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SOME EXISTENCE RESULTS ON PERIODIC SOLUTIONS OF ORDINARY (q,p)–LAPLACIAN SYSTEMS[†]

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Abstract. Some existence theorems are obtained for periodic solutions of nonautonomous second-order differential systems with (q,p)–Laplacian by the minimax methods in critical point theory.

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1. Introduction

In the last years many authors starting with Mawhin and Willem (see [3]) proved the existence of solutions for problem

$$\ddot{u}(t) = \nabla F(t, u(t)) \text{ a.e. } t \in [0,T],$$
$$u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0,$$

under suitable conditions on the potential F (see [12]-[24]). Also in a series of papers (see [4]-[6]) we have generalized some of these results for the case when the potential F is just locally Lipschitz in the second variable x not continuously differentiable. Very recent (see [7] and [9]) we have considered the second order Hamiltonian inclusions systems with p–Laplacian.

The aim of this paper is to show how the results obtained in [25] can be generalized. More exactly our results represent the extensions to second-order differential systems with (q,p)–Laplacian. As far as we know this kind of systems have been considered recently just in a few papers [2], [8] and [10].

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Consider the second order system

$$\begin{cases} -\frac{d}{dt}\big(|\dot u_1(t)|^{q-2}\dot u_1(t)\big) = \nabla_{u_1}F(t,u_1(t),u_2(t)),\\ -\frac{d}{dt}\big(|\dot u_2(t)|^{p-2}\dot u_2(t)\big) = \nabla_{u_2}F(t,u_1(t),u_2(t)) \text{ a.e. } t\in[0,T],\\ u_1(0)-u_1(T)=\dot u_1(0)-\dot u_1(T)=0,\\ u_2(0)-u_2(T)=\dot u_2(0)-\dot u_2(T)=0, \end{cases} \tag{1}$$

where $1<p,q<\infty$, $T>0$, and $F:[0,T]\times\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ satisfy the following assumption (A):

- F is measurable in t for each $(x_1,x_2)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$;
- F is continuously differentiable in (x_1,x_2) for a.e. $t\in[0,T]$;
- there exist $a_1,a_2\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $b\in L^1(0,T;\mathbb{R}_+)$ such that

$$|F(t,x_1,x_2)|,\ |\nabla_{x_1}F(t,x_1,x_2)|,\ |\nabla_{x_2}F(t,x_1,x_2)|\le\big[a_1(|x_1|)+a_2(|x_2|)\big]b(t)$$

for all $(x_1,x_2)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$ and a.e. $t\in[0,T]$.

Following Tang and Wu [23], we generalize subquadratic condition in Rabinowitz's sense, that is, there exist $0<\mu<r=\min(q,p)$, $M>0$ such that

$$(\nabla_{(x_1,x_2)}F(t,x_1,x_2),(x_1,x_2))\le\mu F(t,x_1,x_2) \tag{2}$$

for all $|(x_1,x_2)|\ge M$ and a.e. $t\in[0,T]$. We prove that under condition (2) and some other suitable conditions, the corresponding energy functional also satisfies (C) condition. Then we get some existence results for problem (1) by the Saddle Point Theorem in critical point theory. The main results are the following theorems.

Theorem 1. *Suppose that F satisfies assumptions (A) and (2). Assume that there exists $g\in L^1(0,T)$ such that*

$$F(t,x_1,x_2)\ge g(t) \tag{3}$$

for all $(x_1,x_2)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$, and that there exists a subset E of $[0,T]$ with meas$(E)>0$, such that

$$F(t,x_1,x_2)\to+\infty \text{ as } |x|=\sqrt{|x_1|^2+|x_2|^2}\to\infty \tag{4}$$

for a.e. $t\in E$. Then problem (1) has at least one solution in $W=W_T^{1,q}\times W_T^{1,p}$.

Corollary 2. *Suppose that F satisfies assumptions (A) and (2). Assume that*

$$F(t,x_1,x_2)\to+\infty \text{ as } |x|=\sqrt{|x_1|^2+|x_2|^2}\to\infty$$

uniformly for a.e. $t\in[0,T]$. Then problem (1) has at least one solution in $W=W_T^{1,q}\times W_T^{1,p}$.

Theorem 3. *Suppose that F satisfies assumptions (A), (2) and*

$$\int_0^T F(t,x_1,x_2)\to+\infty \text{ as } |x|=\sqrt{|x_1|^2+|x_2|^2}\to\infty. \tag{5}$$

Assume that $F(t,\cdot,\cdot)$ is (β,γ)–subconvex for a.e. $t \in [0,T]$ with $\beta > 0, \gamma > 0$, that is,

$$F(t, \beta((x_1,x_2)+(y_1,y_2))) \le \gamma\big(F(t,x_1,x_2)+F(t,y_1,y_2)\big) \tag{6}$$

for all $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Then problem (1) has at least one solution in $W = W_T^{1,q} \times W_T^{1,p}$.

Remark 1. *Theorems 1 and 3 generalizes Theorem 1 and 2 of Xu and Tang* [25]*. In fact, it follows from our theorems by letting $F(t,x_1,x_2) = F_1(t,x_1)$.*

2. The proofs of the theorems

We introduce some functional spaces. Let $T > 0$ be a positive number and $1 < q, p < \infty$. We use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^{\mathbb{N}}$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0,T;\mathbb{R}^{\mathbb{N}})$ having a weak derivative $\dot u \in L^p(0,T;\mathbb{R}^{\mathbb{N}})$. The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \Big(\int_0^T \big(|u(t)|^p + |\dot u(t)|^p\big)dt\Big)^{\frac{1}{p}}.$$

It follows from [3] that $W_T^{1,p}$ is a reflexive and uniformly convex Banach space. From [1], we know that a locally uniformly convex Banach space X has the *Kadec-Klee property*, that is, for any sequence $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in X and $\|u_n\| \to \|u\|$, we have $u_n \to u$ strongly in X. We will use this property later.

Moreover, we use the space W defined by

$$W = W_T^{1,q} \times W_T^{1,p}$$

with the norm $\|(u_1,u_2)\|_W = \|u_1\|_{W_T^{1,q}} + \|u_2\|_{W_T^{1,p}}$. It is clear that W is a reflexive Banach space.
We recall that

$$\|u\|_p = \Big(\int_0^T |u(t)|^p dt\Big)^{\frac{1}{p}} \text{ and } \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

For our aims it is necessary to recall some very well know results (for proof and details see [3]).

Proposition 4. *Each $u \in W_T^{1,p}$ can be written as $u(t) = \bar u + \tilde u(t)$ with*

$$\bar u = \frac{1}{T}\int_0^T u(t)dt, \qquad \int_0^T \tilde u(t)dt = 0.$$

We have the Sobolev's inequality

$$\|\tilde u\|_\infty \le C\|\dot u\|_p,\ \|\tilde v\|_\infty \le C\|\dot v\|_q \qquad \text{for each } u \in W_T^{1,p},\ v \in W_T^{1,q},$$

and Wirtinger's inequality

$$\|\tilde u\|_p \le C\|\dot u\|_p,\ \|\tilde v\|_q \le C\|\dot v\|_q \qquad \text{for each } u \in W_T^{1,p},\ v \in W_T^{1,q}.$$

In [16] the authors have proved the following result (see Lemma 3.1) which generalize a very well known result proved by Jean Mawhin and Michel Willem (see Theorem 1.4 in [3]):

Lemma 5. *Let $L : [0,T] \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, $(t, x_1, x_2, y_1, y_2) \to L(t, x_1, x_2, y_1, y_2)$ be measurable in t for each (x_1, x_2, y_1, y_2), and continuously differentiable in (x_1, x_2, y_1, y_2) for a.e. $t \in [0,T]$. If there exist $a_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in L^1(0,T;\mathbb{R}_+)$, and $c_1 \in L^p(0,T;\mathbb{R}_+)$, $c_2 \in L^q(0,T;\mathbb{R}_+)$, $1 < p, q < \infty$, such that for a.e. $t \in [0,T]$ and every $(x_1, x_2, y_1, y_2) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, one has*

$$\begin{aligned}
|L(t, x_1, x_2, y_1, y_2)| &\le \left[a_1(|x_1|) + a_2(|x_2|)\right]\left[b(t) + |y_1|^q + |y_2|^p\right],\\
|D_{x_1} L(t, x_1, x_2, y_1, y_2)| &\le \left[a_1(|x_1|) + a_2(|x_2|)\right]\left[b(t) + |y_2|^p\right],\\
|D_{x_2} L(t, x_1, x_2, y_1, y_2)| &\le \left[a_1(|x_1|) + a_2(|x_2|)\right]\left[b(t) + |y_1|^q\right],\\
|D_{y_1} L(t, x_1, x_2, y_1, y_2)| &\le \left[a_1(|x_1|) + a_2(|x_2|)\right]\left[c_1(t) + |y_1|^{q-1}\right],\\
|D_{y_2} L(t, x_1, x_2, y_1, y_2)| &\le \left[a_1(|x_1|) + a_2(|x_2|)\right]\left[c_2(t) + |y_2|^{p-1}\right],
\end{aligned}$$

then the function $\varphi : W_T^{1,q} \times W_T^{1,p} \to \mathbb{R}$ defined by

$$\varphi(u_1, u_2) = \int_0^T L(t, u_1(t), u_2(t), \dot{u}_1(t), \dot{u}_2(t))dt$$

is continuously differentiable on $W_T^{1,q} \times W_T^{1,p}$ and

$$\begin{aligned}
\langle \varphi'(u_1, u_2), (v_1, v_2)\rangle = \int_0^T &\big[(D_{x_1} L(t, u_1(t), u_2(t), \dot{u}_1(t), \dot{u}_2(t)), v_1(t))\\
&+ (D_{y_1} L(t, u_1(t), u_2(t), \dot{u}_1(t), \dot{u}_2(t)), \dot{v}_1(t))\\
&+ (D_{x_2} L(t, u_1(t), u_2(t), \dot{u}_1(t), \dot{u}_2(t)), v_2(t))\\
&+ (D_{y_2} L(t, u_1(t), u_2(t), \dot{u}_1(t), \dot{u}_2(t)), \dot{v}_2(t))\big]dt.
\end{aligned}$$

Corollary 6. *Let $L : [0,T] \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be defined by*

$$L(t, x_1, x_2, y_1, y_2) = \frac{1}{q}|y_1|^q + \frac{1}{p}|y_2|^p - F(t, x_1, x_2)$$

where $F : [0,T] \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ satisfy condition (A). If $(u_1, u_2) \in W_T^{1,q} \times W_T^{1,p}$ is a solution of the corresponding Euler equation $\varphi'(u_1, u_2) = 0$, then (u_1, u_2) is a solution of (1).

Remark 2. *The function $\varphi : W \to \mathbb{R}$ given by*

$$\varphi(u_1, u_2) = \frac{1}{q}\int_0^T |\dot{u}_1|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2|^p dt - \int_0^T F(t, u_1, u_2)dt$$

for all $(u_1, u_2) \in W$, is weakly lower semi-continuous (w.l.s.c.) on W as the sum of two convex continuous functions and of a weakly continuous one. Moreover, *one has*

$$\langle \varphi'(u_1,u_2),(v_1,v_2)\rangle = \int_0^T \big(|\dot u_1|^{q-2}\dot u_1,\dot v_1\big)dt + \int_0^T \big(|\dot u_2|^{p-2}\dot u_2,\dot v_2\big)dt - \int_0^T \big(\nabla_{(u_1,u_2)}F(t,u_1,u_2),(v_1,v_2)\big)dt$$

for all $(u_1,u_2),(v_1,v_2)\in W$.

Lemma 7. ([25]) *In Sobolev space* $W_T^{1,p}$*, for* $u\in W_T^{1,p}$*,* $\|u\|\to\infty$ *if and only if* $\big(|\bar u|^p+\int_0^T|\dot u|^p dt\big)^{1/p}\to\infty$.

Lemma 8. *Under conditions* (A)*, (2), (3) and (4), the functional* φ *satisfies condition* (C)*, that is,* $\{(u_{1n},u_{2n})\}$ *has a convergent subsequence in* W *whenever* $\varphi(u_{1n},u_{2n})$ *is bounded and* $\|\varphi'(u_{1n},u_{2n})\|\times(1+\|(u_{1n},u_{2n})\|)\to 0$ *as* $n\to\infty$.

Proof. Let $\{(u_{1n},u_{2n})\}$ be a sequence in W such that $\varphi(u_{1n},u_{2n})$ is bounded and $\|\varphi'(u_{1n},u_{2n})\|\times(1+\|(u_{1n},u_{2n})\|)\to 0$ as $n\to\infty$. Then there exists a constant C_1 such that

$$|\varphi(u_{1n},u_{2n})|\le C_1,\ \ \|\varphi'(u_{1n},u_{2n})\|(1+\|(u_{1n},u_{2n})\|)\le C_1 \tag{7}$$

for all $n\in\mathbb{N}$. Let

$$h(t)=(r+M)b(t)\max_{|(x_1,x_2)|\le M}\big[a_1(|x_1|)+a_2(|x_2|)\big].$$

Then, by assumption (A) and (2), one has

$$-h(t)+(\nabla_{(x_1,x_2)}F(t,x_1,x_2),(x_1,x_2))\le \mu F(t,x_1,x_2) \tag{8}$$

for all $(x_1,x_2)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$ and a.e. $t\in[0,T]$. It follows that

$$\begin{aligned}(r+1)C_1 &\ge \|\varphi'(u_{1n},u_{2n})\|(1+\|(u_{1n},u_{2n})\|)-r\varphi(u_{1n},u_{2n})\\ &\ge(\varphi'(u_{1n},u_{2n}),(u_{1n},u_{2n}))-r\varphi(u_{1n},u_{2n})\\ &\ge\int_0^T\big[rF(t,u_{1n},u_{2n})-(\nabla_{(x_1,x_2)}F(t,u_{1n},u_{2n}),(u_{1n},u_{2n}))\big]dt\\ &\ge(r-\mu)\int_0^T F(t,u_{1n},u_{2n})dt-\int_0^T h(t)dt\end{aligned}$$

for all $n\in\mathbb{N}$, which implies that

$$\int_0^T F(t,u_{1n},u_{2n})\le C_2 \tag{9}$$

for all $n\in\mathbb{N}$ and some constant C_2. By (7) and (9), one has

$$C_1\ge\varphi(u_{1n},u_{2n})\ge\frac{1}{q}\int_0^T|\dot u_{1n}|^q dt+\frac{1}{p}\int_0^T|\dot u_{2n}|^p dt-C_2$$

for all $n\in\mathbb{N}$. Hence we have

$$\int_0^T|\dot u_{1n}|^q dt\le C_3 \text{ and } \int_0^T|\dot u_{2n}|^p dt\le C_3 \tag{10}$$

for all $n \in \mathbb{N}$ and some constant C_3. By Sobolev's inequality, we get

$$\|\tilde{u}_{1n}\|_\infty \le C_4 \text{ and } \|\tilde{u}_{2n}\|_\infty \le C_4 \tag{11}$$

for all $n \in \mathbb{N}$ and some constant C_4.

We argue that the sequence $\{(\bar{u}_{1n}, \bar{u}_{2n})\}$ is bounded. Otherwise, there is a subsequence, again denoted by $\{(\bar{u}_{1n}, \bar{u}_{2n})\}$, such that $|(\bar{u}_{1n}, \bar{u}_{2n})| \to \infty$ as $n \to \infty$. Let

$$\begin{aligned}(v_{1n}, v_{2n}) &= \frac{(u_{1n}, u_{2n})}{\|(u_{1n}, u_{2n})\|_W} \\ &= \frac{(\bar{u}_{1n}, \bar{u}_{2n})}{\|(u_{1n}, u_{2n})\|_W} + \frac{(\tilde{u}_{1n}, \tilde{u}_{2n})}{\|(u_{1n}, u_{2n})\|_W} \\ &= (\bar{v}_{1n}, \bar{v}_{2n}) + (\tilde{v}_{1n}, \tilde{v}_{2n}).\end{aligned}$$

Then, $\{(v_{1n}, v_{2n})\}$ is bounded in W and by the compactness of the embedding $W = W_T^{1,q} \times W_T^{1,p} \subset C([0,T];\mathbb{R}^{\mathbb{N}}) \times C([0,T];\mathbb{R}^{\mathbb{N}})$, there is a subsequence, again denoted by $\{(v_{1n}, v_{2n})\}$, such that

$$\begin{aligned}(v_{1n}, v_{2n}) &\rightharpoonup (v_1, v_2) \quad \text{weakly in} \quad W, \\ (v_{1n}, v_{2n}) &\to (v_1, v_2) \quad \text{strongly in} \quad C([0,T];\mathbb{R}^{\mathbb{N}}) \times C([0,T];\mathbb{R}^{\mathbb{N}}).\end{aligned}$$

By (11), $\{(\tilde{u}_{1n}, \tilde{u}_{2n})\}$ is bounded in $C([0,T];\mathbb{R}^{\mathbb{N}}) \times C([0,T];\mathbb{R}^{\mathbb{N}})$, so $(v_1, v_2) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and $(v_1, v_2) \ne (0,0)$. Thus $|(u_{1n}(t), u_{2n}(t))| \to \infty$ as $n \to \infty$, for all $t \in [0,T]$. From (4) and Lebesgue-Fatou Lemma, we have

$$\liminf_{n\to\infty} \int_0^T F(t, u_{1n}, u_{2n})dt \ge \liminf_{n\to\infty} \int_E F(t, u_{1n}, u_{2n})dt - \int_0^T |g(t)|dt = +\infty,$$

which contradicts (9).

Then, by Lemma 7 $\{(u_{1n}, u_{2n})\}$ is bounded in W. By the compactness of the embedding $W_T^{1,q}$(or $W_T^{1,p}$) $\subset C([0,T];\mathbb{R}^{\mathbb{N}})$, the sequence $\{u_{1n}\}$(or $\{u_{2n}\}$) has a subsequence, still denoted by $\{u_{1n}\}$(or $\{u_{2n}\}$) , such that

$$u_{1n}(\text{ or } u_{2n}) \rightharpoonup u_1(\text{ or } u_2) \quad \text{weakly in} \quad W_T^{1,q}(\text{ or } W_T^{1,p}) \,, \tag{12}$$

$$u_{1n} \to u_1 \quad \text{strongly in} \quad C([0,T];\mathbb{R}^{\mathbb{N}}). \tag{13}$$

Note that

$$\begin{aligned}\langle \varphi'(u_{1n}, u_{2n}), (u_1 - u_{1n}, 0)\rangle &= \int_0^T |\dot{u}_{1n}|^{q-2}(\dot{u}_{1n}, \dot{u}_1 - \dot{u}_{1n})dt \\ &\quad - \int_0^T (\nabla_{x_1} F(t, u_{1n}, u_{2n}), u_1 - u_{1n})dt \to 0\end{aligned}$$

as $n \to \infty$. From (13), $\{u_{1n}\}$ is bounded in $C([0,T];\mathbb{R}^{\mathbb{N}})$. Then we have

$$\begin{aligned}\left|\int_0^T (\nabla_{x_1} F(t,u_{1n},u_{2n}), u_1 - u_{1n})dt\right| &\le \int_0^T |\nabla_{x_1} F(t,u_{1n},u_{2n})| \cdot |u_1 - u_{1n}| dt \\ &\le C_5 \int_0^T b(t)|u_1 - u_{1n}|dt \\ &\le C_5 \|b\|_{L^1} \|u_1 - u_{1n}\|_\infty\end{aligned}$$

for some positive constant C_5, which combined with (13) implies that

$$\int_0^T (\nabla_{x_1} F(t,u_{1n},u_{2n}), u_1 - u_{1n})dt \to 0 \quad \text{as } n \to \infty.$$

Hence one has

$$\int_0^T |\dot{u}_{1n}|^{q-2} (\dot{u}_{1n}, \dot{u}_1 - \dot{u}_{1n})dt \to 0 \quad \text{as} \quad n \to \infty \ .$$

Moreover from (13) we obtain

$$\int_0^T |u_{1n}|^{q-2} (u_{1n}, u_1 - u_{1n})dt \to 0 \quad \text{as } n \to \infty \ .$$

Set

$$\psi(u_1,u_2) = \frac{1}{q}\left(\int_0^T |u_1|^q dt + \int_0^T |\dot{u}_1|^q dt\right) + \frac{1}{p}\left(\int_0^T |u_2|^p dt + \int_0^T |\dot{u}_2|^p dt\right).$$

Then one obtains

$$\begin{aligned}\langle \psi'(u_{1n},u_{2n}), (u_1 - u_{1n}, 0)\rangle &= \int_0^T |u_{1n}|^{q-2}(u_{1n}, u_1 - u_{1n})dt \\ &+ \int_0^T |\dot{u}_{1n}|^{q-2}(\dot{u}_{1n}, \dot{u}_1 - \dot{u}_{1n})dt \ ,\end{aligned}$$

and

$$\langle \psi'(u_{1n},u_{2n}), (u_1 - u_{1n}, 0)\rangle \to 0 \quad \text{as } n \to \infty \ . \tag{14}$$

By the Hölder's inequality, we have

$$0 \le (\|u_{1n}\|^{q-1} - \|u_1\|^{q-1})(\|u_{1n}\| - \|u_1\|) \le \langle \psi'(u_{1n},u_{2n}) - \psi'(u_1,u_2), u_{1n} - u_1\rangle$$

which together with (14) yields $\|u_{1n}\| \to \|u_1\|$. It follows that $u_{1n} \to u_1$ strongly in $W_T^{1,q}$ by the uniform convexity of $W_T^{1,q}$. Similarly we have $u_{2n} \to u_2$ strongly in $W_T^{1,p}$. Hence the (C) condition is satisfied. □

Proof of Theorem 1. Let $\widetilde{W} = \widetilde{W}_T^{1,q} \times \widetilde{W}_T^{1,p}$ be the subspace of W given by $\widetilde{W} = \{(u_1,u_2) \in W \ \mid \ (\bar{u}_1, \bar{u}_2) = (0,0)\}$. Then $W = \widetilde{W} + \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. For using the Saddle Point Theorem(see [11] or [3]) we have only to prove that

(φ_1) $\varphi(u_1,u_2) \to +\infty$ as $\|(u_1,u_2)\| \to \infty$ in $\widetilde{W}$,
(φ_2) $\varphi(u_1,u_2) \to -\infty$ as $\|(u_1,u_2)\| \to \infty$ in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$.

For every $|(x_1,x_2)|\geq M$ and a.e. $t\in[0,T]$, let

$$y(s)=F(t,sx_1,sx_2),\qquad Q(s)=y'(s)-\frac{\mu}{s}y(s). \tag{15}$$

Then by (2) we have

$$Q(s)=\frac{1}{s}\big[(\nabla_{(x_1,x_2)}F(t,sx_1,sx_2),(sx_1,sx_2))-\mu F(t,sx_1,sx_2)\big]\leq 0 \tag{16}$$

for all $s\geq M/|(x_1,x_2)|$. It follows from (15) that $y(s)=F(t,sx_1,sx_2)$ is a solution of the first order linear ordinary differential equation

$$y'(s)=\frac{\mu}{s}y(s)+Q(s)$$

which implies that

$$F(t,sx_1,sx_2)=s^{\mu}\Big(\int_1^s r^{-\mu}Q(r)dr+F(t,x_1,x_2)\Big)$$

for $s\geq M/|(x_1,x_2)|$. Moreover, by assumption (A) and (16), we have

$$a_0b(t)\geq F\Big(t,\frac{Mx_1}{|(x_1,x_2)|},\frac{Mx_2}{|(x_1,x_2)|}\Big)\geq\Big(\frac{M}{|(x_1,x_2)|}\Big)^{\mu}F(t,x_1,x_2)$$

for all $|(x_1,x_2)|\geq M$, a.e. $t\in[0,T]$ and some constant $a_0=\max_{|(x_1,x_2)|\leq M}\big[a_1(|x_1|)+a_2(|x_2|)\big]$, which implies that

$$F(t,x_1,x_2)\leq a_0b(t)\Big(\Big(\frac{|(x_1,x_2)|}{M}\Big)^{\mu}+1\Big)$$

for all $(x_1,x_2)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$ and a.e. $t\in[0,T]$. By Sobolev's inequality and Wirtinger's inequality, we have

$$\begin{aligned}\varphi(u_1,u_2)&\geq\frac{1}{q}\|\dot u_1\|_q^q+\frac{1}{p}\|\dot u_2\|_p^p-a_0\Big(\frac{2}{M}\Big)^{\mu}\big(\|u_1\|_\infty^\mu+\|u_2\|_\infty^\mu\big)\int_0^T b(t)dt-a_0\int_0^T b(t)dt\\&\geq\frac{1}{2q}\min(1,C^{-q})\|u_1\|_W^q+\frac{1}{2p}\min(1,C^{-p})\|u_2\|_W^p\\&\quad-a_0\Big(\frac{2C}{M}\Big)^{\mu}\int_0^T b(t)dt\big(\|u_1\|_W^\mu+\|u_2\|_W^\mu\big)-a_0\int_0^T b(t)dt\end{aligned}$$

for all $(u_1,u_2)\in\widetilde{W}$, then we get (φ_1) taking into account that $\mu<r=\min(q,p)$.

By (3) and (4), we have for $(x_1,x_2)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$

$$\begin{aligned}\varphi(x_1,x_2)&=-\int_0^T F(t,x_1,x_2)dt\\&\leq-\int_E F(t,x_1,x_2)dt-\int_{[0,T]\setminus E}g(t)dt\\&\leq-\int_E F(t,x_1,x_2)dt+\int_0^T|g(t)|dt\to-\infty\end{aligned}$$

then we get (φ_2). Now, from the Saddle Point Theorem it follows that Theorem 1 holds. □

Proof of Theorem 3. First, we prove that the functional φ satisfies condition (C). Let $\{(u_{1n},u_{2n})\}$ be a sequence in W such that $\varphi(u_{1n},u_{2n})$ is bounded and $\|\varphi'(u_{1n},u_{2n})\| \times (1+\|(u_{1n},u_{2n})\|) \to 0$ as $n\to\infty$. In a way similar to (9), (10) and (11) in the proof of Lemma 8, we have

$$\int_0^T F(t,u_{1n},u_{2n})dt \le C_2, \quad \|\dot{u}_{1n}\|_q \le C_3, \quad \|\dot{u}_{2n}\|_p \le C_3,$$

$$\|\tilde{u}_{1n}\|_\infty \le C_4, \quad \|\tilde{u}_{2n}\|_\infty \le C_4, \tag{17}$$

for all $n\in\mathbb{N}$. Then we have

$$\begin{aligned} C_2 &\ge \int_0^T F(t,u_{1n},u_{2n})dt \ge \frac{1}{\gamma}\int_0^T F(t,\beta\bar{u}_{1n},\beta\bar{u}_{2n})dt - \int_0^T F(t,-\tilde{u}_{1n},-\tilde{u}_{2n})dt \\ &\ge \frac{1}{\gamma}\int_0^T F(t,\beta\bar{u}_{1n},\beta\bar{u}_{2n})dt - \max_{|x_1|\le 2C_4, |x_2|\le 2C_4}\left[a_1(|x_1|)+a_2(|x_2|)\right]\int_0^T b(t)dt \end{aligned}$$

for all $n\in\mathbb{N}$, which implies that $\{(\bar{u}_{1n},\bar{u}_{2n})\}$ is bounded. From (17) we get $\{(u_{1n},u_{2n})\}$ is bounded. Like in the proof of Lemma 8 we can prove that $\{(u_{1n},u_{2n})\}$ have a convergent subsequence, so φ satisfies condition (C). Also (φ_1) holds for the same reasons like in the proof of Theorem 1, and (φ_2) follows directly from (5). Hence Theorem 3 holds using again the Saddle Point Theorem. □

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