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REAL ROOT ISOLATION OF ZERO-DIMENSIONAL PIECEWISE ALGEBRAIC VARIETY†

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ABSTRACT. As a zero set of some multivariate splines, the piecewise algebraic variety is a kind of generalization of the classical algebraic variety. This paper presents an algorithm for isolating real roots of the zero-dimensional piecewise algebraic variety which is based on interval evaluation and the interval zeros of univariate interval polynomials in Bernstein form. An example is provided to show the proposed algorithm is effective.

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1. Introduction

A polynomial real root isolation algorithm is an algorithm which, given a polynomial $f(x)$ with real coefficients, computes a sequence of disjoint intervals each containing exactly one real root of $f(x)$, and together containing all real roots of $f(x)$. In 1976, Collins and Akritas [2] gave an efficient algorithm for polynomial real root isolation using Descarte's rule of signs, which improved the Uspensky original algorithm. In 2004, Rouillier and Zimmermann [7] gave a generic algorithm which enables one to describe all the known algorithms based on Descartes' rule of sign and the bisection strategy in a unified framework.

Nowadays, isolating the real solutions of an algebraic variety or semi-algebraic set (a system of polynomial equations, inequalities or inequations) has become an important aspect of research in the field of computational real algebra. There have been some works concerning this issue. The classical cylindrical algebraic

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decomposition method proposed by Collins in 1975, which was made more available in their work [1, 3]. In 2002, Xia and Yang [15] proposed an algorithm based on Ritt-Wu method and Uspensky algorithm for isolating the real roots of a semi-algebraic system with integer coefficients. The algorithm was improved in their later work [16], where they gave a complete algorithm by using interval arithmetic and it is faster.

The object we deal with in this paper is called piecewise algebraic variety defined on n-dimensional simplex partition. Here, the piecewise algebraic variety is always assumed to be zero-dimensional, i.e., it consists of a finite number of points. Moreover, all the n-dimensional simplex are assumed to be in "general position", which means none of zeros lie on their boundary.

As the common zeros of a system of multivariate splines, the piecewise algebraic variety is a kind of generalization of the classical algebraic variety. It is important to study the interpolation by multivariate splines and piecewise algebraic variety[9]. One of the important problems is isolating the real zeros of a given univariate spline or a set of multivariate splines. Very recently, Wang and the current author [11] proposed an algorithm for isolating the real zeros of a given univariate spline function, which is primarily based on use of Descartes' rule of signs and de Casteljau algorithm. The zeros of a given univariate spline function can be viewed as a particular piecewise algebraic variety. In this paper, based on interval evaluation and the interval zeros of univariate interval polynomials in Bernstein representation, we present an new algorithm for isolating the real roots of a relatively simplex zero-dimensional piecewise algebraic variety on n-dimensional simplex partition.

The rest paper is organized as follows. In the next section, some basic definition of piecewise algebraic variety and the discussed object are introduced. In section 3, we present an algorithm to isolate the interval zeros of a given interval polynomial in Bernstein form. The main algorithm for isolating the real roots of zero-dimensional piecewise algebraic variety is given in Section 4. Finally, an example is provided to demonstrate the proposed method in Section 5 and conclude this paper in Section 6.

2. Piecewise algebraic variety

The piecewise algebraic variety defined as the common intersection of surfaces represented by multivariate splines is a new topic in algebraic geometry. Moreover, the piecewise algebraic variety will be also important in Computational Geometry, Computer Aided Geometrical Design and Image Processing. Because of the possibility of $\{(x,y)|s|_{\delta_i} = s_i(x,y) = 0\} \cap \delta_i = \emptyset$, it is more difficult to study piecewise algebraic variety. A lot work on (real) piecewise algebraic variety has been done by Wang and his research group [5, 10, 11, 12, 13, 14, 17]. Several basic definitions of piecewise algebraic variety defined on n-dimensional simplex partition are reviewed.

Let Ω be a polygonal region in $\mathbb{R}^n$. Let Δ be a regular n-dimensional simplex partition of Ω and denote by $\Delta = \{\delta_1, \delta_2, \cdots, \delta_T\}$. Then $\delta_i = [V_{i,1} \cdots V_{i,n+1}],\ i = 1, \cdots, T$ is called a n-dimensional simplex or cell, where $V_{i,1}, \cdots V_{i,n+1}$ are the vertices of δ_i. Obviously, the interior of simplex δ_i can be given by

$$\delta_i = \{x \in \mathbb{R}^n \mid H_{i,k}(x) > 0,\ k = 1, \cdots, n+1\},$$

where supporting hyperplane $H_{i,k}(x) = 0,\ k = 1, \cdots, n+1$ are the facets of δ_i.

Definition 1. [14] $P(\Delta) = \{s|_{\delta_i} \in \mathbb{R}[x_1, x_2, \cdots, x_n],\ i = 1, 2, \cdots, T\}$ *is denoted by the piecewise polynomial ring with respect to partition* Δ *on* Ω*, where* $s|_{\delta_i}$ *are polynomials corresponding to* s *on each cell* δ_i*. Moreover,* $S^\mu(\Delta) = \{s \mid s \in C^\mu(\Delta) \cap P(\Delta)\}$ *is called* C^μ *piecewise polynomial ring, where* C^μ *means that* s *possesses* μ *order continuous partial derivatives.*

For $F \subseteq S^\mu(\Delta)$, the zero set of F is defined to be $z(F) = \{x \in \Omega \mid s(x) = 0,\ \forall\, s \in F\}$. Since $S^\mu(\Delta)$ is a Nöther ring, every ideal I has a finite of generators, then $z(F)$ can be expressed as the common zeros of the splines $s_1, s_2, \cdots, s_l$.

Definition 2. [14] *Let* Δ *be a* n*-dimensional simplex partition of the region* Ω*. If there exists* $s_1, s_2, \cdots, s_m \in S^\mu(\Delta)$ *such that* $X = z(s_1, s_2, \cdots, s_m) = \bigcap_{i=1}^{m} z(s_i)$*, then* X *is called a* C^μ *piecewise algebraic variety with respect to* Δ*.*

The object in this paper is zero-dimensional piecewise algebraic variety with real coefficients. That's to say, for each n-dimensional simplex $\delta_i,\ i = 1, \cdots, T$, we ought to discuss the algebraic variety in the interior of δ_i i.e.,

$$z(s_1, \cdots, s_n)|_{\delta_i} : \begin{cases} s_{1,i}(x) = 0,\ s_{2,i}(x) = 0,\ \cdots,\ s_{n,i}(x) = 0, \\ H_{i,k}(x) > 0,\ k = 1, \cdots, n+1. \end{cases} \tag{1}$$

where $s_{k,i} = s_k|_{\delta_i}$ denote polynomials corresponding to s_k on each cell δ_i.

Because the ideal generated by $\{s_1, \cdots, s_n\}$ is zero-dimensional, we can use by Ritt-Wu method, Gröbner basis method or subresultant method [15] to transform the system (1) into one or more systems in triangular form. Therefore, the system (1) can be reduced into one or more systems in the form

$$z(\widetilde{s}_1, \cdots, \widetilde{s}_n)|_{\delta_i} :\ \{\widetilde{s}_{1,i}(x_1) = 0,\ \widetilde{s}_{2,i}(x_1, x_2) = 0, \cdots, \widetilde{s}_{n,i}(x_1, x_2, \cdots, x_n) = 0\} \cap \delta_i \tag{2}$$

Therefore, the system (2) can be represented in Bernstein form as follows(denotes by triangular algebraic variety for short)

$$z(f_1, f_2, \cdots, f_{n+1}) : \begin{cases} f_1(\tau_1) = 0,\ f_2(\tau_1, \tau_2) = 0,\ \cdots,\ f_n(\tau_1, \tau_2, \cdots,\ \tau_n) = 0, \\ f_{n+1}(\tau) := \sum_{i=1}^{n+1} \tau_i - 1 = 0,\ \forall\ \tau_k > 0,\ k = 1, \cdots, n+1, \end{cases} \tag{3}$$

where $\tau = (\tau_1, \tau_2, \cdots, \tau_{n+1})$ is the barycentric coordinates of a point $x \in \mathbb{R}^n$ with respect to δ_i. Here, we omit the subscript i and replace $\widetilde{s}_{k,i}$ with Bernstein polynomial $f_k,\ k = 1, 2, \cdots, n+1$ when the meaning is not confused for later discussion.

3. Interval polynomial

Interval operations have been first introduced by Moore [6]. It is used to tackle the instability, and error analysis of numerical computation. In this section, we firstly brief review interval evaluation, then we introduce the zeros of univariate interval polynomials and present an algorithm to isolate the interval zeros of a given interval polynomial.

Definition 3. [16, 8] *Let f be an arithmetic expression of a polynomial in $\mathbb{R}[x_1, \cdots, x_n]$. We replace all operands of f an intervals and replace all operations of f as interval operations and the result is denote by F. Then, $F : I(\mathbb{R})^n \to I(\mathbb{R})$ is called an interval evaluation, where $I(\mathbb{R})$ denotes the set of all intervals.*

Let F be an interval evaluation. If for all $X, Y \subseteq D$, $X \subset Y$ implies $F(X) \subset F(Y)$, we call F a monotonic interval evaluation.

Theorem 1. [16, 8] *An interval evaluation of any polynomial in $\mathbb{R}[x_1, \cdots, x_n]$ is a monotonic interval evaluation. Especially, this is true for univariate polynomials.*

An interval polynomial of degree k is a polynomial whose coefficients are intervals, the Bernstein form of which is:

$$[f](t) = \sum_{i=0}^{k} [f_i] B_{i,k}(t), \ t \in [0, 1]. \tag{4}$$

where

$$B_{i,k}(t) = \frac{k!}{(k-i)!i!} t^i (1-t)^{k-i}$$

is the ith Bernstein basis function of degree k, and $[f_i] = [\underline{f}_i, \overline{f}_i]$ is an interval.

Since $B_{i,k}(\tau) > 0$, the two boundary polynomials of $[f](t)$

$$[f^l](t) = \sum_{i=0}^{k} \underline{f}_i B_{i,k}(t), \ [f^u](t) = \sum_{i=0}^{k} \overline{f}_i B_{i,k}(t). \tag{5}$$

are called the lower boundary function and upper boundary function of $[f](t)$, respectively.

The set of real zeros of the interval polynomial $[f](t)$ is defined as

$$\mathbb{R}([f]) = \{t_0 \in \mathbb{R} \mid \exists \ f(t) \in [f](t), \ \text{s.t. } f(t_0) = 0\}.$$

Obviously,

$$\mathbb{R}([f]) = \{t_0 \in \mathbb{R} \mid [f^l](t_0) \le 0 \le [f^u](t_0)\},$$

then the zeros set of $[f](t)$ ia composed of several closed intervals. Each of these intervals is called an interval zero of $[f](t)$.

It is from the results in [4] that we can easily have

Proposition 1. *If $[\underline{t}_0, \overline{t}_0]$ is an interval zero of $[f](t)$, then the endpoints $\underline{t}_0$ and $\overline{t}_0$ are the zeros of the lower boundary function $[f^l](t)$ and upper boundary function $[f^u](t)$, respectively.*

Theorem 2. *An interval polynomial $[f](t)$ of degree k has at most k interval zeros.*

Since each interval polynomial $[f](t)$ defined on $[t_1, t_2]$ can be transformed into an interval polynomial $[f](\widetilde{t})$ defined on $[0, 1]$, i.e.,

$$[f](\widetilde{t}) = \sum_{i=0}^{k}[\widetilde{f}_i]B_{i,k}(\widetilde{t}),\ \ \widetilde{t} \in [0, 1] \tag{6}$$

under the coordinate transformation $\widetilde{t} = \frac{t-t_1}{t_2-t_1},\ t \in [t_1, t_2]$. So from now on, we use interval polynomial (4) for the later discussion without the meaning confused.

Since Bernstein basis function satisfies the property of partition of unity, it is obvious that

$$[f^l]([0, 1]) \subset [m[f^l], M[f^l]],$$

where $m[f^l] = \min\limits_{1\le i\le k} \underline{f}_i$ and $M[f^l] = \max\limits_{1\le i\le k} \underline{f}_i$.

Similarly,

$$[f^u]([0, 1]) \subset [m[f^u], M[f^u]],$$

where $m[f^u] = \min\limits_{1\le i\le k} \overline{f}_i$ and $M[f^u] = \max\limits_{1\le i\le k} \overline{f}_i$.

Therefore,

$$[f]([0, 1]) \subset [m[f^l], M[f^u]].$$

If an interval polynomial $[f](t),\ t \in [0, 1]$ completely lie below or above x-axis, then $f[t]$ has no real zero. If the interval polynomial $[f^l](t),\ t \in [0, 1]$ lie below the x-axis and $[f^u](t),\ t \in [0, 1]$ lie above the x-axis, then $[0, 1]$ is an interval zero of $f[t]$. These facts conclude the following proposition which is the foundation of the algorithm to isolate the interval zeros of a given interval polynomial.

Proposition 2. *With the above notations, we have the results:*
(1)If $0 \notin [m[f^l], M[f^u]]$, then [0,1] is not a zero of $[f](t)$.
(2)If $M[f^l] < 0$, $m[f^u] > 0$, then $[0, 1]$ is an interval zero of $[f](t)$.

For any interval, there are three cases to be considered. In case (1), we discard the interval. In case (2), we save the interval. Otherwise, we bisection the interval and test each interval until each interval is sufficiently small.

The numerical algorithm to find a set of intervals which bound the interval zeros of a given interval polynomial $[f](t)$ is presented as follows.

Algorithm 1. *Computing the interval zeros of interval polynomial on* $[0, 1]$

Input *An interval polynomial $[f](t) = \sum\limits_{i=0}^{k}[\underline{f}_i, \overline{f}_i]B_{i,k}(t),\ t \in [0, 1]$, and a sufficiently small positive tolerance ε.*
Output *A set S containing all the interval zeros of $[f](t)$.*
Step 1 *Set the initial interval $I := [0, 1]$ and let S be an empty set.*
Step 2 *If $0 \notin [m[f^l], M[f^u]]$, discard this interval and process the next interval. Otherwise go to Step 3.*

Step 3 *If $M[f^l] < 0$, $m[f^u] > 0$, or the width of I is less than the given tolerance ε, then set $S := S \cup I$. Otherwise divide I into two intervals at midpoints, transform $[f](t)$ into form* (6) *on each subinterval, and go to Step 2.*

Step 4 *Union all the neighboring intervals in S.*

4. Real root isolation of zero-dimensional piecewise algebraic variety

A natural idea is to isolate the real zeros of the first equation of the system (3) on the interval $[0, 1]$ and substitute each resulting interval in the rest of the equations and repeat the above computation. Therefore, we deal with polynomials with "interval coefficients ".

Definition 4. *Let $f(\tau_1, \cdots, \tau_i, \tau_{i+1})$ can be represented as*

$$f(\tau_1, \cdots, \tau_i, \tau_{i+1}) = \sum_{j=0}^{k} f_j(\tau_1, \cdots, \tau_i) B_{j,k}(\tau_{i+1}).$$

For any $X = ([\underline{\tau_1}, \overline{\tau_1}], \cdots, [\underline{\tau_i}, \overline{\tau_i}]) \in I(\mathbb{R})^i$, let F_j, $j = 1, \cdots, k$ be an interval evaluation of f_j in X and

$$[f](\tau_{i+1}) = \sum_{j=0}^{k} F_j([\underline{\tau_1}, \overline{\tau_1}], \cdots, [\underline{\tau_i}, \overline{\tau_i}]) B_{j,k}(\tau_{i+1}).$$

Then $[f](\tau_{i+1})$ is called an interval evaluation polynomial of $f(\tau_1, \cdots, \tau_i, \tau_{i+1})$.

The algorithm for isolating the real roots of the system in the form of (3) is outlined below.

Algorithm 2. *Real root isolation of triangular algebraic variety*

Input *A given triangular algebraic variety $z(f_1, f_2, \cdots, f_{n+1})$ in the form of* (3)*, and a given sufficiently small positive tolerance ε.*

Output *A set S of all the isolating intervals of barycentric coordinates for $z(f_1, f_2, \cdots, f_{n+1})$.*

Step 1 *Compute the set of interval zeros of $[f_1](\tau_1)$ and denote by $S^{(1)} = \{[\underline{\tau_{1,j}}, \overline{\tau_{1,j}}],\ j = 1, \cdots, n_1\}$, where $[f_1](\tau_1)$ is an interval evaluation polynomial of $f_1(\tau_1)$.*

Step 2 *If $S^{(1)} = \emptyset$, then stop and TAV has no real zero. Otherwise, substitute $[\underline{\tau_{1,j}}, \overline{\tau_{1,j}}]$ into $f_2(\tau_1, \tau_2)$ and obtain an interval evaluation polynomial $[f_2]^{(j)}(\tau_2)$. Compute the set of interval zeros of $[f_2]^{(j)}(\tau_2)$ and denote by $S_j^{(2)} = \Big\{[\underline{\tau_{1,j_k}}, \overline{\tau_{1,j_k}}] | k = 1, 2, \cdots, k_j\Big\}$. Hence, $S^{(2)}$ is defined by*

$$S^{(2)} = \bigcup_{j=1}^{n_1} S_j^{(2)} = \Big\{[\underline{\tau_{2,j}}, \overline{\tau_{2,j}}] | j = 1, 2, \cdots, n_2\Big\}.$$

Step 3 *Inductively, we continue the similar procedure as Step 2 to obtain the sequence $\{S^{(3)}, S^{(4)}, \cdots, S^{(n+1)}\}$.*

Therefore, the set of isolating intervals of barycentric coordinates for TAV is $S = S^{(1)} \times S^{(2)} \times \cdots \times S^{(n+1)}$.

With the above preparations, we can easily give the algorithm for isolating the real roots of zero-dimensional piecewise algebraic variety on n-dimensional simplex partition.

Algorithm 3. *Real root isolation of piecewise algebraic variety*

Input *A given piecewise algebraic variety* $z(s_1, s_2, \cdots, s_n)$ *in the form of* (1).

Output *A set* S *of all the isolating intervals of barycentric coordinates for* $z(s_1, s_2, \cdots, s_n)$.

Step 1 *Set* $i := 1$ *and* $S = \emptyset$.

Step 2 *Transform* $z(s_{1,i}, s_{2,i}, \cdots, s_{n,i})$ *into the form* (2) *and thehe resulting polynomials are expressed in Bernstein form of* (3).

Step 3 *Compute* $z(s_{1,i}, s_{2,i}, \cdots, s_{m,i})$ *in the interior of* δ_i *by using Algorithm* 2 *and denote by* $S^{(i)}$ *the isolating intervals and set* $S := S \cup S^{(i)}$.

Step 4 *Let* $i := i + 1$. *If* $i \le T$, *then go to Step 2. Otherwise, stop.*

Therefore, S *is the set of isolating intervals of barycentric coordinates for zero-dimensional piecewise algebraic variety* $z(s_1, \cdots, s_n)$.

5. Numerical example

In this section, an example is provided to demonstrate the proposed algorithm for isolating the zeros of a given zero-dimensional piecewise algebraic variety.

Example 1. *Let* $\Delta = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ *be a triangulation of a quadrangle* $V_A V_B V_C V_D$ *in* $\mathbb{R}^2$, *where* $\delta_1 = [V_D V_O V_A]$, $\delta_2 = [V_D V_C V_O]$, $\delta_3 = [V_C V_B V_O]$, $\delta_4 = [V_B V_A V_O]$, $V_A = (1, 0)$, $V_B = (0, -1)$, $V_C = (-1, 0)$, $V_D = (0, 1)$ *and* $V_O = (0, 0)$ *(see Figure 1).*

Suppose that $f, g \in S_3^1(\Delta)$ *are defined as follows:*

- *on cell* δ_1: $\begin{cases} f_1(x,y) = f|_{\delta_1} = x^3 + y^2 - \frac{5}{8} \\ g_1(x,y) = g|_{\delta_1} = y^3 - x \end{cases}$
- *on cell* δ_2: $\begin{cases} f_2(x,y) = f|_{\delta_2} = f_1(x,y) + x^2(x+y) \\ g_2(x,y) = g|_{\delta_2} = g_1(x,y) - x^2(2y) \end{cases}$
- *on cell* δ_3: $\begin{cases} f_3(x,y) = f|_{\delta_3} = f_2(x,y) + y^2(x-y+2) \\ g_3(x,y) = g|_{\delta_3} = g_2(x,y) - y^2(y-3) \end{cases}$
- *on cell* δ_4: $\begin{cases} f_4(x,y) = f|_{\delta_4} = f_1(x,y) + y^2(x-y+2) \\ g_4(x,y) = g|_{\delta_4} = g_1(x,y) - y^2(y-3) \end{cases}$

In order to illustrate the proposed algorithm, we take the algebraic variety $z(f_4, g_4)$ *in the interior of the triangle* δ_4 *for example.*

Set $\varepsilon = 10^{-3}$. *Then* $\{f_4(x,y), g_4(x,y)\}$ *can be transformed into* $\{\widetilde{f}_4(y), \widetilde{g}_4(x,y)\}$ *by using Gröbner method, where* $\widetilde{f}_4 = -5+24y^2-8y^3+24y^4+216y^6$, $\widetilde{g}_4 = x-3y^2$.

Obviously, $z(\{\widetilde{f}_4, \widetilde{g}_4\})$ *in the interior of* δ_4 *can be expressed as common zeros of polynomials in Bernstein form:* $z(h_1(\tau_1), h_2((\tau_1, \tau_2), h_3(\tau_1, \tau_2, \tau_3))$, *where* $h_1(\tau_1) =$

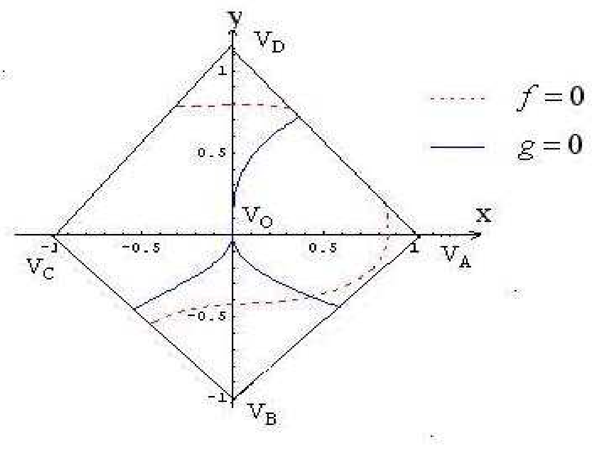

Fig. 1. Two piecewise algebraic curves $f=0$ and $g=0$

$-5+24\tau_1^2+8\tau_1^3+24\tau_1^4+216\tau_1^6$, *$h_2(\tau_1,\tau_2)=\tau_2-3\tau_1^2$, and $h_3(\tau_1,\tau_2,\tau_3)=\tau_1+\tau_2+\tau_3-1$, where (τ_1,τ_2,τ_3) is the barycentric coordinates with respect to δ_4.*

Firstly, compute the set $S^{(1)}$ of interval zeros of $[h_1](\tau_1)$ and denote by $S^{(1)}=\{[0.3784,0.3789]\}$, where $[h_1](\tau_1)$ is an interval evaluation polynomial of $h_1(\tau_1)$. Secondly, the set $S^{(2)}$ of interval zeros of $[h_2](\tau_2)$ is computed and denote by $S^{(2)}=\{[0.4296,0.4305]\}$, where $[h_2](\tau_2)$ is an interval evaluation polynomial of $h_2(\tau_1,\tau_2)$ on $S^{(1)}$. Lastly, we obtain $S^{(3)}=\{[0.191,0.192]\}$.

Hence, $S^{(1)}\times S^{(2)}\times S^{(3)}$ is the isolating interval of $z(f_4,g_4)$ inside δ_4. In other words, the real zero of $z(f_4,g_4)|_{\delta_4}$ is $([0.4296,0.4305],[-0.3789,-0.3784])$.

Similarly, $z(f_1,g_1), z(f_2,g_2)$ and $z(f_3,g_3)$ have no common zeros in the interior of triangles δ_1,δ_2 and δ_3, respectively.

6. Conclusion

This paper presents an algorithm for isolating the real roots of a given piecewise algebraic variety defined on n-dimensional partition, i.e., determining a sequence of disjoint hyperrectangles such that each of them contains exactly one real root of the piecewise algebraic variety. It is primarily based on the interval zeros of univariate interval polynomial in Bernstein form. Numerical example is provided to demonstrate the propose algorithm is flexible.

It is from the results in [12] that the number of common zeros of multivariate splines not only depend on the degree of splines, but also heavily depend on the geometrical structure of the partition. Therefore, it is difficult but important to count the real zeros of zero-dimensional piecewise algebraic variety, and determine whether there exists real root of algebraic variety in the interior of a given cell. It remains to be our future work.

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