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NEW HYBRID ALGORITHM FOR WEAK RELATIVELY NONEXPANSIVE MAPPING AND INVERSE-STRONGLY MONOTONE MAPPING IN BANACH SPACE

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Abstract. The purpose of this paper is to prove strong convergence theorems for finding a common element of the set of fixed points of a weak relatively nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping by a new hybrid method in a Banach space. We shall give an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping in Banach space l^2. Our results improve and extend the corresponding results announced by Ying Liu[Ying Liu, Strong convergence theorem for relatively nonexpansive mapping and inverse-strongly-monotone mapping in a Banach space, Appl. Math. Mech. -Engl. Ed. 30(7)(2009), 925-932] and some others.

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1. Introduction

Let E be a Banach space with dual E^*, $\|\cdot\|$ denote the norm, and $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Suppose that C is a nonempty closed convex subset of E and A is a monotone operator of C into E^*. Then, we study a variational inequality problem [1]: Find a point $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. An operator A of C into E^* is said to be α-inverse-strongly-monotone [2-3], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

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If A is an α-inverse-strongly-monotone mapping, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for every $x, y \in C$.

In 2005, Iiduka and Takahashi [4] proved strong convergence theorems for finding a common element of the set of solutions of the variational inequality problem for an inverse-strongly- monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space. In the meantime, Matsushita and Takahashi [5] proved a strong convergence theorem for relatively nonexpansive mappings in a Banach space using the hybrid method. Later, Iiduka and Takahashi [3] proved a weak convergence theorem for finding a solution of the variational inequality problem with an operator A that satisfies the following conditions in a two-uniformly convex and uniformly smooth Banach space E:
(A1) A is a α-inverse-strongly-monotone;
(A2) $VI(C, A) \neq \emptyset$;
(A3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C, A)$.

Recently, Iiduka and Takahashi [2] also introduced a hybrid type method for finding a solution of the variational inequality problem with an operator A satisfying (A1)-(A3) in a two-uniformly convex and uniformly smooth Banach space.

In 2009, Ying Liu [9] established a hybrid method for finding a common element of the set of solutions of a variational inequality problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Inspired and motivated by these results above, The purpose of this paper is to prove strong convergence theorems for finding a common element of the set of fixed points of a weak relatively nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping by a new hybrid method in a Banach space. We shall give an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping in Banach space l^2. Our results improve and extend the corresponding results announced by Ying Liu[Ying Liu, Strong convergence theorem for relatively nonexpansive mapping and inverse-strongly-monotone mapping in a Banach space, Appl. Math. Mech. -Engl. Ed. 30(7), 925-932 (2009) DOI: 10.1007/s10483-009-0711-y] and some others.

2. Preliminaries

A multi-valued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ and range $R(T) = \cup\{Tz \in E^* : z \in D(T)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i, i = 1, 2$. A monotone operator T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator.

We define a function $\delta : [0,2] \to [0,1]$, called the modulus of convexity of E, as follows:

$$\delta(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in U, \|x-y\| \geq \epsilon\}.$$

Then, E is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0,2]$.

It is also very well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces C and consequently, it is not available in more general Banach spaces. In this connection, Alber [22] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a real smooth Banach space. Let us consider the functional defined as [20,21] by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{2.1}$$

for all $x, y \in E$. Observe that, in a Hilbert space H, (2.1) reduces to $\phi(y,x) = \|x-y\|^2, x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y,x), \tag{2.2}$$

existence and uniqueness of the operator Π_C follow from the properties of the C functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, for example, [22-24]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2 \tag{2.3}$$

for all $x, y \in E$.

Remark 1. If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of J, we have $Jx = Jy$. That is, $x = y$; see [23,24] for more details.

A Banach space E is said to have the Kadec-Klee property if a weakly convergent sequence $\{x_n\}$ in E with limit $x_0 \in E$ satisfies that $\lim_{n\to\infty} \|x_n\| = \|x_0\|$, then $\{x_n\}$ converges strongly to x_0. It is obvious that if E is uniformly convex, E has the Kadec-Klee property.

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T.

A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of asymptotic fixed point of T will be denoted by b $\widehat{F}(T)$.

A mapping T of C into itself is said to be relatively nonexpansive [5,10,11] if the following conditions are satisfied:

(1)$F(T)$ is nonempty;
(2)$\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T), x \in C$;
(3)$\widehat{F}(T) = F(T)$.

The hybrid algorithms for fixed point of relatively nonexpansive mappings and applications have been studied by many authors, for example [10-15].

A point p in C is said to be a strong asymptotic fixed point of T [16,17] if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called weak relatively nonexpansive if

(1)$F(T)$ is nonempty;
(2)$\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T), x \in C$;
(3)$\widetilde{F}(T) = F(T)$.

Remark 2. In [17], the weak relatively nonexpansive mapping is also said to be relatively weak nonexpansive mapping.

Remark 3. In [18], the authors have given the definition of hemi-relatively nonexpansive mapping as follows. A mapping T from C into itself is called hemi-relatively nonexpansive if

(1)$F(T)$ is nonempty;
(2)$\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T), x \in C$.

The following conclusion is obvious.

Conclusion 1. *A mapping is closed hemi-relatively nonexpansive if and only if it is weak relatively nonexpansive.*

It is obvious that, if $T : E \to E$ is relatively nonexpansive then using the definition of ϕ one can show that $F(T)$ is closed and convex. It is also obvious that, relatively nonexpansive mapping is a weak relatively nonexpansive mapping and a weak relatively nonexpansive mapping is a hemi-relatively nonexpansive mapping. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

In recent years, the definition of weak relatively nonexpansive mapping has been presented and studied by many authors [16-19], but they have not given the example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping. In the following, we give an example in Banach space l^2.

Example. Let $E = l^2$, where

$$l^2 = \{\xi = (\xi_1, \xi_2, \xi_3, ..., \xi_n, ...) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\},$$

$$\|\xi\| = (\sum_{n=1}^{\infty} |\xi_n|^2)^{\frac{1}{2}}, \ \forall\, \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \ \forall\, \xi = (\xi_1, \xi_2, \xi_3, ..., \xi_n, ...), \ \eta = (\eta_1, \eta_2, \eta_3, ..., \eta_n....) \in l^2.$$

It is well known that, l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 =&(1, 0, 0, 0, ...)\\ x_1 =&(1, 1, 0, 0, ...)\\ x_2 =&(1, 0, 1, 0, 0, ...)\\ x_3 =&(1, 0, 0, 1, 0, 0, ...)\\ &......................................\\ x_n =&(\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, ..., \xi_{n,k}, ...)\\ &......................................,\end{aligned}$$

where

$$\xi_{n,k} = \begin{cases} 1 & if \qquad k = 1, \ n+1, \\ 0 & if \qquad k \neq 1, k \neq n+1, \end{cases}$$

for all $n \geq 1$. Define a mapping $T : E \to E$ as follows

$$T(x) = \begin{cases} \frac{n}{n+1} x_n & if \qquad x = x_n (\exists\ n \geq 1), \\ -x & if \qquad x \neq x_n (\forall\ n \geq 1). \end{cases}$$

Conclusion 2.1. *$\{x_n\}$ converges weakly to x_0.*

Proof. For any $f = (\zeta_1, \zeta_2, \zeta_3, ..., \zeta_k, ...) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \to 0,$$

as $n \to \infty$. That is, $\{x_n\}$ converges weakly to x_0. □

Conclusion 2.2. *$\{x_n\}$ is not a Cauchy sequence, so that, it does not converges strongly to any element of l^2.*

Proof. In fact, we have $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$. Then $\{x_n\}$ is not a Cauchy sequence. □

Conclusion 2.3. *T has a unique fixed point 0, that is $F(T) = \{0\}$.*

Proof. The conclusion is obvious. □

Conclusion 2.4. *x_0 is an asymptotic fixed point of T .*

Proof. Since $\{x_n\}$ converges weakly to x_0 and

$$\|Tx_n - x_n\| = \|\frac{n}{n+1}x_n - x_n\| = \frac{1}{n+1}\|x_n\| \to 0$$

as $n \to \infty$, so that, x_0 is an asymptotic fixed point of T. □

Conclusion 2.5. *T has a unique strong asymptotic fixed point 0, so that, $F(T) = \widetilde{F}(T)$.*

Proof. In fact that, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, from conclusion 2.2, there exist sufficiently large nature number N such that $z_n \neq x_m$, for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, it follows from $\|z_n - Tz_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. □

Conclusion 2.6. *T is a weak relatively nonexpansive mapping.*

Proof. Since $E = L^2$ is a Hilbert space, we have

$$\phi(0, Tx) = \|0 - Tx\|^2 = \|Tx\|^2 \leq \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall \ \ x \in E.$$

From conclusion 2.5, we have $F(T) = \widetilde{F}(T)$, then T is a weak relatively nonexpansive mapping. □

Conclusion 2.7. *T is not a relatively nonexpansive mapping.*

Proof. From conclusion 2.3 and 2.4, we have $F(T) \neq \widehat{F}(T)$, so that, T is not a relatively nonexpansive mapping. □

Let E be a smooth, strictly convex, and reflexive Banach space and J be the duality mapping from E into E^*. Then, J^{-1} is also single-valued, one-to-one and surjective, and it is the duality mapping from E^* into E. We define the following mapping V:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \quad \forall x \in E, x^* \in E^*. \tag{2.4}$$

We also need the following lemmas for the proof of our main results.

Lemma 2.1 ([6]). *Let p be a real number with $p \geq 2$ and E be a Banach space. Then, E is p-uniformly convex if and only if there exists a constant $0 < c \leq 1$ such that*

$$\frac{1}{2}(\|x + y\|^p + \|x - y\|^p) \geq \|x\|^p + c^p\|y\|^p, \quad \forall x, y \in E. \tag{2.5}$$

The best constant $\frac{1}{c}$ in Lemma 2.1 is called the p-uniformly convexity constant of E. Putting $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$ in (2.5), we get that, for all $u, v \in E$,

$$\frac{1}{2}(\|u\|^p + \|v\|^p) \geq \|\frac{u+v}{2}\|^p + c^p\|\frac{u-v}{2}\|^p. \tag{2.6}$$

Suppose $p > 1$, the (generalized) duality mapping J_p from E into 2^{E^*} is defined as

$$J_p x = \{v \in E^* : \langle x, v\rangle = \|x\|^p, \|v\| = \|x\|^{p-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping, which has the following properties:
(1) If E is smooth, J is single-valued.
(2) If E is strictly convex, J is one-to-one.
(3) If E is reflexive, J is surjective.
(4) If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E.

Lemma 2.2 ([3]). *Let p be a given real number with $p \geq 2$ and E be a p-uniformly convex Banach space. Then, for all $x, y \in E, j_x \in J_p x$ and $j_y \in J_p y$, there is*

$$\langle x - y, j_x - j_y\rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p,$$

where $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.3 ([5]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $\|y_n - z_n\| \to 0$.*

Lemma 2.4 ([22]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.5 ([22]). *Let E be a reflexive , strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.6 ([5]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.7 ([3]). *Let E be a reflexive, strictly convex and smooth Banach space and V be defined as in (2.4). Then,*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*.$$

We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \geq 0 \quad for\ all\ y \in C\}.$$

Lemma 2.8 ([8]). *Let C be a nonempty closed convex subset of a Banach space E and A be a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined by*

$$Tv = \begin{cases} Av + N_C(v), v \in C, \\ \emptyset, \qquad\qquad \notin C. \end{cases}$$

Then, T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.9 ([3]). *Let C be a nonempty closed convex subset of a Banach space E and A be a monotone, hemicontinuous operator of C into E^*. Then,*

$$VI(C, A) = \{u \in C : \langle v - u, Av\rangle \geq 0, \quad \forall v \in C\}.$$

It is obvious from Lemma 2.9 that the set $VI(C, A)$ is a closed convex subset of C.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a two-uniformly convex and uniformly smooth Banach space E. Assume that A is an operator of C into E^* that satisfies the conditions (A1)-(A3) and T is a weak relatively nonexpansive mapping from C into itself such that $F = F(T) \cap VI(C, A) \neq \emptyset$. The sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ w_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)J\Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n))), \\ z_n = \Pi_C w_n, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTz_n), \\ C_0 = C, \\ C_n = \{v \in C_{n-1} : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_n}(x_0), \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1) such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\limsup_{n\to\infty} \beta_n < 1$. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ with $0 < a < b < \frac{c^2\alpha}{2}$, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\frac{1}{c}$ is the two-uniformly convexity constant of E.

Proof. Firstly, we show that C_n is closed and convex for each $n \geq 0$.
From the definition of C_n, it is obvious that C_n is closed for each $n \geq 0$. We show that C_n is convex for each $n \geq 0$. Since $\phi(v, y_n) \leq \phi(v, x_n)$ is equivalent to

$$2\langle v, Jx_n - Jy_n\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0,$$

thus C_n is convex for every $n \geq 0$.
Secondly, we prove that $F \subset C_n$, for all $n \geq 0$.
Put $u_n = J^{-1}(Jx_n - \lambda_n Ax_n)$ for every $n \geq 0$. Let $p \in F$. From Lemmas 2.5 and 2.7, it holds

$$\begin{aligned}\phi(p, \Pi_C u_n) &\leq \phi(p, u_n) = V(p, Jx_n - \lambda_n Ax_n) \\ &\leq V(p, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) \\ &\quad - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, \lambda_n Ax_n\rangle \\ &= V(p, Jx_n) - 2\lambda_n\langle u_n - p, Ax_n\rangle \\ &= \phi(p, x_n) - 2\lambda_n\langle x_n - p, Ax_n\rangle + 2\langle u_n - x_n, -\lambda_n Ax_n\rangle,\end{aligned} \tag{3.2}$$

for every $n \geq 0$. From condition (A1) and $p \in VI(C, A)$, we have

$$\begin{aligned}-2\lambda_n\langle x_n - p, Ax_n\rangle &= -2\lambda_n\langle x_n - p, Ax_n - Ap\rangle - 2\lambda_n\langle x_n - p, Ap\rangle \\ &\leq -2\lambda_n\alpha\|Ax_n - Ap\|^2\end{aligned} \tag{3.3}$$

for every $n \geq 0$. By Lemma 2.2 and condition (A3), we also have

$$\begin{aligned}2\langle u_n - x_n, -\lambda_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n, -\lambda_n Ax_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n\|\|\lambda_n Ax_n\| \\ &\leq \frac{4}{c^2}\|Jx_n - \lambda_n Ax_n - Jx_n\|\|\lambda_n Ax_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2.\end{aligned} \tag{3.4}$$

Therefore, from (3.2)-(3.4), we have

$$\phi(p, \Pi_C u_n) \leq \phi(p, x_n) + 2a(\frac{2b}{c^2} - \alpha)\|Ax_n - Ap\|^2.$$

By the convexity of $\|\cdot\|^2$ and Lemma 2.5, we have

$$\begin{aligned}\phi(p, z_n) &\leq \phi(p, w_n) \\ &= \|p\|^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)J\Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n))\rangle \\ &\quad + \|\beta_n Jx_n + (1-\beta_n)J\Pi_C u_n\|^2 \\ &\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle p, J\Pi_C u_n\rangle \\ &\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|\Pi_C u_n\|^2 \\ &= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, \Pi_C u_n) \\ &\leq \phi(p, x_n) + (1-\beta_n)2a(\frac{2b}{c^2} - \alpha)\|Ax_n - Ap\|^2.\end{aligned}$$

Then,

$$\begin{aligned}
\phi(p, y_n) &= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JTz_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JTz_n\|^2 \\
&\le \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JTz_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tz_n\|^2 \\
&= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, Tz_n) \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n) \\
&\le \phi(p, x_n) + (1-\alpha_n)(1-\beta_n)2a(\frac{2b}{c^2} - \alpha)\|Ax_n - Ap\|^2 \\
&\le \phi(p, x_n).
\end{aligned} \tag{3.5}$$

Therefore $p \in C_n$, for all $n \ge 0$ and hence $F \subset C_n$, for all $n \ge 0$. Since F is nonempty, C_n is a nonempty closed convex subset of E and thus Π_{C_n} exists for every $n \ge 0$. Hence $\{x_n\}$ is well defined.

Thirdly, we shall show that $\lim_{n\to\infty} x_n = \overline{x} \in F(T)$.

Since $x_{n+1} = \Pi_{C_n}(x_0)$, one has

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall z \in C_n$$

and

$$\langle x_{n+1} - p, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall p \in F. \tag{3.6}$$

From Lemma 2.5, one has

$$\phi(x_{n+1}, x_0) = \phi(\Pi_{C_n}(x_0), x_0) \le \phi(p, x_0) - \phi(p, x_{n+1}) \le \phi(p, x_0),$$

for each $p \in F \subset C_n$ and $n \ge 0$. Then the sequence $\{\phi(x_{n+1}, x_0)\}$ is bounded. Moreover from (2.3), we have that $\{x_n\}$ is bounded. Since $x_{n+1} = \Pi_{C_n}(x_0)$, one has

$$\phi(x_n, x_0) \le \phi(x_{n+m}, x_0), \quad \forall n \ge 0.$$

Therefore, $\{\phi(x_n, x_0)\}$ is non-decreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. From Lemma 2.5, we have, for each $n \ge 0$,

$$\phi(x_{n+m}, x_n) \le \phi(x_{n+m}, x_0) - \phi(x_n, x_0).$$

This implies that $\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0$. It follows from Lemma 2.3, that $x_{n+m} - x_n \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \to \overline{x} \in C$ as $n \to \infty$.

Since $x_{n+1} = \Pi_{C_n}(x_0) \in C_n$, from the denition of C_n, we also have, for each $n \ge 0$,

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n).$$

Taking $n \to \infty$, we have $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. Using Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From $\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, we have

$$\|x_n - y_n\| \to 0, \quad n \to \infty.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \quad (3.7)$$

Therefore, for each $p \in F$, we have

$$\begin{aligned} \phi(p, x_n) - \phi(p, y_n) &= 2\langle p, Jy_n - Jx_n\rangle + \|x_n\|^2 - \|y_n\|^2 \\ &\leq 2\|p\|\|Jy_n - Jx_n\| \\ &\quad + (\|x_n\| - \|y_n\|)(\|x_n\| + \|y_n\|) \to 0. \end{aligned} \quad (3.8)$$

On the other hand, we have, for each $n \geq 0$,

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JTz_n)\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JTz_n\| - \alpha_n\|Jx_{n+1} - Jx_n\|. \end{aligned}$$

Therefore,

$$\|Jx_{n+1} - JTz_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_{n+1} - Jx_n\|).$$

From (3.7) and $\limsup_{n\to\infty} \alpha_n < 1$, we obtain

$$\|Jx_{n+1} - JTz_n\| \to 0, \quad n \to \infty.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have $\lim_{n\to\infty} \|x_{n+1} - Tz_n\| = 0$. From $\|x_n - Tz_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\|$, we have

$$\lim_{n\to\infty} \|x_n - Tz_n\| = 0. \quad (3.9)$$

From (3.5), we have

$$-(1-\alpha_n)(1-\beta_n)2a(\frac{2b}{c^2} - \alpha)\|Ax_n - Ap\|^2 \leq \phi(p, x_n) - \phi(p, y_n).$$

By (3.8), $\limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$, we have

$$\|Ax_n - Ap\| \to 0, \quad n \to \infty. \quad (3.10)$$

From Lemmas 2.5 and 2.7, and (3.4), for each $n \geq 0$, we have

$$\begin{aligned} \phi(x_n, \Pi_C u_n) &\leq \phi(x_n, u_n) = V(x_n, Jx_n - \lambda_n Ax_n) \\ &\leq V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\ &= V(x_n, Jx_n) - 2\lambda_n\langle u_n - x_n, Ax_n\rangle \\ &= \phi(x_n, x_n) + 2\langle u_n - x_n, -\lambda_n Ax_n\rangle, \\ &= 2\langle u_n - x_n, -\lambda_n Ax_n\rangle, \\ &\leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2. \end{aligned}$$

By (3.10), we get

$$\phi(x_n, \Pi_C u_n) \to 0, \quad n \to \infty. \quad (3.11)$$

Applying Lemma 2.3, from(3.11), we obtain that

$$\|x_n - \Pi_C u_n\| \to 0, \quad n \to \infty. \tag{3.12}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\|J\Pi_C u_n - Jx_n\| \to 0, \quad n \to \infty. \tag{3.13}$$

From (3.1) and (3.13), we have

$$\|Jw_n - Jx_n\| = (1 - \beta_n)\|J\Pi_C u_n - Jx_n\| \to 0, \quad n \to \infty.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have $\lim_{n\to\infty} \|w_n - x_n\| = 0$. Since

$$\begin{aligned}\phi(x_n, z_n) \le \phi(x_n, w_n) &= \langle x_n, Jx_n - Jw_n\rangle + \langle w_n - x_n, Jw_n\rangle \\ &\le \|x_n\|\|Jx_n - Jw_n\| + \|w_n - x_n\|\|w_n\| \to 0, \quad n \to \infty.\end{aligned}$$

from Lemma 2.3, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.14}$$

From (3.9) and (3.14), we have

$$\|z_n - Tz_n\| \le \|z_n - x_n\| + \|x_n - Tz_n\| \to 0, \quad n \to \infty. \tag{3.15}$$

It follows from (3.14) that $z_n \to \overline{x}$ as $n \to \infty$. From (3.15) and the definition of T, we have $\overline{x} \in F(T)$.

Fourthly, we aim to prove $\overline{x} \in VI(C, A)$.

From(3.12), we have $\Pi_C u_n \to \overline{x}$. Let $S \subset E \times E^*$ be an operator as follows:

$$Sv = \begin{cases} Av + N_C(v), v \in C, \\ \emptyset, \qquad\qquad v \notin C. \end{cases}$$

By Lemma 2.8, S is maximal monotone and $S^{-1}0 = VI(C, A)$. Let $(v, w) \in G(S)$. Since $w \in Sv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $\Pi_C u_n \in C$, we get

$$\langle v - \Pi_C u_n, w - Av\rangle \ge 0. \tag{3.16}$$

On the other hand, from Lemma 2.4, we have $\langle v - \Pi_C u_n, J\Pi_C u_n - Ju_n\rangle \ge 0$. Hence, there is

$$\langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n} - Ax_n\rangle \le 0. \tag{3.17}$$

Then, it holds from (3.16) and (3.17) that, for every $n \geq 0$,

$$\begin{aligned}
\langle v - \Pi_C u_n, w\rangle &\geq \langle v - \Pi_C u_n, Av\rangle \\
&\geq \langle v - \Pi_C u_n, Av\rangle + \langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n} - Ax_n\rangle \\
&= \langle v - \Pi_C u_n, Av - Ax_n\rangle + \langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n}\rangle \\
&= \langle v - \Pi_C u_n, Av - A\Pi_C u_n\rangle + \langle v - \Pi_C u_n, A\Pi_C u_n - Ax_n\rangle \\
&\quad + \langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n}\rangle \\
&\geq -\|v - \Pi_C u_n\| \frac{\|\Pi_C u_n - x_n\|}{\alpha} - \|v - \Pi_C u_n\| \frac{\|J\Pi_C u_n - Jx_n\|}{a} \\
&\geq -M(\frac{\|\Pi_C u_n - x_n\|}{\alpha} + \frac{\|J\Pi_C u_n - Jx_n\|}{a}),
\end{aligned}$$

where $M = \sup\{\|v - \Pi_C u_n\| : n \geq 0\}$. From (3.12) and (3.13), we have $\langle v - \overline{x}, w\rangle \geq 0$ as $n \to \infty$. By the maximality of S, we obtain $\overline{x} \in S^{-1}0$, that is $\overline{x} \in VI(C, A)$. Therefore, $\overline{x} \in F$.
Finally, we prove $\overline{x} = \Pi_F x_0$. By taking limit in (3.6), one has

$$\langle \overline{x} - p, Jx_0 - J\overline{x}\rangle \geq 0, \quad \forall p \in F.$$

at this point, in view of Lemma 2.4, one sees that $\overline{x} = \Pi_F x_0$. This completes the proof. □

Taking $A = 0$, Theorem 3.1 reduces to the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a two-uniformly convex and uniformly smooth Banach space E. Assume that T is a weak relatively nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. The sequence $\{x_n\}$ is defined by*

$$\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
w_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)J\Pi_C x_n), \\
z_n = \Pi_C w_n, \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTz_n), \\
C_0 = C, \\
C_n = \{v \in C_{n-1} : \phi(v, y_n) \leq \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_n}(x_0),
\end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1) such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\limsup_{n\to\infty} \beta_n < 1$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\frac{1}{c}$ is the two-uniformly convexity constant of E.

Taking $\alpha_n \equiv 0$, $\beta_n \equiv 0$, $T = I$, Theorem 3.1 reduces to the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a two-uniformly convex and uniformly smooth Banach space E. Assume that A is an operator of*

C into E^* that satisfies the conditions (A1)-(A3) and that $VI(C,A) \neq \emptyset$. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ w_n = \Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n)), \\ y_n = \Pi_C w_n, \\ C_0 = C, \\ C_n = \{v \in C_{n-1} : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_n}(x_0), \end{cases}$$

where $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ with $0 < a < b < \frac{c^2\alpha}{2}$, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\frac{1}{c}$ is the two-uniformly convexity constant of E.

We can also get the following result.

Theorem 3.4. *Let C be a nonempty closed convex subset of a two-uniformly convex and uniformly smooth Banach space E. Assume that A is an operator of C into E^* that satisfies the conditions (A1)-(A3) and T is a relatively nonexpansive mapping from C into itself such that $F = F(T) \cap VI(C,A) \neq \emptyset$. The sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ w_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)J\Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n))), \\ z_n = \Pi_C w_n, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ C_0 = C, \\ C_n = \{v \in C_{n-1} : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_n}(x_0), \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1) such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\limsup_{n\to\infty} \beta_n < 1$. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ with $0 < a < b < \frac{c^2\alpha}{2}$, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\frac{1}{c}$ is the two-uniformly convexity constant of E.

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