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# STRONG LAW OF LARGE NUMBERS FOR ASYMPTOTICALLY NEGATIVE DEPENDENT RANDOM VARIABLES WITH APPLICATIONS†

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Abstract. In this paper, we obtain the Hàjeck-Rènyi type inequality and the strong law of large numbers for asymptotically linear negative quadrant dependent random variables by using this inequality. We also give the strong law of large numbers for the linear process under asymptotically linear negative quadrant dependence assumption.

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## 1. Introduction

Two random variables $X$ and $Y$ are negatively quadrant dependent(NQD) if $Cov(f(X), g(Y)) \leq 0$ for every coordinatewise nondecreasing functions $f$ and $g$. A sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is linear negative quadrant dependent(LNQD) if $Cov(f(\sum_{i\in A} a_i X_i), g(\sum_{j\in B} a_j X_j)) \leq 0$ for every pair of disjoint subsets $A, B$ of $\{1, 2, \cdots, n\}$, $a_i, a_j \geq 0$ and every coordinatewise nondecreasing functions $f$ and $g$.

For two nonempty disjoint sets $S, T \subset \mathbf{R}$, let $dist(S,T)$ be $\inf\{|s-t|, s \in S, t \in T\}$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be $\rho^*$-mixing if $\rho^*(r) = \sup\{\rho(S,T), dist(S,T) \geq r, S, T \subset \mathbf{R}\ \} \to 0$ as $r \to \infty$, where $\rho(S,T) = \sup\{|Cov(f,g)|(Var f)^{-\frac{1}{2}}(Var\ g)^{-\frac{1}{2}}, f \in L_2(\sigma(S)), g \in L_2(\sigma(T))\}$ and $\sigma(S)$ is the $\sigma$-field generated by $\{X_k, k \in S\}$, and $\sigma(T)$ similarly.

For any disjoint subsets $S, T \subset \mathbf{N}$, let $\rho^-(S,T) = \sup\{\rho^-(X,Y), X \in F(S), Y \in F(T)\}$, where $F(S) = \{\sum_{k\in S} a_k X_k, a_k \geq 0$ and $a_k \neq 0$ for finitely many $k's\}$ and $F(T)$ similarly.

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Let $\rho^-(X,Y) = 0 \vee \sup\{Cov(f(X),g(Y))/(Varf(X))^{1/2}(Varg(Y))^{1/2}\}$, for all coordinatewise non-decreasing functions $f,g$ such that $E[f(X)]^2 < \infty$ and $E[g(Y)]^2 < \infty$.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be asymptotically linear negative quadrant dependent(ALNQD) if

$$\rho^-(r) = \sup\{\rho^-(S,T); dist(S,T) \geq r, S,T \subset \mathbf{N} \text{ are finite}\} \to 0 \text{ as } r \to \infty.$$

The concept of ALNQD random variables was introduced by Zhang(2000). It is obvious that either a linear negative quadrant dependent(LNQD)sequence or a $\rho^*$-mixing sequence is an ALNQD sequence.

Zhang(2000) pointed out that $\rho^-(f(X),g(Y)) \leq \rho^-(X,Y)$ for any coordinatewise non-decreasing functions $f$ and $g$ and showed some examples of an ALNQD sequence which is possibly neither LNQD nor $\rho^*$-mixing.

Hàjeck-Rènyi(1955) proved the following important inequality : If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$ and $\{b_n, n \geq 1\}$ is a sequence of positive nondecreasing real numbers, then for any $\epsilon > 0$, any positive integer $m < n$,

$$P(\max_{m \leq k \leq n} |\frac{\sum_{j=1}^k X_j}{b_k}| \geq \epsilon) \leq \epsilon^{-2}(\sum_{j=m+1}^n \frac{EX_j^2}{b_j^2} + \sum_{j=1}^m \frac{EX_j^2}{b_m^2}).$$

Since then, this inequality has been studied by many authors. For example, Liu, Gan and Chen(1999) proved the Hàjeck-Rènyi inequaliy for NA random variables, Christofides(2000) studied the Hàjeck-Rènyi inequalities and strong law of large numbers for demimartingales, Fazekas and Klesov(2001) and Hu, S. and Hu, M.(2006) investigated the general method for the Hàjeck-Rènyi type maximal inequality, and Hu, Wang, Yang and Zhao(2009) proved the Hàjeck-Rènyi type inequaliy for associated random variables.

In this paper, we will prove the Hàjeck-Rènyi type inequality, the strong law of large numbers, and integrability of supremum for ALNQD random variables, which have not been established previously in the literature. We also consider the strong law of large numbers for a linear process generated by ALNQD random variables.

## 2. Hàjeck-Rènyi type inequality for ALNQD random variables

The following lemmas is needed to prove the Hàjeck-Rènyi type inequality for ALNQD random variables.

**Lemma 2.1(Fazekas, Klesov(2000)))** Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables, $\{b_n, n \geq 1\}$ a non-decreasing sequence of positive numbers, and $\{\alpha_n, n \geq 1\}$ a sequence of nonnegative numbers. Let $r$ be a fixed positive number and $S_n = X_1 + \cdots + X_n$. Assume that for each $m$ with $1 \leq m \leq n$

$$E(\max_{1 \leq k \leq m} |S_k|^r) \leq \sum_{i=1}^m \alpha_i. \tag{2.1}$$

Then

$$E(\max_{1\le k\le m}|\frac{S_k}{b_k}|^r)\le 4\sum_{i=1}^{m}\frac{\alpha_i}{b_i^r}. \tag{2.2}$$

**Lemma 2.2(Zhang(2000))** Let $\{X_n,\ n\ge 1\}$ be a sequence of ALNQD random variables with $EX_n=0, n\ge 1$. Then, for any $r>2$ there exists a positive constant $D_r=D(r,\rho^-(\cdot))$ such that

$$E\max_{1\le k\le n}|S_k|^r\le D_r n^{\frac{r}{2}}\max_{1\le k\le n}E|X_k|^r.$$

**Theorem 2.3** Let $\{X_n,\ n\ge 1\}$ be a sequence of ALNQD random variables with $EX_n=0$ and $EX_n^2<\infty$, $n\ge 1$ and $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers. Assume $\sup_{n\ge 1}E|X_n|^r<\infty$ for any $r>2$. Then, for all $\epsilon>0$, there exists a positive constant $C$ such that

$$P(\max_{1\le k\le n}|\frac{1}{b_k}S_k|\ge\epsilon)\le C\sum_{k=1}^{n}\frac{k^{\frac{r}{2}}-(k-1)^{\frac{r}{2}}}{b_k^r}$$

where $C=4\epsilon^{-r}D_r\max_{1\le k\le n}E|X_k|^r$.

**Proof** By Markov inequality and Lemmas 2.1 and 2.2 we have

$$\begin{aligned}P(\max_{1\le k\le n}|\frac{S_k}{b_k}|\ge\epsilon) &\le \epsilon^{-r}D_r\max_{1\le k\le n}E|X_k|^r\sum_{k=1}^{n}\frac{k^{\frac{r}{2}}-(k-1)^{\frac{r}{2}}}{b_k^r}\\ &\le C\sum_{k=1}^{n}\frac{k^{\frac{r}{2}}-(k-1)^{\frac{r}{2}}}{b_k^r},\ r>2,\end{aligned}$$

where $D_r=D(r,\rho^-(\cdot))$ and $C=4\epsilon^{-r}D_r\max_{1\le k\le n}E|X_k|^r$.

From Theorem 2.3 we get the following more generalized Hàjeck-Rènyi type inequality.

**Theorem 2.4** Let $\{b_n, n\ge 1\}$ be a sequence of nondecreasing positive numbers and let $\{X_n,\ n\ge 1\}$ be a sequence of centered ALNQD random variables. Assume $\sup_{n\ge 1}E|X_n|^r<\infty$ for any $r>2$. Then, for any $r>2$, there exists a positive constant $C$ such that

$$P(\max_{m\le k\le n}|\frac{S_k}{b_n}|\ge\epsilon)\le C(\sum_{k=m+1}^{n}\frac{(k-m)^{\frac{r}{2}}-(k-1-m)^{\frac{r}{2}}}{b_k^r}+\frac{m^{\frac{r}{2}}}{b_m^r}),$$

where $C=4\epsilon^{-r}D_r\sup_{n\ge 1}E|X_n|^r$.

## 3. SLLNs and integrability of supremum for ALNQD rvs

**Theorem 3.1** Let $\{X_n,\ n \geq 1\}$ be a sequence of centered ALNQD random variables with $\sup_{n\geq 1} E|X_n|^r < \infty$ for any $r > 2$ and $\{b_n, n \geq 1\}$ be a sequence of nondecreasing unbounded positive numbers. If, for any $r > 2$,

$$\sum_{k=1}^{\infty} \frac{k^{\frac{1}{r}} - (k-1)^{\frac{r}{2}}}{b_k^r} < \infty, \tag{3.1}$$

then we have

$$\frac{S_n}{b_n} \to 0 \text{ a.s. as } n \to \infty. \tag{3.2}$$

**Proof** By Theorem 2.4 we have

$$P(\max_{m\leq k\leq n} |\frac{S_k}{b_k}| \geq \epsilon) \leq C(\sum_{k=m+1}^{n} \frac{(k-m)^{\frac{r}{2}} - (k-1-m)^{\frac{r}{2}}}{b_k^r} + \frac{m^{\frac{r}{2}}}{b_m^r}).$$

But

$$\begin{aligned} P(\sup_{n\geq m} |\frac{S_n}{b_n}| \geq \epsilon) &= \lim_{n\to\infty} P(\max_{m\leq k\leq n} |\frac{S_k}{b_k}| \geq \epsilon) \\ &\leq C(\sum_{k=m+1}^{\infty} \frac{(k-m)^{\frac{r}{2}} - (k-1-m)^{\frac{r}{2}}}{b_k^r} + \frac{m^{\frac{r}{2}}}{b_m^r}). \end{aligned}$$

By the Kronecker lemma, we get

$$\lim_{m\to\infty} P(\sup_{n\geq m} |\frac{S_n}{b_n}| \geq \epsilon) = 0.$$

Then, $S_n/b_n \to 0$ $a.s.$ follows.

**Remark** By Lemma 2.1 and Markov inequality we have

$$\begin{aligned} \sum_{n=1}^{\infty} P(\frac{|S_n|}{b_n} \geq \epsilon) &\leq \sum_{n=1}^{\infty} P(\max_{1\leq k\leq n} \frac{|S_k|}{b_k} \geq \epsilon) \\ &\leq 4\epsilon^{-r} D_r \sup_n E|X_n|^r \sum_{k=1}^{\infty} \frac{k^{\frac{r}{2}} - (k-1)^{\frac{r}{2}}}{b_k^r} \\ &\leq C \sum_{n=1}^{\infty} \frac{k^{\frac{r}{2}} - (k-1)^{\frac{r}{2}}}{b_k^r} < \infty \text{ by (3.1).} \end{aligned}$$

Hence, by Borel-Cantelli lemma (3.2) follows.

In Theorem 3.1, by taking $b_n = n$ we obtain the following corollary.

**Corollary 3.2** Let $\{X_n,\ n \geq 1\}$ be a sequence of ALNQD random variables with $EX_n = 0$. Assume $\sup_{n\geq 1} E|X_n|^r < \infty$ for any $r > 2$. Then,

$$\frac{S_n}{n} \to 0 \text{ a.s. as } n \to \infty. \tag{3.3}$$

**Theorem 3.3** Let $\{b_n,\ n \geq 1\}$ be a sequence of nondecreasing positive numbers satisfying (3.1) and let $\{X_n,\ n \geq 1\}$ be a sequence of centered ALNQD random variables satisfying $\sup_{n\geq 1} E|X_n|^r < \infty$ for any $r > 2$. Then, for any $0 < p < 2$

$$E(\sup_{n\geq 1} |\frac{S_n}{b_n}|^p) < \infty. \tag{3.4}$$

**Proof** Note that

$$E(\sup_{n\geq 1} |\frac{S_n}{b_n}|^p) < \infty. \iff \int_1^\infty P(\sup_{n\geq 1} |\frac{S_n}{b_n}| > t^{\frac{1}{p}})dt < \infty$$

For any $r > 2$

$$\begin{aligned}
\int_1^\infty P(\sup_{n\geq 1} |\frac{S_n}{b_n}| > t^{\frac{1}{p}})dt &\leq C\int_1^\infty t^{-\frac{r}{p}}(\sum_{n=1}^\infty \frac{n^{\frac{r}{2}} - (n-1)^{\frac{r}{2}}}{b_n^r})dt \\
&= C\sum_{n=1}^\infty \frac{n^{\frac{r}{2}} - (n-1)^{\frac{r}{2}}}{b_n^r}\int_1^\infty t^{-\frac{r}{p}}dt \\
&< \infty, \text{ since } \frac{r}{p} > 1.
\end{aligned}$$

From Theorem 3.3 we obtain the following result.

**Corollary 3.4** Let $\{X_n,\ n \geq 1\}$ be a sequence of centered ALNQD random variables with $\sup_{n\geq 1} E|X_n|^r < \infty$ for any $r > 2$. Then, for any $0 < p < 2$ we have

$$E\sup_{n\geq 1}(\frac{|S_n|}{n})^p < \infty.$$

Now we apply Theorem 3.1 to the linear process:

Let $\{Y_n,\ n \geq 1\}$ be a stationary linear process of the form below defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$

$$Y_n = \sum_{j=0}^\infty a_j X_{n-j}, \tag{3.5}$$

where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^\infty |a_j| < \infty$ and $\{X_n\}$ is a strictly stationary sequence of ALNQD random variables with $EX_1 = 0$ and $0 < EX_1^2 < \infty$.

**Lemma 3.5** Let $\{X_i, -\infty < i < \infty\}$ be a sequence of ALNQD random variables with $EX_i = 0$, $-\infty < i < \infty$. Then, for any $r > 2$ there exists a positive constant $D(r, \rho^-(\cdot))$ such that

$$E|\max_{1\leq k\leq n} X_{i+1} + \cdots + X_{i+k}|^r \leq D_r n^{\frac{r}{2}} \max_{-\infty<i<\infty} E|X_i|^r.$$

**Proof** The proof is similar to that of Lemma 2.2.

**Lemma 3.6** Let $\{X_i, -\infty < i < \infty\}$ be a strictly stationary sequence of ALNQD random variables with $EX_1 = 0$, $0 < EX_1^2 < \infty$ and $E|X_1|^r < \infty$ for any $r > 2$. Let $Y_n = \sum_{j=0}^{\infty} a_j X_{n-j}$, $T_k = \sum_{i=1}^{k} Y_i$, $\widetilde{Y}_i = (\sum_{j=0}^{\infty} a_j) X_i$, and $\widetilde{T}_k = \sum_{i=1}^{k} \widetilde{Y}_i$, where $\{a_j\}$ is a sequence of positive numbers with $\sum_{j=0}^{\infty} a_j < \infty$. If $\{b_n\}$ is a sequence of nondecreasing unbounded positive numbers and (3.1) holds, then

$$b_n^{-1} \max_{1 \le k \le n} |\tilde{T}_k - T_k| \to 0 \text{ a.s. as } n \to \infty.$$

**Proof** See the Appendix.

**Theorem 3.7** Let $Y_n = \sum_{j=0}^{\infty} a_j X_{n-j}$ be a linear process, where $\{a_j\}$ is a sequence of positive constants with $\sum_{j=0}^{\infty} a_j < \infty$ and $\{X_i, -\infty < i < \infty\}$ is a strictly stationary sequence of centered ALNQD random variables with $E|X_1|^r < \infty$ for any $r > 2$. Let $\{b_n, n \ge 1\}$ be a sequence of nondecreasing unbounded positive numbers. Assume that (3.1) holds. Then

$$\frac{T_n}{b_n} \to 0 \ a.s. \tag{3.6}$$

where $T_n = \sum_{i=1}^{n} Y_i$.
**Proof** As in Lemma 3.6, set

$$\widetilde{Y}_n = \sum_{j=0}^{\infty} a_j X_n$$

and

$$\widetilde{T}_n = \sum_{i=1}^{n} \widetilde{Y}_i = (\sum_{j=0}^{\infty} a_j) \sum_{i=1}^{n} X_i.$$

Then by Theorem 3.1

$$\frac{\widetilde{T}_n}{b_n} \to 0 \ a.s.$$

According to Lemma 3.6, we also have

$$\frac{|\widetilde{T}_n - T_n|}{b_n} \to 0 \text{ a.s. as } n \to \infty. \tag{3.7}$$

Hence, from (3.7), (3.6) follows.

## Appendix

**Proof of Lemma 3.6** Clearly $\widetilde{T}_n/b_n \to 0$ a.s. as $n \to \infty$ by Theorem 3.1. As in the Appendix of Kim and Baek(2001), we have

$$\begin{aligned}\widetilde{T}_k &= \sum_{t=1}^{k}(\sum_{j=0}^{k-t} a_j)X_t + \sum_{t=1}^{k}(\sum_{j=k-t+1}^{\infty} a_j)X_t \\ &= \sum_{t=1}^{k}(\sum_{j=0}^{t-1} a_j X_{t-j}) + \sum_{t=1}^{k}(\sum_{j=k-t+1}^{\infty} a_j)X_t.\end{aligned}$$

and

$$\begin{aligned}\widetilde{T}_k - T_k &= -\sum_{t=1}^{k}(\sum_{j=t}^{\infty} a_j X_{t-j}) + \sum_{t=1}^{k}(\sum_{j=k-t+1}^{\infty} a_j)X_t \\ &= I + II \text{ (say)}.\end{aligned}$$

It suffices to show that

$$(A.1) \qquad b_n^{-1} \max_{1\le k\le n} |I| \to 0 \text{ a.s.}$$

and

$$(A.2) \qquad b_n^{-1} \max_{1\le k\le n} |II| \to 0 \text{ a.s.}.$$

First we have for $r > 2$

$$\begin{aligned}(A.3)\, b_n^{-r} E \max_{1\le k\le n} |\sum_{t=1}^{k}\sum_{j=t}^{\infty} a_j X_{t-j}|^r &= b_n^{-r} E \max_{1\le k\le n} |\sum_{j=1}^{\infty}\sum_{t=1}^{j\wedge k} a_j X_{t-j}|^r \\ &\le b_n^{-r}(\sum_{j=1}^{\infty}|a_j|\{E \max_{1\le k\le n}|\sum_{t=1}^{j\wedge k} X_{t-j}|^r\}^{\frac{1}{r}})^r \\ &\quad \text{(by Minkowski's inequality)} \\ &\le b_n^{-r}(\sum_{j=1}^{\infty}|a_j| D^{\frac{1}{r}}(j\wedge n)^{\frac{1}{2}})^r \\ &\quad \text{(by Lemma 3.6)} \\ &= (\sum_{j=1}^{\infty}|a_j| D^{\frac{1}{r}}\frac{(j\wedge n)^{\frac{1}{2}}}{b_n})^r \\ &\quad \text{(by the dominated convergence theorem)} \\ &\le D\frac{n^{\frac{r}{2}}}{b_n^r}.\end{aligned}$$

Hence, by Markov inequality we have

$$\sum_{n=1}^{\infty} P(\max_{1\le k\le n} |I| > \epsilon b_n) \le D \sum_{n=1}^{\infty} \frac{n^{\frac{r}{2}}}{b_n^r} < \infty,$$

To prove (A.2) write $II = II_{k_1} + II_{k_2}$, where $II_{k_1} = a_1 X_k + a_2(X_k + X_{k-1}) + \cdots + a_k(X_k + \cdots + X_1)$ and $II_{k_2} = (a_{k+1} + a_{k+2} + \cdots)(X_k + \cdots + X_1)$, and let $\{p_n\}$ be a sequence of positive integers such that

$$(A.4) \qquad p_n \to \infty \text{ and } \frac{p_n}{n} \to 0 \text{ as } n \to \infty.$$

Then

$$\begin{aligned}(A.5) \qquad b_n^{-1} \max_{1\le k\le n} |II_{k_2}| &\le (\sum_{j=0}^{\infty} |a_j|) b_n^{-1} \max_{1\le k\le n} |X_1 + \cdots + X_k| \\ &+ \quad (\sum_{j>p_n} |a_j|) b_n^{-1} \max_{1\le k\le n} |X_1 + \cdots + X_k|. \\ &= \quad III + IV \text{ (say)}.\end{aligned}$$

It follows from Lemma 2.1 and (A.4) that, for $r > 2$ and a constant $D$,

$$\begin{aligned}(\sum_{j=0}^{\infty} |a_j|)^r b_n^{-r} E \max_{1\le k\le p_n} |X_1 + \cdots + X_k|^r &\le (\sum_{j=0}^{\infty} |a_j|)^r D (\frac{p_n}{n})^{\frac{r}{2}} (\frac{n^{\frac{r}{2}}}{b_n^r}) \\ &\le D(\frac{n^{\frac{r}{2}}}{b_n^r}),\end{aligned}$$

from which it follows that

$$\sum_{n=1}^{\infty} P(|III| > b_n \frac{\epsilon}{2}) \le D \sum_{n=1}^{\infty} \frac{n^{\frac{r}{2}}}{b_n^r} < \infty$$

by the Marlov inequality, that is, $III \to 0$ a.s.

Similary, we have

$$(\sum_{j>p_n} |a_j|)^r b_n^{-r} E \max_{1\le k\le n} |X_1 + \cdots + X_k|^r \le D(\sum_{j>p_n} |a_j|)^r \frac{n^{\frac{r}{2}}}{b_n^r},$$

which yields

$$\sum_{n=1}^{\infty} P(IV > b_n \frac{\epsilon}{2}) \le D \sum_{n=1}^{\infty} \frac{n^{\frac{r}{2}}}{b_n^r} < \infty,$$

that is, $IV \to 0$ a.s. Hence, $b_n^{-1} \max_{1\le k\le n} |II_{k_2}| \to 0$ a.s.

It remains to show that $L_n = b_n^{-1} \max_{1\le k\le n} |II_{k_1}| \to 0$ a.s.

For each $m \ge 1$, define $II_{k_1,m} = c_1 X_k + c_2(X_k + X_{k-1}) + \cdots + c_k(X_k + \cdots + X_1)$, where $c_k = a_k$ for $k \le n$ and $c_k = 0$ otherwise, and let $L_{n,m} = b_n^{-1} \max_{1\le k\le n} |II_{k_1,m}|$ . Then

$$L_{n,m} \le b_n^{-1}(|a_1| + \cdots + |a_m|)(|X_1| + \cdots + |X_m|),$$

and

$$\sum_{n=1}^{\infty} P(L_{n,m} \geq \epsilon) \leq \sum_{n=1}^{\infty} E(L_{n,m})^r \leq D \sum_{n=1}^{\infty} \frac{m^{\frac{r}{2}}}{b_n^r} < \infty$$

for each $m$, that is,

$$L_{n,m} \to 0 \text{ a.s. as } n \to \infty. \tag{A.6}$$

Note that

$$|L_{n,m} - L_n| \leq b_n^{-1} \max_{1\leq k\leq n} |\sum_{i=1}^{k} (a_i - b_i)(X_k + \cdots + X_{k-j+1})|. \tag{A.7}$$

Since

$$\begin{aligned} |\sum_{i=1}^{k}(a_i - b_i)(X_k \quad &+ \quad \cdots + X_{k-j+1})| \\ &= \begin{cases} 0, & k \leq m, \\ |\sum_{i=m+1}^{k} a_i(X_k + \cdots + X_{k-i+1})|, & \text{otherwise} \end{cases} \end{aligned}$$

the right-hand side of (A.7) is

$$\begin{aligned} (A.8) \quad &\leq b_n^{-1} \max_{m<k\leq n} (\sum_{i=m+1}^{k} |a_i||X_k + \cdots + X_{k-i+1}) \\ &\leq b_n^{-1} \max_{m<k\leq n} \sum_{i=m+1}^{k} |a_i| \max_{m<i\leq k} |X_k + \cdots + X_{k-i+1}| \\ &\leq b_n^{-1} \sum_{i>m} |a_i| \max_{m<k\leq n} \max_{m<i\leq k} (|X_1 + \cdots + X_k| + |X_1 + \cdots + X_{k-i}|) \\ &\leq b_n^{-1} \sum_{i>m} |a_i| (\max_{m<k\leq n} |X_1 + \cdots + X_k| + \max_{m<k\leq n} \max_{m<i\leq k} |X_1 + \cdots + X_{k-i}|) \\ &\leq b_n^{-1} \sum_{i>m} |a_i| (\max_{1\leq j\leq n} |X_1 + \cdots + X_j| + \max_{1\leq j\leq n} |X_1 + \cdots + X_j|) \\ &\leq 2b_n^{-1} \sum_{i>m} |a_i| \max_{1\leq j\leq n} |X_1 + \cdots + X_j|. \end{aligned}$$

By Lemma 2.1, (A.7), (A.8) and Markov inequality, we have

$$\begin{aligned} \sum_{n=1}^{\infty} P(|L_{n,m} - L_n| > \delta) &\leq \sum_{n=1}^{\infty} P(2b_n^{-1} \sum_{i>m} |a_i| \max_{1\leq j\leq n} |X_1 + \cdots + X_j| > \delta) \\ &\leq D \sum_{n=1}^{\infty} \frac{n^{\frac{r}{2}}}{b_n^r} (\sum_{i>m} |a_i|)^r \\ &= 0 \text{ as } m \to \infty, \end{aligned}$$

which yields

$$(A.9) \qquad |L_{n,m} - L_n| \to 0 \text{ a.s.}$$

Hence, $L_n \to 0$ a.s. as $m \to \infty$ by (A.6) and (A.9). The proof of the lemma is now completed.

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