailand.

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