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STABILITY AND BIFURCATION ANALYSIS OF A LOTKA-VOLTERRA MODEL WITH TIME DELAYS

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Abstract. In this paper, a Lotka-Volterra model with time delays is considered. A set of sufficient conditions for the existence of Hopf bifurcation are obtained via analyzing the associated characteristic transcendental equation. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by applying the normal form method and center manifold theory. Finally, the main results are illustrated by some numerical simulations.

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1. Introduction

The dynamic relationship between predators and preys has long been, and will continue to be, one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [14]. The dynamics properties (including stable, unstable, oscillatory and chaotic behavior) of predator-prey system have been studied extensively since the theoretical work of Lotka (1926), Volterra (1931), Nicholson and Bailey (1935) and the experimental work of Gause (1934). For example, Lu and Takenuchi [15] have proved that a two species Lotka-Volterra delayed competition system is permanent under any delay effect provided that the corresponding undelayed system has a globally stable positive equilibrium. They have also obtained conditions for global stability of positive equilibrium. Mukherjee and Roy [16] proposed a generalized prey-predator system with time delay and find the conditions for uniform persistence and global stability. For more research on predator-prey systems, one can see the references cited therein.

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Recently, Zhang and Chen [2] investigated the permanence and the global attractivity of positive periodic solution for the following non-autonomous delayed predator-prey system

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}(t)x_1(t-\tau_{11}) - a_{13}x_3(t-\tau_{13})], \\ \dot{x}_2(t) = x_2(t)[r_2(t) - a_{22}(t)x_2(t-\tau_{22}) - a_{23}(t)x_3(t-\tau_{23})], \\ \dot{x}_3(t) = x_3(t)[-r_3(t) + a_{31}(t)x_1(t-\tau_{31}) + a_{32}(t)x_2(t-\tau_{32}) \\ \qquad\qquad - a_{33}(t)x_3(t-\tau_{33})] \end{cases} \tag{1}$$

with the initial conditions

$$x_i(s) = \phi_i(s) \geq 0,\ s \in (-\tau, 0),\ \phi_i(0) > 0, i = 1, 2, 3,$$

where $x_i (i = 1, 2)$ denote the density of the i-th pery at time t, x_3 denotes the density of the predator at time t, all the coefficients in system (1) are continuous strictly positive bounded functions and $\tau_{11}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{31}$, τ_{32}, τ_{33} are all positive constants.

It is worth pointing out that an important and ubiquitous problem in predator-prey theory and related topics in mathematical ecology concerns the periodic phenomena occur in species. For a long time, it has been recognized that periodic solutions can arise through the Hopf bifurcation in delay equation [13].

In the present paper, we will provide a detailed analysis on the local Hopf bifurcation of system (1) under the following assumptions:
$(H1)$ $r_i(t)(i = 1, 2, 3), a_{11}(t), a_{13}(t), a_{22}(t), a_{23}(t), a_{31}(t), a_{32}(t), a_{33}(t)$ are all positive constants, i.e., $r_i(t) = r_i, a_{11}(t) = a_{11}, a_{13}(t) = a_{13}, a_{22}(t) = a_{22}, a_{23}(t) = a_{23}, a_{31}(t) = a_{31}, a_{32}(t) = a_{32}, a_{33}(t) = a_{33}$;
$(H2)$ $\tau_{11} = \tau_{13} = \tau_{22} = \tau_{23} = \tau_{31} = \tau_{32} = \tau_{33} = \tau$.

Based on the assumptions above, system (1) can be transformed as the following system

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1 - a_{11}x_1(t-\tau) - a_{13}x_3(t-\tau)], \\ \dot{x}_2(t) = x_2(t)[r_2 - a_{22}x_2(t-\tau) - a_{23}x_3(t-\tau)], \\ \dot{x}_3(t) = x_3(t)[-r_3 + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) - a_{33}x_3(t-\tau)]. \end{cases} \tag{2}$$

The purpose of this paper is to discuss the stability and the properties of Hopf bifurcation of model (2). More specifically, we will prove that, as the delay τ increases , the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur. Furthermore, using the normal form and center manifold theory[8], we derive an explicit algorithm and sufficient conditions for the stability of the bifurcating periodic solutions.

This paper is organized as follows. In Section 2, the stability of the positive equilibrium and the existence of Hopf bifurcation at the positive equilibrium are studied. In Section 3, the direction of Hopf bifurcation and the stability and periodic of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2. Stability of the equilibrium and local Hopf bifurcations

In this section, we shall study the stability of the positive equilibrium and the existence of local Hopf bifurcations.

Obviously, system (2) has a unique equilibrium $E_0(x_1^*, x_2^*, x_3^*)$, where

$$x_1^* = \frac{r_1a_{22}a_{33} - r_2a_{13}a_{32} - r_3a_{22}a_{13} - r_1a_{23}a_{32}}{a_{11}a_{22}a_{33} - a_{22}a_{13}a_{31} - a_{11}a_{23}a_{32}},$$
$$x_2^* = \frac{r_2a_{11}a_{33} + r_1a_{23}a_{31} - r_2a_{13}a_{31} - r_3a_{11}a_{32}}{a_{11}a_{22}a_{33} - a_{22}a_{13}a_{31} - a_{11}a_{23}a_{32}},$$
$$x_3^* = \frac{r_3a_{11}a_{22} - r_1a_{22}a_{31} - r_2a_{11}a_{32}}{a_{11}a_{22}a_{33} - a_{22}a_{13}a_{31} - a_{11}a_{23}a_{32}},$$

Throughout the paper, we make the following assumptions:

$$(H3) \qquad \begin{aligned} &sign\{r_1a_{22}a_{33} - r_2a_{13}a_{32} - r_3a_{22}a_{13} - r_1a_{23}a_{32}\} \\ &= sign\{r_2a_{11}a_{33} + r_1a_{23}a_{31} - r_2a_{13}a_{31} - r_3a_{11}a_{32}\} \\ &= sign\{r_3a_{11}a_{22} - r_1a_{22}a_{31} - r_2a_{11}a_{32}\} \\ &= sign\{a_{11}a_{22}a_{33} - a_{22}a_{13}a_{31} - a_{11}a_{23}a_{32}\}. \end{aligned}$$

It is easy to check that the equilibrium $E_0(x_1^*, x_2^*, x_3^*)$ of Eq. (2) is a positive equilibrium if the condition $(H3)$ holds.

The linearized system of (2) around $E_0(x_1^*, x_2^*, x_3^*)$ takes the form:

$$\begin{cases} \dot{x}_1(t) = m_1x_1(t) + m_2x_1(t-\tau) + m_3x_3(t-\tau), \\ \dot{x}_2(t) = n_1x_2(t) + n_2x_2(t-\tau) + n_3x_3(t-\tau), \\ \dot{x}_3(t) = p_1x_3(t) + p_2x_1(t-\tau) + p_3x_2(t-\tau) + p_4x_3(t-\tau), \end{cases} \tag{3}$$

where

$$\begin{aligned} &m_1 = r_1 - a_{22}x_1^* - a_{13}x_3^*, m_2 = -a_{11}x_1^*, m_3 = -a_{13}x_1^*, \\ &n_1 = r_2 - a_{22}x_2^* - a_{23}x_3^*, n_2 = -a_{22}x_2^*, n_3 = -a_{23}x_2^*, \\ &p_1 = -r_3 + a_{31}x_1^* + a_{32}x_2^* - a_{33}x_3^*,\ p_2 = a_{31}x_3^*,\ p_3 = a_{32}x_3^*,\ p_4 = -a_{33}x_3^*. \end{aligned}$$

Then the associated characteristic equation of (3) is

$$\det \begin{pmatrix} \lambda - m_1 - m_2e^{-\lambda\tau} & 0 & -m_3e^{-\lambda\tau} \\ 0 & \lambda - m_1 - m_2e^{-\lambda\tau} & -n_3e^{-\lambda\tau} \\ -p_2e^{-\lambda\tau} & -p_3e^{-\lambda\tau} & \lambda - m_1 - m_2e^{-\lambda\tau} \end{pmatrix} \tag{4}$$

which leads to the the following form:

$$p_1(\lambda)e^{\lambda\tau} + p_2(\lambda) + p_3(\lambda)e^{-\lambda\tau} + p_4(\lambda)e^{-2\lambda\tau} = 0. \tag{5}$$

where

$$
\begin{aligned}
p_1(\lambda) &= (\lambda - m_1)(\lambda - n_1)(\lambda - p_1),\\
p_2(\lambda) &= (p_1 - \lambda)[n_2(\lambda - m_1) + m_2(\lambda - n_1)] - p_4(\lambda - m_1)(\lambda - n_1),\\
p_3(\lambda) &= p_4[n_2(\lambda - m_1) + m_2(\lambda - n_1)] - \lambda(m_3p_2 + n_3p_3)\\
&\quad + m_2n_2\lambda - m_2n_2p_1 + m_3n_1p_2 + n_1n_3p_3,\\
p_4(\lambda) &= m_3n_2p_2 + m_2n_3p_3 - m_2n_2p_4.
\end{aligned}
$$

Let $\lambda = i\omega_0, \tau = \tau_0$, and substituting this into (5), for the sake of simplicity, denote ω_0 and τ_0 by ω, τ, respectively, then (5) becomes

$$
\begin{aligned}
&(u_1 + iv_1)(\cos\omega\tau + i\sin\omega\tau) + u_2 + iv_2\\
&+ (u_3 + iv_3) \times (\cos\omega\tau - i\sin\omega\tau) + u_4(\cos 2\omega\tau - i\sin 2\omega\tau) = 0,
\end{aligned}
\tag{6}
$$

where

$$
u_i = Re\{p_i(i\omega)\},\ v_i = Im\{p_i(i\omega)\},\ (i = 1, 2, 3, 4). \tag{7}
$$

Separating the real and imaginary parts, we have

$$
(u_1 + u_3)\cos\omega\tau + (v_3 - v_1)\sin\omega\tau + u_2 = -u_4\cos 2\omega\tau, \tag{8}
$$

$$
(v_1 + v_3)\cos\omega\tau + (u_1 - u_3)\sin\omega\tau + v_2 = u_4\sin 2\omega\tau. \tag{9}
$$

Squaring both sides of (8) and (9), and adding them up gives

$$
\begin{aligned}
&[(u_1 + u_3)\cos\omega\tau + (v_3 - v_1)\sin\omega\tau + u_2]^2\\
&+ [(v_1 + v_3)\cos\omega\tau + (u_1 - u_3)\sin\omega\tau + v_2]^2 = u_4^2.
\end{aligned}
\tag{10}
$$

According to $\sin\omega\tau = \pm\sqrt{1 - \cos^2\omega\tau}$, we consider the two cases:
(I) If $\sin\omega\tau = \sqrt{1 - \cos^2\omega\tau}$, then (10) takes the following form:

$$
\begin{aligned}
&\left[(u_1 + u_3)\cos\omega\tau + (v_3 - v_1)\sqrt{1 - \cos^2\omega\tau} + u_2\right]^2\\
&+ \left[(v_1 + v_3)\cos\omega\tau + (u_1 - u_3)\sqrt{1 - \cos^2\omega\tau} + v_2\right]^2 = u_4^2.
\end{aligned}
\tag{11}
$$

It is easy to see that (11) is equivalent to

$$
q_1\cos^4\omega\tau + q_2\cos^3\omega\tau + q_3\cos^2\omega\tau + q_4\cos\omega\tau + q_5 = 0, \tag{12}
$$

where

$$
\begin{aligned}
q_1 &= [2(u_1+u_3)(v_3-v_1)+2(v_1+v_3)(u_1-u_3)]^2+[4(u_1u_3+v_1v_3)]^2,\\
q_2 &= 2\{[2(u_1+u_3)(v_3-v_1)+2(v_1+v_3)(u_1-u_3)][2u_2(v_3-v_1)\\
&\quad +2v_2(u_1-u_3)]+[2u_2(v_3-v_1)+2v_2(u_1-u_3)][4(u_1u_3+v_1v_3)]\},\\
q_3 &= [2u_2(u_1+u_3)+2v_2(v_1+v_3)]^2+[2u_2(v_3-v_1)+2v_2(u_1-u_3)]^2\\
&\quad -[2(u_1+u_3)(v_3-v_1)+2(v_1+v_3)(u_1-u_3)]^2\\
&\quad -2[u_4^2+v_4^2-u_2^2-v_2^2-(v_3-v_1)^2-(u_1-u_3)^2][4(u_1u_3+v_1v_3)],\\
q_4 &= 2\{[2(u_1+u_3)(v_3-v_1)+2(v_1+v_3)(u_1-u_3)][2u_2(v_3-v_1)\\
&\quad +2B_2(u_1-u_3)]+[u_4^2+v_4^2-u_2^2-v_2^2-(v_3-v_1)^2-(u_1-u_3)^2]\\
&\quad \times[2u_2(v_3-v_1)+2v_2(u_1-u_3)]\},\\
q_5 &= [u_4^2+v_4^2-u_2^2-v_2^2-(v_3-v_1)^2-(u_1-u_3)^2]^2\\
&\quad -[2u_2(v_3-v_1)+2v_2(u_1-u_3)]^2.
\end{aligned}
$$

By (12), we can obtain the expression of $\cos\omega\tau$, say

$$\cos\omega\tau = f_1(\omega), \tag{13}$$

where $f_1(\omega)$ is a function with respect to ω. Substitute (13) into (11), then we can easily get the expression of $\sin\omega\tau$, say

$$\sin\omega\tau = f_2(\omega), \tag{14}$$

where $f_2(\omega)$ is a function with respect to ω. Thus we obtain

$${f_1}^2(\omega)+{f_2}^2(\omega)=1. \tag{15}$$

If the coefficients of the system (2) are given, it is easy to use computer to calculate the roots of (14) (say ω). Then from (15), we derive

$$\tau_1^{(k)} = \frac{1}{\omega}[\arccos f_1(\omega)+2k\pi]\ (k=0,1,2,...). \tag{16}$$

(II) If $\sin\omega\tau = -\sqrt{1-\cos^2\omega\tau}$, then (10) takes the following form:

$$
\begin{aligned}
&\Big[(u_1+u_3)\cos\omega\tau-(v_3-v_1)\sqrt{1-\cos^2\omega\tau}+u_2\Big]^2\\
&+\Big[(v_1+v_3)\cos\omega\tau-(u_1-u_3)\sqrt{1-\cos^2\omega\tau}+v_2\Big]^2=u_4^2.
\end{aligned} \tag{17}
$$

It is easy to see that (17) is equivalent to

$$q_1^*\cos^4\omega\tau+q_2^*\cos^3\omega\tau+q_3^*\cos^2\omega\tau+q_4^*\cos\omega\tau+q_5^*=0, \tag{18}$$

where

$$\begin{aligned}
q_1^* &= [2(u_1+u_3)(v_3-v_1)+2(v_1+v_3)(u_1-u_3)]^2+[4(u_1u_3+v_1v_3)]^2,\\
q_2^* &= 2\{[-2(u_1+u_3)(v_3-v_1)-2(v_1+v_3)(u_1-u_3)][2u_2(v_3-v_1)\\
&\quad +2v_2(u_1-u_3)]+[2u_2(v_3-v_1)+2v_2(u_1-u_3)][4(u_1u_3+v_1v_3)]\},\\
q_3^* &= [2u_2(u_1+u_3)+2v_2(v_1+v_3)]^2+[2u_2(v_3-v_1)+2v_2(u_1-u_3)]^2\\
&\quad -[2(u_1+u_3)(v_3-v_1)+2(v_1+v_3)(u_1-u_3)]^2\\
&\quad -2[u_4^2+v_4^2-u_2^2-v_2^2-(v_3-v_1)^2-(u_1-u_3)^2][4(u_1u_3+v_1v_3)],\\
q_4^* &= 2\{-[2(u_1+u_3)(v_3-v_1)+2(v_1+v_3)(u_1-u_3)][2u_2(v_3-v_1)\\
&\quad +2v_2(u_1-u_3)]+[u_4^2+v_4^2-u_2^2-v_2^2-(v_3-v_1)^2-(u_1-u_3)^2]\\
&\quad \times[2u_2(v_3-v_1)+2v_2(u_1-u_3)]\},\\
q_5^* &= [u_4^2+v_4^2-u_2^2-v_2^2-(v_3-v_1)^2-(u_1-u_3)^2]^2\\
&\quad -[2u_2(v_3-v_1)+2v_2(u_1-u_3)]^2.
\end{aligned}$$

Similar process as case (I), we can easily obtain

$$\tau_2^{(k)}=\frac{1}{\omega^*}[\arccos f_1^*(\omega)+2k\pi]\ (k=0,1,2,...). \tag{19}$$

where $f_1^*(\omega)$ is a function with respect to ω and $f_1^*(\omega)=\cos\omega\tau$, ω^* ie the roots of the following equation

$$f_1^{*2}(\omega)+f_2^{*2}(\omega)=1, \tag{20}$$

where

$$f_2^*(\omega)=\sin\omega\tau. \tag{21}$$

We assume that (15) and (20) has at least one positive real root. Define $\tau_0=\min\{\tau_1^{(k)},\tau_2^{(k)}\},(k=0,1,2,...)$ where $\tau_1^{(k)}$ and $\tau_2^{(k)}$ is defined by (16) and (19), respectively. Note that when $\tau=0$, (5) becomes

$$\lambda^3+d_1\lambda^2+d_2\lambda+d_3=0, \tag{22}$$

where

$$\begin{aligned}
d_1 &= -(p_1+m_1n_1+m_2+n_2=p_4),\\
d_2 &= m_1n_1+m_1p_1+n_1p_1+m_1p_4+n_1p_4+2m_2n_1+n_1n_2+m_1n_2\\
&\quad +m_2n_2-m_3p_2-n_3p_3+m_2p_4+n_4p_4+m_2n_2-m_3p_2-n_3p_3,\\
d_3 &= -m_1n_1p_1-m_1n_1p_4+m_1n_1n_2-m_2n_1^2+m_3n_1p_2+n_1n_3p_3\\
&\quad -m_2n_2p_1-m_1n_4p_4-m-2n_1p_4+m_3n_2p_4+m_2n_3p_3-m_2n_2p_4.
\end{aligned}$$

A set of necessary and sufficient conditions that all roots of (22) have a negative real part is given by the well-known Routh-Hurwitz criteria in the following form:

$$D_1=d_1>0, \tag{23}$$

$$D_2 = \det \begin{pmatrix} d_1 & 1 \\ d_3 & d_2 \end{pmatrix} > 0, \tag{24}$$

$$D_3 = \det \begin{pmatrix} d_1 & 1 & 0 \\ d_3 & d_2 & d_1 \\ 0 & 0 & d_3 \end{pmatrix} > 0. \tag{25}$$

In order to obtain the main results in this paper, it is necessary to make the following assumptions:
(H4) If (23)-(25) hold, (22) have three roots with negative real parts when $\tau = 0$, (3) is stable near the equilibrium.
(H5) $Re\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0} \neq 0$.

Taking the derivative of λ with respect to τ in (5), it is easy to obtain:

$$\frac{d\lambda(\tau)}{d\tau} = \frac{P}{Q}, \tag{26}$$

where

$$\begin{aligned} P = & -\lambda e^{\lambda\tau} + \lambda[\lambda(m_2p_4 + n_4p_4 + m_2n_2 - m_3p_2 - n_3p_3) \\ & + m_3n_1p_2 + n_1n_3p_3 - m_2n_2p_1 - m_1n_4p_4 - m_2n_1p_4]e^{-\lambda\tau} \\ & + 2\lambda e^{-2\lambda\tau}(m_3n_2p_4 + m_2n_3p_3 - m_2n_2p_4), \end{aligned} \tag{27}$$

$$\begin{aligned} Q = & [3\lambda - 2(m_1 + n_1 + p_1)\lambda + m_1n_1 + m_1p_1 + n_1p_1]e^{-\lambda\tau} + \tau e^{\lambda\tau} \\ & - 2\lambda(m_2 + n_2 + p_4) + m_1p_4 + n_1p_4 + 2m_2n_1 + n_1n_2 \\ & + m_1n_2 + (m_2p_4 + n_4p_4 + m_2n_2 - m_3p_2 - n_3p_3)e^{-\lambda\tau} \\ & - \tau e^{-\lambda\tau}[(m_2p_4 + n_4p_4 + m_2n_2 - m_3p_2 - n_3p_3)\lambda \\ & + m_3n_1p_2 + n_1n_3p_3 - m_2n_2p_1 - m_1n_4p_4 - m_2n_1p_4] \\ & - 2\tau(m_3n_2p_4 + m_2n_3p_3 - m_2n_2p_4)\times e^{-\lambda\tau}. \end{aligned} \tag{28}$$

For the sake of simplicity, denote ω_0 and τ_0 by ω, τ, respectively, then

$$Re\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0} = \frac{P_1Q_1 + P_2Q_2}{Q_1^2 + Q_2^2}, \tag{29}$$

where

$$\begin{aligned} P_1 = & \,\omega \sin\omega\tau - \omega^2 \cos\omega\tau(m_2p_4 + n_4p_4 + m_2n_2 - m_3p_2 - n_3p_3) \\ & + \omega \sin\omega\tau(m_3n_1p_2 + n_1n_3p_3 - m_2n_2p_1 - m_1n_4p_4 - m - 2n - 1p_4) \\ & + 2\omega \sin\omega\tau(m_3n_2p_4 + m_2n_3p_3 - m_2n_2p_4), \\ P_2 = & -\omega \cos\omega\tau + \omega^2 \sin\omega\tau(m_2p_4 + n_4p_4 + m_2n_2 - m_3p_2 - n_3p_3) \\ & + \omega \cos\omega\tau(m_3n_1p_2 + n_1n_3p_3 - m_2n_2p_1 - m_1n_4p_4 - m_2n_1p_4) \\ & + 2\omega \cos\omega\tau(m_3n_2p_4 + m_2n_3p_3 - m_2n_2p_4), \end{aligned}$$

$$\begin{aligned}Q_1 = {} & (m_1n_1 + m_1p_1 + n_1p_1 - 3\omega^2)\cos\omega\tau - 2\omega\sin\omega\tau(p_1 + m_1 + n_1) \\ & + \tau\cos\omega\tau + m_1p_4 + n_1p_4 + 2m_2n_1 + n_1n_2 + m_1n_2 \\ & + (m_2p_4 + n_4p_4 + m_2n_2 - m_3p_2 - n_3p_3)\cos\omega\tau - \tau\cos\omega\tau(m_3n_1p_2 \\ & + n_1n_3p_3 - m_2n_2p_1 - m_1n_4p_4 - m_2n_1p_4) - \tau\omega\sin\omega\tau(m_2p_4 + n_4p_4 \\ & + m_2n_2 - m_3p_2 - n_3p_3), \\ Q_2 = {} & -\sin\omega\tau(m_1n_1 + m_1p_1 + n_1p_1 - 3\omega^2) - 2\omega\cos\omega\tau(p_1 + m_1 + n_1) \\ & + \tau\sin\omega\tau - 2\omega(m_1 + n_2 + p_4) - \sin\omega\tau(m_2p_4 + n_4p_4 + m - 2n_2 - m_3p_2 \\ & - n_3p_3) - \tau\cos\omega\tau(m_2p_4 + n_4p_4 + m - 2n_2 - m_3p_2 - n_3p_3) + \tau\sin\omega\tau \\ & \times (m_3n_1p_2 + n_1n_3p_3 - m_2n_2p_1 - n_1n_4p - 4 - m_2n_1p_4).\end{aligned}$$

In order to investigate the distribution of roots of the transcendental equation (5), the following Lemma that is stated in [1] is useful.

Lemma 2.1. [1] *For the transcendental equation*

$$\begin{aligned}P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) = {} & \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ & + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots \\ & + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0,\end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ *vary, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

From Lemma 2.1, it is easy to obtain the following results:

Theorem 2.2. *If (H1)-(H5) hold, then*
(I) For system (2), its zero solution is asymptotically stable for $\tau \in [0, \tau_0)$;
(II) system (2) undergoes a Hopf bifurcation at the origin when $\tau = \tau_0$, *i.e, system (2) has a branch of periodic solutions bifurcating from the zero solution near* $\tau = \tau_0$.

3. Direction and stability of the Hopf bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_0$. In this section, we shall derived the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $E_0(x_1^*, x_2^*, x_3^*)$ at these critical value of τ, by using techniques from normal form and center manifold theory [8], Throughout this section, we always assume that system (2) undergoes Hopf bifurcation at the equilibrium $E_0(x_1^*, x_2^*, x_3^*)$ for $\tau = \tau_0$, and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation at the equilibrium $E_0(x_1^*, x_2^*, x_3^*)$.

For convenience, Let $t = s\tau, \bar{x}_i(t) = x_i(\tau t),\ (i = 1, 2, 3), \tau = \tau_0 + \mu, \mu \in R$ and drop the bars for simplification of notations. Then system (2) becomes

$$\begin{cases} \dot{x}_1(t) = (\tau_0+\mu)x_1(t)\{[r_1 - a_{11}x_1(t-1) - a_{13}x_3(t-1)]\}, \\ \dot{x}_2(t) = (\tau_0+\mu)x_2(t)\{[r_2 - a_{22}x_2(t-1) - a_{23}x_3(t-1)]\}, \\ \dot{x}_3(t) = (\tau_0+\mu)x_3(t)\{[-r_3 - a_{31}x_1(t-1) + a_{32}x_2(t-1) \\ \qquad\qquad - a_{33}x_3(t-1)]\}. \end{cases} \tag{30}$$

Its linear part is given by

$$\begin{cases} \dot{x}_1(t) = (\tau_0+\mu)[m_1x_1(t) + m_2x_1(t-\tau) + m_3x_3(t-\tau)], \\ \dot{x}_2(t) = (\tau_0+\mu)[n_1x_2(t) + n_2x_2(t-\tau) + n_3x_3(t-\tau)], \\ \dot{x}_3(t) = (\tau_0+\mu)[p_1x_3(t) + p_2x_1(t-\tau) + p_3x_2(t-\tau) \\ \qquad\qquad + p_4x_3(t-\tau)]. \end{cases} \tag{31}$$

Its non-linear part is given by

$$f(\mu,u_t) = (\tau_0+\mu)\begin{pmatrix} -a_{11}x_1(t)x_1(t-1) - a_{13}x_2(t)x_3(t-1) \\ -a_{22}x_2(t)x_2(t-1) - a_{23}x_2(t)x_3(t-1) \\ a_{31}x_3(t)x_1(t-1) + a_{32}x_3(t)x_2(t-1) \\ -a_{33}x_3(t)x_3(t-1) \end{pmatrix}. \tag{32}$$

Denote

$$C^k[-1,0] = \{\varphi|\varphi : [-1,0] \to R^3, \text{each component of } \varphi \\ \text{has } k \text{ order continuous derivative}\}.$$

For convenience, denote $C[-1,0]$ by $C^0[-1,0]$.
For $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta))^T \in C([-1,0], R^3)$, define a family of operators

$$\begin{aligned} L_\mu\varphi =& (\tau_0+\mu)\begin{pmatrix} m_1 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} \\ &+ (\tau_0+\mu)\begin{pmatrix} m_2 & 0 & m_3 \\ 0 & n_2 & n_3 \\ p_2 & p_3 & p_4 \end{pmatrix}\begin{pmatrix} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \end{pmatrix}, \end{aligned} \tag{33}$$

where L_μ is a one-parameter family of bounded linear operators in $C([-1,0], R^3) \to R^3$. By the Riesz representation theorem, there exists a matric whose components are bounded variation functions $\eta(\theta,\mu)$ in $[-1,0] \to R^{3^2}$, such that

$$L_\mu\varphi = \int_{-1}^{0} d\eta(\theta,\mu)\varphi(\theta). \tag{34}$$

In fact, choosing

$$\begin{aligned} \eta(\theta,\mu) =& (\tau_0+\mu)\begin{pmatrix} m_1 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix}\delta(\theta) \\ &- (\tau_0+\mu)\begin{pmatrix} m_2 & 0 & m_3 \\ 0 & n_2 & n_3 \\ p_2 & p_3 & p_4 \end{pmatrix}\begin{pmatrix} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \end{pmatrix}\delta(\theta+1). \end{aligned} \tag{35}$$

where $\delta(\theta)$ is Dirac function, then (34) is satisfied. For $(\varphi_1, \varphi_2, \varphi_3) \in (C^1[-1,0], R^3)$, define

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(s,\mu)\varphi(s), & \theta = 0 \end{cases} \tag{36}$$

and

$$R\varphi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\mu, \varphi), & \theta = 0. \end{cases} \tag{37}$$

Then (30) is equivalent to the abstract differential equation

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{38}$$

where $x = (x_1, x_2, x_3)^T,\ x_t(\theta) = x(t+\theta), \theta \in [-1,0]$.

For $\psi \in C([0,1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{39}$$

For $\phi \in C([-1,0], R^3)$ and $\psi \in C([0,1], (R^3)^*)$, define the bilinear form

$$< \psi, \phi >= \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{40}$$

where $\eta(\theta) = \eta(\theta, 0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3.1. *$A = A(0)$ and A^* are adjoint operators.*

Proof. Let $\phi \in C^1([-1,0], R^3)$ and $\psi \in C^1([0,1], (R^3)^*)$. It follows from (40) and the definitions of $A = A(0)$ and A^* that

$$\begin{aligned} A(0)\phi(\theta) > &= \bar{\psi}(0)A(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\ &= \bar{\psi}(0)\int_{-1}^{0} d\eta(\theta)\phi(\theta) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\ &= \bar{\psi}(0)\int_{-1}^{0} d\eta(\theta)\phi(\theta) - \int_{-1}^{0} [\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)]_{\xi=0}^{\theta} \\ &\quad + \int_{-1}^{0}\int_{\xi=0}^{\theta} \frac{d\bar{\psi}(\xi-\theta)}{d\xi} d\eta(\theta)\phi(\xi)d\xi \\ &= \int_{-1}^{0} \bar{\psi}(-\theta)d\eta(\theta)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \left[-\frac{d\bar{\psi}(\xi-\theta)}{d\xi}\right] d\eta(\theta)\phi(\xi)d\xi \\ &= A^*\bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} A^*\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \\ &=< A^*\psi(s), \phi(\theta) > . \end{aligned}$$

This shows that $A = A(0)$ and A^* are adjoint operators and the proof is complete.

By the discussions in the Section 2, we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. We have the following result.

Lemma 3.2. *The vector*

$$q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\tau_0\theta},\ \theta \in [-1, 0],$$

is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_0$, and

$$q^*(s) = D(1, a_1^*, a_2^*)e^{i\omega_0\tau_0 s},\ s \in [0, 1],$$

is the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_0$, moreover, $< q^*(s), q(\theta) >= 1$, where

$$\bar{D} = 1 + \sum_{i=1}^{2} \bar{a}_i a_i^* + \tau_0 \left[(m_2 + a_2^* p_2)e^{i\omega_0\tau_0} + a_1(a_1^* n_2 + a_2^* p_2)e^{i\omega_0\tau_0} \right.$$
$$\left. + a_3(m_3 + n_3 a_1^* + p_4 a_2^*)e^{i\omega_0\tau_0}\right].$$

and a_1, a_2 and a_1^*, a_2^* are defined by (43) and (46), respectively.

Proof. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_0$, namely, $A(0)q(\theta) = i\omega_0\tau_0 q(\theta)$ and $A^*q(s) = -i\omega_0\tau_0 q^*(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q(s) = -dq^*(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$ and $q^*(s) = q(0)e^{i\omega_0\tau_0 s}$. In addition,

$$\int_{-1}^{0} d\eta(\theta)q(\theta) = \tau_0 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix} q(0) + \tau_0 \begin{pmatrix} m_2 & 0 & m_3 \\ 0 & n_2 & n_3 \\ p_2 & p_3 & p_4 \end{pmatrix} q(-1)$$
$$= A(0)q(0) = i\omega_0\tau_0 q(0). \tag{41}$$

That is

$$\begin{pmatrix} m_1 + m_2 e^{-i\omega_0\tau_0} + m_3 a_2 e^{-i\omega_0\tau_0} \\ n_1 a_1 + n_2 a_1 e^{-i\omega_0\tau_0} + n_3 a_2 e^{-i\omega_0\tau_0} \\ p_1 a_2 + p_2 e^{-i\omega_0\tau_0} + p_3 a_1 e^{-i\omega_0\tau_0} + p_4 a_2 e^{-i\omega_0\tau_0} \end{pmatrix} = \begin{pmatrix} i\omega_0 \\ ia_1\omega_0 \\ ia_2\omega_0 \end{pmatrix} \tag{42}$$

Therefore, we can easily obtain

$$a_1 = \frac{n_3(m_1 + m_2 e^{-i\omega_0} - i\omega_0)}{m_3(n_1 - i\omega_0 + n_2 e^{-i\omega_0})},\ a_2 = \frac{(i\omega_0 - m_1)e^{i\omega_0} - m_2}{m_3}. \tag{43}$$

On the other hand,

$$\int_{-1}^{0} q^*(-t)d\eta(t) = \tau_0 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}^T q^*(0) + \tau_0 \begin{pmatrix} m_2 & 0 & m_3 \\ 0 & n_2 & n_3 \\ p_2 & p_3 & p_4 \end{pmatrix}^T q^*(-1)$$
$$= A^*q^*(0) = -i\omega_0\tau_0 q^*(0). \tag{44}$$

Namely,

$$\begin{pmatrix} m_1+m_2e^{-i\omega_0\tau_0}+p_2a_2^*e^{-i\omega_0\tau_0} \\ n_1a_1^*+n_2a_1^*e^{-i\omega_0\tau_0}+p_3a_2^*e^{-i\omega_0\tau_0} \\ p_1a_2^*+m_3e^{-i\omega_0\tau_0}+n_3a_1^*e^{-i\omega_0\tau_0}+p_4a_2^*e^{-i\omega_0\tau_0} \end{pmatrix} = \begin{pmatrix} -i\omega_0 \\ -ia_1^*\omega_0 \\ -ia_2^*\omega_0 \end{pmatrix}. \tag{45}$$

Therefore, we can easily obtain

$$a_1^* = \frac{p_3(i\omega_0+m_1+m_2e^{-i\omega_0\tau_0})}{p_2(n_1+n_2e^{-i\omega_0\tau_0}+i\omega_0)},\ a_2^* = -\frac{(-i\omega_0+m_1)e^{-i\omega_0\tau_0}+m_2}{p_2} \tag{46}$$

In the sequel, we shall verify that $< q^*(s), q(\theta) >= 1$. In fact, from (40), we have $< q^*(s), q(\theta) >$

$$\begin{aligned} < q^*(s), q(\theta) > &= \bar{D}(1, \bar{a_1^*}, \bar{a_2^*})(1, a_1, a_2)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{a_1^*}, \bar{a_2^*})e^{-i\omega_0\tau_0(\xi-\theta)}d\eta(\theta)(1, a_1, a_2)^T e^{i\omega_0\tau_0\xi}d\xi \\ &= \bar{D}\left[1+\sum_{i=1}^{2}a_i\bar{a_i^*} - \int_{-1}^{0}(1, \bar{a_1^*}, \bar{a_2^*})\theta e^{i\omega_0\tau_0\theta}d\eta(\theta)(1, a_1, a_2)^T\right] \\ &= \bar{D}\left\{1+\sum_{i=1}^{2}a_i\bar{a_i^*} + (1, \bar{a_1^*}, \bar{a_2^*})\left[\tau_0 G e^{-i\omega_0\tau_0}\right](1, a_1, a_2)^T\right\} \\ &= \bar{D}\Bigg\{1+\sum_{i=1}^{2}a_i\bar{a_i^*} + \tau_0\Big[(m_2+\bar{a_2^*}p_2)e^{-i\omega_0\tau_0} + a_1(\bar{a_1^*}n_2+\bar{a_2^*}p_2)e^{-i\omega_0\tau_0} \\ &\quad + a_3(m_3+n_3\bar{a_1^*}+p_4\bar{a_2^*})e^{-i\omega_0\tau_0}\Big]\Bigg\} = 1. \end{aligned}$$

where

$$G = \begin{pmatrix} m_2 & 0 & m_3 \\ 0 & n_2 & n_3 \\ p_2 & p_3 & p_4 \end{pmatrix}. \tag{47}$$

Next, we use the same notations as those in Hassard, Kazarinoff and Wan [8], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq. (30) when $\mu = 0$.

Define

$$z(t) =< q^*, x_t >,\ W(t,\theta) = x_t(\theta) - 2Re\{z(t)q(\theta)\} \tag{48}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{49}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{50}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (30),

$$\begin{aligned}\dot{z}(t) &=< q^*(s), \dot{x}_t >=< q^*(s), A(0)u_t + R(0)x_t > \\ &=< q^*(s), A(0)x_t > + < q^*(s), R(0)x_t > \\ &=< A^* q^*(s), x_t > + \bar{q^*}(0)R(0)x_t \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q^*}(\xi-\theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\ &=< i\omega_0\tau_0 q^*(s), x_t > + \bar{q^*}(0)f(0, x_t(\theta) \\ &\stackrel{\text{def}}{=} i\omega_0\tau_0 z(t) + \bar{q^*}(0)f_0(z(t), \bar{z}(t)).\end{aligned} \tag{51}$$

That is

$$\dot{z}(t) = i\omega_0\tau_0 z + g(z, \bar{z}), \tag{52}$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots . \tag{53}$$

Hence, we have

$$\begin{aligned}g(z, \bar{z}) &= \bar{q}^*(0)f_0(z, \bar{z}) \\ &= f(0, x_t) \\ &= \tau_0\bar{D}(1, \bar{a}_1^*, \bar{a}_2^*) \times (f_1(0, x_t), f_2(0, x_t), f_3(0, x_t))^T,\end{aligned} \tag{54}$$

where

$$\begin{aligned}f_1(0, x_t) &= -a_{11}x_{1t}(0)x_{1t}(-1) - a_{13}x_{2t}(0)x_{3t}(-1), \\ f_2(0, x_t) &= -a_{22}x_{2t}(0)x_{2t}(-1) - a_{23}x_{2t}(0)x_{3t}(-1), \\ f_3(0, x_t) &= a_{31}x_{3t}(0)x_{1t}(-1) + a_{32}x_{3t}(0)x_{2t}(-1) - a_{33}x_{3t}(0)x_{3t}(-1).\end{aligned}$$

Noticing that

$$x_t(\theta) = (x_{1t}(\theta),\ x_{2t}(\theta),\ x_{3t}(\theta))^T = W(t, \theta) + zq(\theta) + \bar{z}\bar{q}$$

and

$$q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\tau_0\theta},$$

we have

$$\begin{aligned}x_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots , \\ x_{2t}(0) &= a_1 z + \bar{a}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots , \\ x_{3t}(0) &= a_2 z + \bar{a}_2\bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + \cdots ,\end{aligned}$$

$$x_{1t}(-1) = e^{-i\omega_0\tau_0}z + e^{i\omega_0\tau_0}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots,$$
$$x_{2t}(-1) = a_1e^{-i\omega_0\tau_0}z + \bar{a}_1e^{i\omega_0\tau_0}\bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots,$$
$$x_{3t}(-1) = a_2e^{-i\omega_0\tau_0}z + \bar{a}_2e^{i\omega_0\tau_0}\bar{z} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + \cdots.$$

From (53) and (54), we can obtain

$$\begin{aligned}
g(z,\bar{z}) = {} & \tau_0\bar{D}\Big[-a_{11}e^{-i\omega_0\tau_0} - a_{13}a_1a_2e^{-i\omega_0\tau_0} + \bar{a_1^*}(-a_{22}a_1^2e^{-i\omega_0\tau_0} - a_{23}a_1a_2e^{-i\omega_0\tau_0}) \\
& +\bar{a_2^*}(a_{31}a_2e^{-i\omega_0\tau_0} + a_{32}a_1a_2e^{-i\omega_0\tau_0} - a_{33}a_2^2e^{-i\omega_0\tau_0})\Big]z^2 \\
& + \tau_0\bar{D}\Big\{-2a_{11}Re\{e^{i\omega_0\tau_0}\} - 2a_{13}Re\{\bar{a_1}a_2e^{-i\omega_0\tau_0}\} \\
& + \bar{a_1^*}\Big[2a_{22}|a_1|^2Re\{e^{i\omega_0\tau_0}\} - 2a_{23}Re\{\bar{a_1}a_2e^{-i\omega_0\tau_0}\}\Big] \\
& + \bar{a_2^*}\Big[2a_{31}Re\{a_2e^{i\omega_0\tau_0}\} + 2a_{32}Re\{\bar{a_1}a_2e^{i\omega_0\tau_0}\} - 2a_{33}|a_2|^2Re\{e^{i\omega_0\tau_0}\}\Big]\Big\}z\bar{z} \\
& + \tau_0\bar{D}[(-a_{11} - a_{13}\bar{a_1}\bar{a_2})e^{i\omega_0\tau_0} + \bar{a_1^*}(-a_{22}\bar{a_1}^2 - a_{23}\bar{a_1}\bar{a_2})e^{i\omega_0\tau_0}) \\
& + \bar{a_2^*}(a_{31}\bar{a_2} + a_{32}\bar{a_1}\bar{a_2} - a_{33}\bar{a_2}^2)e^{i\omega_0\tau_0}]\bar{z}^2 \\
& + \tau_0\bar{D}\Big\{-a_{11}\Big[\frac{1}{2}W_{20}^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(0)e^{i\omega_0\tau_0} + W_{11}^{(1)}(-1) + W_{11}^{(1)}(0)e^{-i\omega_0\tau_0}\Big] \\
& - a_{13}\Big[\frac{1}{2}W_{20}^{(2)}(0)\bar{a_2}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(3)}(-1)\bar{a_1} + W_{11}^{(2)}(0)a_2e^{-i\omega_0\tau_0} + W_{11}^{(3)}(-1)a_1\Big] \\
& + \bar{a_1^*}\Big[-a_{22}\Big(\frac{1}{2}W_{20}^{(2)}(0)\bar{a_1}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(2)}(-1)\bar{a_1} + W_{11}^{(2)}(0)a_1e^{-i\omega_0\tau_0} \\
& +W_{11}^{(2)}(-1)a_1\Big) - a_{23}\Big(\frac{1}{2}W_{20}^{(2)}(0)\bar{a_2}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(3)}(0)\bar{a_1} + W_{11}^{(2)}(-1)a_2e^{-i\omega_0\tau_0} \\
& +W_{11}^{(3)}(-1)a_1\Big) + \bar{a_2^*}\Big[a_{31}\Big(\frac{1}{2}W_{20}^{(3)}(0)e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(1)}(-1)\bar{a_2} + W_{11}^{(1)}(-1)a_2 \\
& +W_{11}^{(3)}(0)e^{-i\omega_0\tau_0}\Big) + a_{32}\Big(\frac{1}{2}W_{20}^{(2)}(0)\bar{a_1}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(2)}(-1)\bar{a_2} + W_{11}^{(3)}(0)a_1e^{-i\omega_0\tau_0} \\
& +W_{11}^{(2)}(-1)a_2\Big) - a_{33}\Big(\frac{1}{2}W_{20}^{(3)}(0)\bar{a_2}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(3)}(-1)\bar{a_2} + W_{11}^{(3)}(0)a_2e^{-i\omega_0\tau_0} \\
& +W_{20}^{(3)}(-1)\bar{a_2}\Big)\Big]\Big\}z^2\bar{z} + \cdots,
\end{aligned}$$

then we get

$$\begin{aligned}
g_{20} = {} & 2\tau_0\bar{D}\Big[-a_{11}e^{-i\omega_0\tau_0} - a_{13}a_1a_2e^{-i\omega_0\tau_0} + \bar{a_1^*}(-a_{22}a_1^2e^{-i\omega_0\tau_0} - a_{23}a_1a_2e^{-i\omega_0\tau_0}) \\
& +\bar{a_2^*}(a_{31}a_2e^{-i\omega_0\tau_0} + a_{32}a_1a_2e^{-i\omega_0\tau_0} - a_{33}a_2^2e^{-i\omega_0\tau_0})\Big], \\
g_{11} = {} & 2\tau_0\bar{D}\Big\{-a_{11}Re\{e^{i\omega_0\tau_0}\} - a_{13}Re\{\bar{a_1}a_2e^{-i\omega_0\tau_0}\} \\
& + \bar{a_1^*}\Big[a_{22}|a_1|^2Re\{e^{i\omega_0\tau_0}\} - a_{23}Re\{\bar{a_1}a_2e^{-i\omega_0\tau_0}\}\Big] \\
& + \bar{a_2^*}\Big[a_{31}Re\{a_2e^{i\omega_0\tau_0}\} + a_{32}Re\{\bar{a_1}a_2e^{i\omega_0\tau_0}\} - a_{33}|a_2|^2Re\{e^{i\omega_0\tau_0}\}\Big]\Big\},
\end{aligned}$$

$$g_{02} = 2\tau_0 \bar{D}[(-a_{11} - a_{13}\bar{a_1}\bar{a_2})e^{i\omega_0\tau_0} + \bar{a_1^*}(-a_{22}\bar{a_1}^2 - a_{23}\bar{a_1}\bar{a_2})e^{i\omega_0\tau_0})$$
$$+ \bar{a_2^*}(a_{31}\bar{a_2} + a_{32}\bar{a_1}\bar{a_2} - a_{33}\bar{a_2}^2)e^{i\omega_0\tau_0}],$$
$$g_{21} = 2\tau_0\bar{D}\Bigg\{ - a_{11}\left[\frac{1}{2}W_{20}^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(0)e^{i\omega_0\tau_0} + W_{11}^{(1)}(-1) + W_{11}^{(1)}(0)e^{-i\omega_0\tau_0}\right]$$
$$- a_{13}\left[\frac{1}{2}W_{20}^{(2)}(0)\bar{a_2}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(3)}(-1)\bar{a_1} + W_{11}^{(2)}(0)a_2e^{-i\omega_0\tau_0} + W_{11}^{(3)}(-1)a_1\right]$$
$$+ \bar{a_1^*}\left[-a_{22}\left(\frac{1}{2}W_{20}^{(2)}(0)\bar{a_1}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(2)}(-1)\bar{a_1} + W_{11}^{(2)}(0)a_1e^{-i\omega_0\tau_0} + W_{11}^{(2)}(-1)a_1\right)\right.$$
$$- a_{23}\left(\frac{1}{2}W_{20}^{(2)}(0)\bar{a_2}e^{i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(3)}(0)\bar{a_1} + W_{11}^{(2)}(-1)a_2e^{-i\omega_0\tau_0} + W_{11}^{(3)}(-1)a_1\right)$$

For unknown

$$W_{20}^{(1)}(0), W_{11}^{(0)}(0), W_{20}^{(2)}(0), W_{20}^{(3)}(0), W_{11}^{(2)}(0), W_{11}^{(3)}(0),$$
$$W_{20}^{(1)}(-1), W_{11}^{(1)}(-1), W_{11}^{(3)}(-1), W_{20}^{(2)}(-1), W_{11}^{(2)}(-1), W_{20}^{(3)}(-1)$$

in g_{21}, we still need to compute them.
Form (38), (40), we have

$$W' = \begin{cases} AW - 2Re\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -1 \le \theta < 0, \\ AW - 2Re\{\bar{q}^*(0)\bar{f}q(\theta)\} + \bar{f}, & \theta = 0. \end{cases}$$
$$\stackrel{\text{def}}{=} AW + H(z, \bar{z}, \theta), \tag{55}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots . \tag{56}$$

Comparing the coefficients, we obtain

$$(A - 2i\tau_0\omega_0)W_{20} = -H_{20}(\theta), \tag{57}$$

$$AW_{11}(\theta) = -H_{11}(\theta), \ \cdots\cdots . \tag{58}$$

And we know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \tag{59}$$

Comparing the coefficients of (56) with (59) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{60}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{61}$$

From (57),(60) and the definition of A , we get

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \tag{62}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1e^{2i\omega_0\tau_0\theta}, \tag{63}$$

where E_1 is a constant vector.

Similarly, from (58), (61) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \tag{64}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_2. \tag{65}$$

where E_2 is a constant vector.

In what follows, we shall seek appropriate E_1, E_2 in (63), (65), respectively. It follows from the definition of A and (60), (61) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0W_{20}(0) - H_{20}(0) \tag{66}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{67}$$

where $\eta(\theta) = \eta(0,\theta)$.

From (57), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\tau_0(H_1, H_2, H_3)^T, \tag{68}$$

where

$$\begin{aligned}
H_1 &= -a_{11}e^{-i\omega_0\tau_0} - a_{13}a_1a_2e^{-i\omega_0\tau_0},\\
H_2 &= -a_{22}a_1^2e^{-i\omega_0\tau_0} - a_{23}a_1a_2e^{-i\omega_0\tau_0},\\
H_3 &= a_{31}a_2e^{-i\omega_0\tau_0} + a_{32}a_1a_2e^{-i\omega_0\tau_0} - a_{33}a_2^2e^{-i\omega_0\tau_0}.
\end{aligned}$$

From (58), we have

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + \tau_0(P_1, P_2, P_3)^T, \tag{69}$$

where

$$\begin{aligned}
P_1 &= -a_{11}Re\{e^{i\omega_0\tau_0}\} - a_{13}Re\{\bar{a_1}a_2e^{-i\omega_0\tau_0},\\
P_2 &= a_{22}|a_1|^2Re\{e^{i\omega_0\tau_0}\} - a_{23}Re\{\bar{a_1}a_2e^{-i\omega_0\tau_0},\\
P_3 &= a_{31}Re\{a_2e^{i\omega_0\tau_0}\} + a_{32}Re\{\bar{a_1}a_2e^{i\omega_0\tau_0}\} - a_{33}|a_2|^2Re\{e^{i\omega_0\tau_0}\}.
\end{aligned}$$

Noting that

$$\left(i\omega_0\tau_0I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\right)q(0) = 0, \tag{70}$$

$$\left(-i\omega_0\tau_0I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right)\bar{q}(0) = \tau_0(H_1, H_2, H_3)^T \tag{71}$$

and substituting (63) and (68) into (66), we have

$$\left(2i\omega_0\tau_0I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta}d\eta(\theta)\right)E_1 = \tau_0(H_1, H_2, H_3)^T. \tag{72}$$

That is

$$\begin{pmatrix} 2i\omega_0 - m_1 - m_2 e^{-2i\omega_0\tau_0\theta} & 0 & -m_3 e^{-2i\omega_0\tau_0\theta} \\ 0 & 2i\omega_0 - n_1 - n_2 e^{-2i\omega_0\tau_0\theta} & -n_3 e^{-2i\omega_0\tau_0\theta} \\ -p_2 e^{-2i\omega_0\tau_0\theta} & -p_3 e^{-2i\omega_0\tau_0\theta} & 2i\omega_0 - p_1 - p_4 e^{-2i\omega_0\tau_0\theta} \end{pmatrix} \times (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T = (H_1, H_2, H_3)^T. \tag{73}$$

Hence,

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1}, \tag{74}$$

where

$$\Delta_1 = \det\begin{pmatrix} 2i\omega_0 - m_1 - m_2 e^{-2i\omega_0\tau_0\theta} & 0 & -m_3 e^{-2i\omega_0\tau_0\theta} \\ 0 & 2i\omega_0 - n_1 - n_2 e^{-2i\omega_0\tau_0\theta} & -n_3 e^{-2i\omega_0\tau_0\theta} \\ -p_2 e^{-2i\omega_0\tau_0\theta} & -p_3 e^{-2i\omega_0\tau_0\theta} & 2i\omega_0 - p_1 - p_4 e^{-2i\omega_0\tau_0\theta} \end{pmatrix},$$

$$\Delta_{11} = \det\begin{pmatrix} H_1 & 0 & -m_3 e^{-2i\omega_0\tau_0\theta} \\ H_2 & 2i\omega_0 - n_1 - n_2 e^{-2i\omega_0\tau_0\theta} & -n_3 e^{-2i\omega_0\tau_0\theta} \\ H_3 & -p_3 e^{-2i\omega_0\tau_0\theta} & 2i\omega_0 - p_1 - p_4 e^{-2i\omega_0\tau_0\theta} \end{pmatrix},$$

$$\Delta_{12} = \det\begin{pmatrix} 2i\omega_0 - m_1 - m_2 e^{-2i\omega_0\tau_0\theta} & H_1 & -m_3 e^{-2i\omega_0\tau_0\theta} \\ 0 & H_2 & -n_3 e^{-2i\omega_0\tau_0\theta} \\ -p_2 e^{-2i\omega_0\tau_0\theta} & H_3 & 2i\omega_0 - p_1 - p_4 e^{-2i\omega_0\tau_0\theta} \end{pmatrix},$$

$$\Delta_{13} = \det\begin{pmatrix} 2i\omega_0 - m_1 - m_2 e^{-2i\omega_0\tau_0\theta} & 0 & H_1 \\ 0 & 2i\omega_0 - n_1 - n_2 e^{-2i\omega_0\tau_0\theta} & H_2 \\ -p_2 e^{-2i\omega_0\tau_0\theta} & -p_3 e^{-2i\omega_0\tau_0\theta} & H_3 \end{pmatrix}.$$

Similarly, substituting (64) and (69) into (67), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right) E_2 = (P_1, P_2, P_3)^T. \tag{75}$$

That is

$$\begin{pmatrix} m_1 + m_2 & 0 & m_3 \\ 0 & n_1 + n_2 & n_3 \\ p_2 & p_3 & p_1 + p_4 \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = \begin{pmatrix} -P_1 \\ -P_2 \\ -P_3 \end{pmatrix}. \tag{76}$$

Hence,

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2} \tag{77}$$

where

$$\Delta_2 = \det\begin{pmatrix} m_1 + m_2 & 0 & m_3 \\ 0 & n_1 + n_2 & n_3 \\ p_2 & p_3 & p_1 + p_4 \end{pmatrix},\ \Delta_{21} = \det\begin{pmatrix} -P_1 & 0 & m_3 \\ -P_2 & n_1 + n_2 & n_3 \\ -P_3 & p_3 & p_1 + p_4 \end{pmatrix},$$

$$\Delta_{22} = \det\begin{pmatrix} m_1 + m_2 & -P_1 & m_3 \\ 0 & -P_2 & n_3 \\ p_2 & -P_3 & p_1 + p_4 \end{pmatrix},\quad \Delta_{23} = \det\begin{pmatrix} m_1 + m_2 & 0 & -P_1 \\ 0 & n_1 + n_2 & -P_2 \\ p_2 & p_3 & -P_3 \end{pmatrix}.$$

From (63), (65), (74), (77), we can calculate g_{21} and derive the following values:

$$c_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{Re\{c_1(0)\}}{Re\{\lambda'(\tau_0)\}},$$
$$\beta_2 = 2Re(c_1(0)),$$
$$T_2 = -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0\tau_0}.$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (30) at $\tau = \tau_0$, on the center manifold. From the discussion above, we have the following result:

Theorem 3.3. *For system (2), if (H1)-(H5) hold, the periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); The periodic of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

Remark 3.4. *A $\tau_0 T$-periodic solution of (30) is a T-periodic solution of (2).*

4. Numerical examples

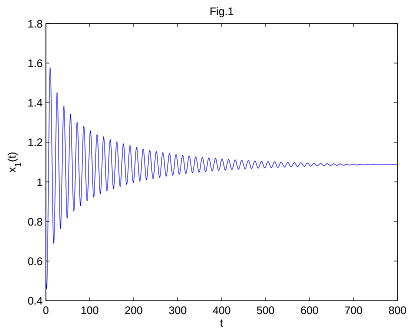
In this section, we present some numerical results of system (2) to verify the analytical predictions obtained in the previous section. From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following special case of system (2)

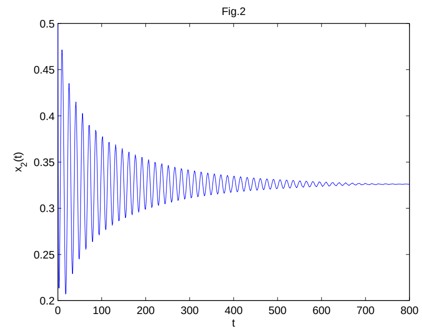
$$\begin{cases} \dot{x}_1(t) = x_1(t)[0.5 - 0.3x_1(t-\tau) - 0.8x_3(t-\tau)], \\ \dot{x}_2(t) = x_2(t)[0.5 - x_2(t-\tau) - 0.8x_3(t-\tau)], \\ \dot{x}_3(t) = x_3(t)[-0.5 + 0.5x_1(t-\tau) + 0.4x_2(t-\tau) - 0.8x_3(t-\tau)]. \end{cases} \tag{78}$$

It is easy to see that system (4.1) has a unique positive equilibrium $E_*(x_1^*, x_2^*, x_3^*) = (1.0870, 0.3261, 0.2174)$. By some complicated computation by means of Matlab 7.0, we get $\tau_0 \approx 1.53, \lambda'(\tau_0) \approx 2.0522 - 3.1345i$. Thus we can calculate the following values:

$$c_1(0) \approx -1.1220 - 11.0437i, \mu_2 \approx 0.7855, \beta_2 \approx -3.4533, T_2 \approx 12.1125.$$

we obtain the conditions indicated in Theorem 2.2 are satisfied. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$ is stable when $\tau < \tau_0$ as is illustrated by the computer simulations (see Fig.1-6). When τ passes through the critical value τ_0, the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $E_*(x_1^*, x_2^*, x_3^*)$ at τ_0 are stable, which are depicted in Fig.7-12.





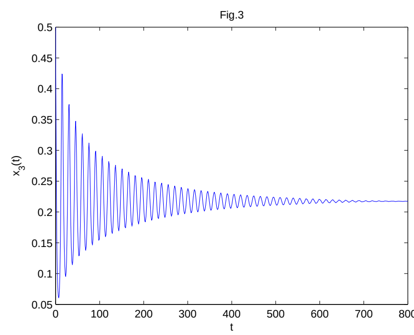
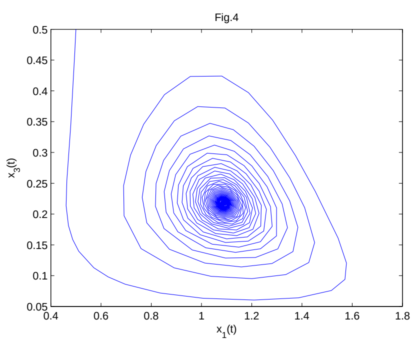


Fig.1-3 Dynamic behavior of system (4.1): times series of $x_i (i = 1, 2, 3)$. A Matlab simulation of the asymptotically stable positive equilibrium to system (4.1) with $\tau_1 = 1.5 < \tau_0 \approx 1.53$. The initial value is (0.5, 0.5, 0.5).



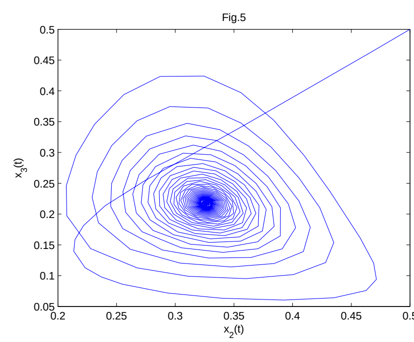


Fig.4-5 Dynamic behavior of system (4.1): projection on $x_1 - x_3, x_2 - x_3$ plane, respectively. A Matlab simulation of the asymptotically stable positive equilibrium to system (4.1) with $\tau = 1.5 < \tau_0 \approx 1.53$. The initial value is (0.5, 0.5, 0.5).

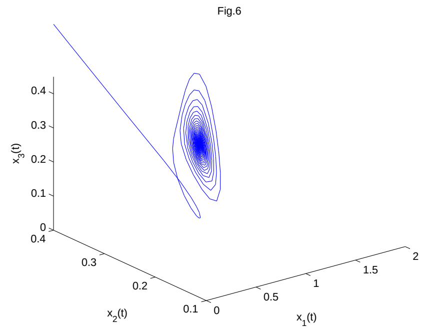
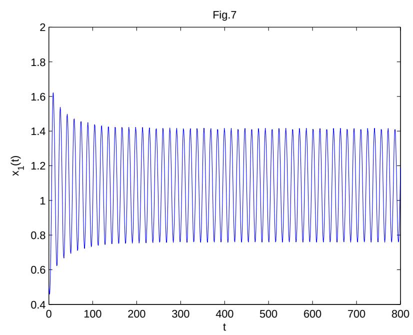
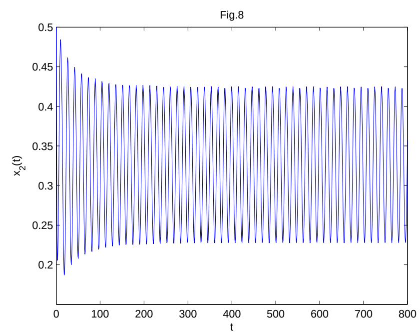


Fig.6 Dynamic behavior of system (4.1): projection on $x_1 - x_2 - x_3$ space. A Matlab simulation of the asymptotically stable positive equilibrium to system (4.1) with $\tau = 1.5 < \tau_0 \approx 1.53$. The initial value is (0.5, 0.5, 0.5).





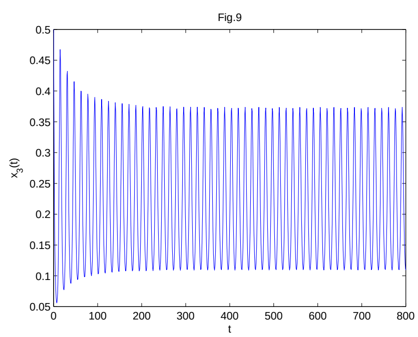
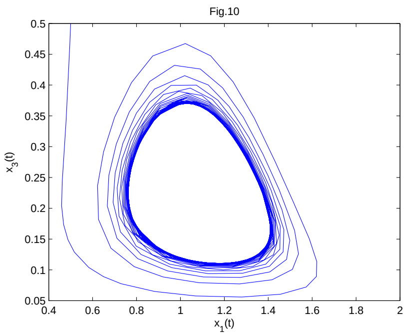


Fig.7-9 Dynamic behavior of system (4.1): times series of $x_i (i = 1, 2, 3)$. A Matlab simulation of the asymptotically stable positive equilibrium to system (4.1) with $\tau = 1.6 > \tau_0 \approx 1.53$. The initial value is (0.5, 0.5, 0.5).



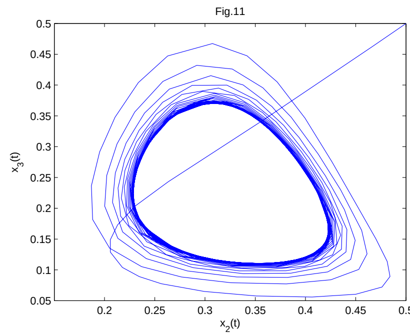


Fig.10-11 Dynamic behavior of system (4.1): projection on $x_1 - x_3, x_2 - x_3$ plane, respectively. A Matlab simulation of the asymptotically stable positive equilibrium to system (4.1) with $\tau = 1.6 > \tau_0 \approx 1.53$. The initial value is (0.5, 0.5, 0.5).

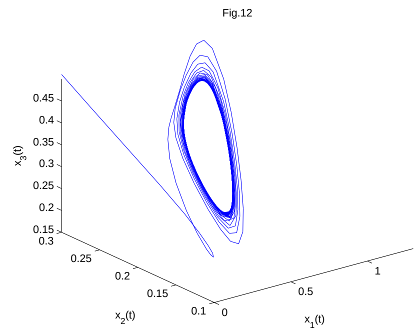


Fig.12 Dynamic behavior of system (4.1): projection on $x_1 - x_2 - x_3$ space, respectively. A Matlab simulation of the asymptotically stable positive equilibrium to system (4.1) with $\tau = 1.6 > \tau_0 \approx 1.53$. The initial value is (0.5, 0.5, 0.5).

5. Conclusions

In this paper, we have investigated local stability of the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$ and local Hopf bifurcation in a Lotka-Volterra model with time delays. We have showed that if the conditions $(H1) - (H5)$ hold, the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$ of system (1.2) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. We have also showed that, if the conditions $(H1) - (H5)$ hold, as the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur at $E_*(x_1^*, x_2^*, x_3^*)$, i.e., a family of periodic orbits bifurcates from the the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem.

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