International Journal of Fuzzy Logic and Intelligent Systems, vol. 11, no. 1, March 2011, pp. 38−43
DOI : 10.591/IJFIS.2011.11.1.038

On Fixed Point Theorem of Weak Compatible Maps of Type(γ) in Complete Intuitionistic Fuzzy Metric Space

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Abstract

In this paper, we give definitions of compatible mappings of type(γ) in intuitionistic fuzzy metric space and obtain common fixed point theorem under the conditions of weak compatible mappings of type(γ) in complete intuitionistic fuzzy metric space. Our research generalize, extend and improve the results given by Sedghi et.al.[12].

Key words : Common fixed point theorem, weak compatible maps of type(γ), complete intuitionistic fuzzy metric space.

1. Introduction

Zadeh[13] was introduced the concept of fuzzy sets. Fang[1], Kaleva and Seikkala[3], George and Veeramani[2] have introduced the concept of fuzzy metric space for each different methods, and some authors have been improved generalized and extended several properties in this space. Also, Kutukcu et.al.[4] obtained the common fixed points of compatible maps of tyle(β) on fuzzy metric spaces, and Sedghi et.al.[12] studied the common fixed point of compatible maps of type(γ) in complete fuzzy metric spaces.

Recently, Park[5] and Park et.al.[9] defined the intuitionistic fuzzy metric space. Many authors([7], [8], [9] etc) obtained a fixed point theorems in this space. Also, Park[6], Park et.al.[10] introduced the concept of compatible mappings of type(α) and type(β), and obtained common fixed point theorems in intuitionistic fuzzy metric space.

In this paper, we give definitions of compatible mappings of type(γ) in intuitionistic fuzzy metric space and obtain common fixed point theorem under the conditions of weak compatible mappings of type(γ) in complete intuitionistic fuzzy metric space.

2. Preliminaries

Throughout this paper, $\mathbf{N}$ denote the set of all positive integers. Now, we begin with some definitions, properties in intuitionistic fuzzy metric space as following:

Let us recall(see [11]) that a continuous $t-$norm is a operation $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a * 1 = a$ for all $a \in [0,1]$, (d)$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$. Also, a continuous $t-$conorm is a operation $\diamond : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (a)$\diamond$ is commutative and associative, (b)$\diamond$ is continuous, (c)$a \diamond 0 = a$ for all $a \in [0,1]$, (d)$a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$.

Definition 2.1. ([9])The $5-$tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a)$M(x, y, t) > 0$,
(b)$M(x, y, t) = 1 \iff x = y$,
(c)$M(x, y, t) = M(y, x, t)$,
(d)$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(e)$M(x, y, \cdot) : (0,\infty) \to (0, 1]$ is continuous,
(f)$N(x, y, t) > 0$,
(g)$N(x, y, t) = 0 \iff x = y$,
(h)$N(x, y, t) = N(y, x, t)$,
(i)$N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
(j)$N(x, y, \cdot) : (0,\infty) \to (0, 1]$ is continuous.

Note that (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Let X be an intuitionistic fuzzy metric space. For any $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and ra-

Manuscript received May. 19, 2010; revised Feb. 3, 2011;
Accepted Feb. 6, 2011.
*This paper is supported by the Chinju National University of Edu -cation Research Fund in 2010

dius $0 < r < 1$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1-r,\ \ N(x,y,t) < r\}$$

Let X be an intuitionistic fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Then τ is a topology on X(induced by the intuitionistic fuzzy metric (M,N)). A sequence $\{x_n\} \subset X$ converges to x if and only if $M(x_n,x,t) \to 1$, $N(x_n,x,t) \to 0$ as $n \to \infty$, for all $t > 0$. It is called a Cauchy sequence if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n,x_m,t) > 1-\epsilon$, $N(x_n,x_m,t) < \epsilon$ for any $m,n \geq n_0$. The intuitionistic fuzzy metric space X is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x,y,t) > 1-r$, $N(x,y,t) < r$ for all $x,y \in A$.

Lemma 2.2. ([7])Let X be an intuitionistic fuzzy metric space. Then $M(x,y,t)$ is nondecreasing, $N(x,y,t)$ is nonincreasing with respect to t for all $x,y \in X$

Definition 2.3. Let X be an intuitionistic fuzzy metric space. Then M,N are said to be continuous on $X^2 \times (0,\infty)$ if

$$\lim_{n\to\infty} M(x_n,y_n,t_n) = M(x,y,t),$$
$$\lim_{n\to\infty} N(x_n,y_n,t_n) = N(x,y,t)$$

whenever a sequence $\{(x_n,y_n,t_n)\} \subset X^2 \times (0,\infty)$ converges to a point $(x,y,t) \in X^2 \times (0,\infty)$.

Definition 2.4. Let A and B be mappings from an intuitionistic fuzzy metric space X into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Bx$ implies that $ABx = BAx$.

Definition 2.5. Let A and B be mappings from an intuitionistic fuzzy metric space X into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1,\ \ \forall t > 0$$
$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

Proposition 2.6. The self mappings A and B of an intuitionistic fuzzy metric space X are compatible, then they are weak compatible

The converse is not true as seen in following example:

Example 2.7. Let X be an intuitionistic fuzzy metric space, where $X = [0,2]$, $*,\diamond$ defined $a * b = \min\{a,b\}$, $a \diamond b = \max\{a,b\}$ for all $a,b \in [0,1]$ and $M(x,y,t) = \frac{t}{t+d(x,y)}$, $N(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$ for all $t > 0$ and $x,y \in X$. Define self maps A and B on X as follows:

$$Ax = \begin{cases} 2 \ \text{ if } \ 0 \leq x \leq 1 \\ \dfrac{x}{2} \ \text{ if } \ 1 < x \leq 2 \end{cases}$$
$$Bx = \begin{cases} 2 \ \text{ if } \ x = 1 \\ \dfrac{x+3}{5} \ \ \text{otherwise} \end{cases}$$

and $x_n = 2 - \frac{1}{2n}$. Then we have $B1 = 2 = A1$ and $B2 = 1 = A2$. Also, $BA2 = B1 = 2$, $AB2 = A1 = 2(BA2 = AB2 = 2)$. thus A and B are weak compatible. Also, since $Ax_n = \frac{1}{2}(2-\frac{1}{2n}) = 1 - \frac{1}{4n}$, $Bx_n = \frac{1}{5}(2 - \frac{1}{2n} + 3) = 1 - \frac{1}{10n}$. Thus $\lim_{n\to\infty} Ax_n = 1 = \lim_{n\to\infty} Bx_n$. Furthermore, $BAx_n = B(1-\frac{1}{4n}) = \frac{1}{5}(1-\frac{1}{4n}+3) = \frac{4}{5} - \frac{1}{20n}$, $ABx_n = A(1-\frac{1}{10n}) = 2$.

Now,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t)$$
$$= \lim_{n\to\infty} M(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{5t}{5t+6},$$
$$\lim_{n\to\infty} N(ABx_n, BAx_n, t)$$
$$= \lim_{n\to\infty} N(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{6}{5t+6}.$$

Hence A and B is not compatible

3. Weak compatible mappings of type(γ)

Definition 3.1. [7]Let A and B be mappings from an intuitionistic fuzzy metric space X into itself. Then the mappings A and B are said to be compatible maps of type(γ) if satisfying:

(i)A and B are compatible, that is,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0 \ \forall t > 0$$

whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

(ii)they are continuous at x.

On the other hand, we have

$$Ax = A(\lim_{n\to\infty} Ax_n) = A(\lim_{n\to\infty} Bx_n)$$
$$= \lim_{n\to\infty} BAx_n = B(\lim_{n\to\infty} Ax_n) = Bx$$

Definition 3.2. [7]Let A and B be mappings from an intuitionistic fuzzy metric space X into itself. The mappings A and B are said to be weak-compatible of type(γ) if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$ implies that $Ax = Bx$.

Remark 3.3. If self maps A and B of an intuitionistic fuzzy metric space X are compatible of type(γ), then they are weak compatible of type(γ). But the converse is not true.

Lemma 3.4. [12]Let X be an intuitionistic fuzzy metric space.

(i)If we define $E_\lambda : X^2 \to [0,\infty)$ by

$$E_\lambda(x,y) = \inf\{t > 0; M(x,y,t) > 1-\lambda, N(x,y,t) < \lambda\}$$

for each $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$E_\lambda(x_1,x_n) \le E_\lambda(x_1,x_2)+E_\lambda(x_2,x_3)+\cdots E_\lambda(x_{n-1},x_n)$$

for any $x_1, x_2, \cdots, x_n \in X$.

(ii)The sequence $\{x_n\}_{n\in N}$ is convergent in intuitionistic fuzzy metric space X if and only if $E_\lambda(x_n,x) \to 0$. Also, the sequence $\{x_n\}_{n\in N}$ is Cauchy sequence if and only if it is Cauchy sequence with E_λ.

Lemma 3.5. [12]Let X be an intuitionistic fuzzy metric space.

$$M(x_n,x_{n+1},t) \ge M(x_0,x_1,k^n t),$$
$$N(x_n,x_{n+1},t) \le N(x_0,x_1,k^n t)$$

for some $k > 1$ and for every $n \in N$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Lemma 3.6. ([7])Let X be an intuitionistic fuzzy metric space. If there exists a number $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,

$$M(x,y,kt) \ge M(x,y,t),\ N(x,y,kt) \le N(x,y,t),$$

then $x = y$.

4. Main Results

Lemma 4.1. Let A and B be self-mappings of a complete intuitionistic fuzzy metric space X satisfying:

(i)There exists a constant $k \in (0,1)$ such that

$$\begin{aligned}
&M^2(Ax,By,kt) * [M(x,Ax,kt)M(y,By,kt)]\\
&\quad *M^2(y,By,kt) + aM(y,By,kt)M(x,By,2kt)\\
&\ge [pM(x,Ax,t) + qM(x,y,t)]M(x,By,2kt),\\
&N^2(Ax,By,kt) \diamond [N(x,Ax,kt)N(y,By,kt)]\\
&\quad \diamond N^2(y,By,kt) + aN(y,By,kt)N(x,By,2kt)\\
&\le [pN(x,Ax,t) + qN(x,y,t)]N(x,By,2kt)
\end{aligned}$$

for every $x, y \in X$ and $t > 0$, where $0 < p, q < 1$, $0 \le a < 1$ such that $p + q - a = 1$. Then A and B have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point, there exist $x_1 \in X$ such that $Ax_0 = x_1$, $Bx_1 = x_2$. Inductively, construct the sequences $\{x_n\} \subset X$ such that $x_{2n+1} = Ax_{2n}$, $x_{2n+2} = Bx_{2n+1}$ for $n = 0, 1, 2, \cdots$. Then we prove that $\{x_n\}$ is a Cauchy sequence. For $x = x_{2n}, y = x_{2n+1}$ by (i) we have

$$\begin{aligned}
&M^2(Ax_{2n},Bx_{2n+1},kt)\\
&\quad *[M(x_{2n},Ax_{2n},kt)M(x_{2n+1},Bx_{2n+1},kt)]\\
&\quad *M^2(x_{2n+1},Bx_{2n+1},kt)\\
&+aM(x_{2n+1},Bx_{2n+1},kt)M(x_{2n},Bx_{2n+1},2kt)\\
&\ge [pM(x_{2n},Ax_{2n},t) + qM(x_{2n},x_{2n+1},t)]\\
&\quad \times M(x_{2n},Bx_{2n+1},2kt)
\end{aligned}$$

and

$$\begin{aligned}
&M^2(x_{2n+1},x_{2n+2},kt)\\
&\quad *[M(x_{2n},x_{2n+1},kt)M(x_{2n+1},x_{2n+2},kt)]\\
&\quad *M^2(x_{2n+1},x_{2n+2},kt)\\
&+aM(x_{2n+1},x_{2n+2},kt)M(x_{2n},x_{2n+2},2kt)\\
&\ge [pM(x_{2n},x_{2n+1},t) + qM(x_{2n},x_{2n+1},t)]\\
&\quad \times M(x_{2n},x_{2n+2},2kt),
\end{aligned}$$

$$\begin{aligned}
&N^2(Ax_{2n},Bx_{2n+1},kt)\\
&\quad \diamond[N(x_{2n},Ax_{2n},kt)N(x_{2n+1},Bx_{2n+1},kt)]\\
&\quad \diamond N^2(x_{2n+1},Bx_{2n+1},kt)\\
&+aN(x_{2n+1},Bx_{2n+1},kt)N(x_{2n},Bx_{2n+1},2kt)\\
&\le [pN(x_{2n},Ax_{2n},t) + qN(x_{2n},x_{2n+1},t)]\\
&\quad \times N(x_{2n},Bx_{2n+1},2kt)
\end{aligned}$$

and

$$\begin{aligned}
&N^2(x_{2n+1},x_{2n+2},kt)\\
&\quad \diamond[N(x_{2n},x_{2n+1},kt)N(x_{2n+1},x_{2n+2},kt)]\\
&\quad \diamond N^2(x_{2n+1},x_{2n+2},kt)\\
&+aN(x_{2n+1},x_{2n+2},kt)N(x_{2n},x_{2n+2},2kt)\\
&\le [pN(x_{2n},x_{2n+1},t) + qN(x_{2n},x_{2n+1},t)]\\
&\quad \times N(x_{2n},x_{2n+2},2kt).
\end{aligned}$$

Then

$$\begin{aligned}
&M^2(x_{2n+1},x_{2n+2},kt)\\
&\quad *[M(x_{2n},x_{2n+1},kt)M(x_{2n+1},x_{2n+2},kt)]\\
&+aM(x_{2n+1},x_{2n+2},kt)M(x_{2n},x_{2n+2},2kt)\\
&\ge (p+q)M(x_{2n},x_{2n+1},t)M(x_{2n},x_{2n+2},2kt),
\end{aligned}$$

$$\begin{aligned}
&N^2(x_{2n+1},x_{2n+2},kt)\\
&\quad \diamond[N(x_{2n},x_{2n+1},kt)N(x_{2n+1},x_{2n+2},kt)]\\
&+aN(x_{2n+1},x_{2n+2},kt)N(x_{2n},x_{2n+2},2kt)\\
&\le (p+q)N(x_{2n},x_{2n+1},t)N(x_{2n},x_{2n+2},2kt).
\end{aligned}$$

So

$$M(x_{2n+1}, x_{2n+2}, kt) + aM(x_{2n+1}, x_{2n+2}, kt) \geq (p+q)M(x_{2n}, x_{2n+1}, t),$$
$$N(x_{2n+1}, x_{2n+2}, kt) + aN(x_{2n+1}, x_{2n+2}, kt) \leq (p+q)N(x_{2n}, x_{2n+1}, t).$$

Therefore

$$M(x_{2n+1}, x_{2n+2}, kt) \geq M(x_{2n}, x_{2n+1}, t),$$
$$N(x_{2n+1}, x_{2n+2}, kt) \leq N(x_{2n}, x_{2n+1}, t).$$

Similarly, we also have

$$M(x_{2n+2}, x_{2n+3}, kt) \geq M(x_{2n+1}, x_{2n+2}, t),$$
$$N(x_{2n+2}, x_{2n+3}, kt) \leq N(x_{2n+1}, x_{2n+2}, t).$$

For $k \in (0,1)$ if $k_1 = \frac{1}{k} > 1$ and $t = k_1 t_1$, then we have

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k_1^n t_1),$$
$$N(x_n, x_{n+1}, t) \leq N(x_0, x_1, k_1^n t_1).$$

By Lemma 3.5, since $\{x_n\}$ is a Cauchy sequence in X which is complete, $\{x_n\}$ converges to z in X. Hence $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Bx_{2n+1} = z$.

Now, taking $x = z$ and $y = x_{2n+1}$ in (i), we have as $n \to \infty$,

$$M^2(Az, z, kt) * [M(z, Az, kt)M(z, z, kt)] * M^2(z, z, kt) + aM(z, z, kt)M(z, z, 2kt) \geq [pM(z, Az, t) + qM(z, z, t)]M(z, z, 2kt),$$
$$N^2(Az, z, kt) \diamond [N(z, Az, kt)N(z, z, kt)] \diamond N^2(z, z, kt) + aN(z, z, kt)N(z, z, 2kt) \leq [pN(z, Az, t) + qN(z, z, t)]N(z, z, 2kt).$$

Therefore

$$M(Az, z, kt) + a \geq pM(z, Az, t) + q,$$
$$N(Az, z, kt) \leq 0$$

for all $t > 0$, so $Az = z$. Taking $x = x_{2n}$ and $y = z$ in (i), we have as $n \to \infty$,

$$M(z, Bz, t) + a \geq p + q,$$
$$N(z, Bz, t) + aN(z, Bz, t) \leq 0$$

for all $t > 0$, so $Bz = z$. Thus z is a common fixed point of A and B.

Let w be another common fixed point of A and B. Then using (i), we have

$$M^2(z, w, kt) + aM(z, w, 2kt) \geq [p + qM(z, w, t)]M(z, w, 2kt),$$
$$N^2(z, w, kt) \leq qN(z, w, t)N(z, w, 2kt)$$

and

$$M(z, w, t)M(z, w, 2kt) + aM(z, w, 2kt) \geq [p + qM(z, w, t)]M(z, w, 2kt),$$
$$N(z, w, t)N(z, w, 2kt) \leq qN(z, w, t)N(z, w, 2kt)$$

Thus, it follows that

$$M(z, w, t) \geq \frac{p-a}{1-q} = 1, \quad N(z, w, t) \leq 0$$

for all $t > 0$, so $z = w$. Hence A and B have a unique common fixed point in X. □

Theorem 4.2. Let A, B, C and D be self mappings of a complete intuitionistic fuzzy metric space X satisfying

(i)$A(X) \subseteq D(X)$, $B(X) \subseteq C(X)$,

(ii)There exists a constant $k \in (0,1)$ such that

$$M^2(Ax, By, kt) * [M(Cx, Ax, kt)M(Dy, By, kt)] * M^2(Dy, By, kt) + aM(Dy, By, kt)M(Cx, By, 2kt) \geq [pM(Cx, Ax, t) + qM(Cx, Dy, t)]M(Cx, By, 2kt),$$
$$N^2(Ax, By, kt) \diamond [N(Cx, Ax, kt)N(Dy, By, kt)] \diamond N^2(Dy, By, kt) + aN(Dy, By, kt)N(Cx, By, 2kt) \leq [pN(Cx, Ax, t) + qN(Cx, Dy, t)]N(Cx, By, 2kt)$$

for every $x, y \in X$ and $t > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,

(iii)The pairs (A, C) and (B, D) are weak compatible of type(γ).

Then A, B, C and D have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $A(X) \subseteq D(X)$ and $B(X) \subseteq C(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Dx_1 = y_1$, $Bx_1 = Cx_2 = y_2$. Because we can construct the sequences $\{x_n\}, \{y_n\} \subset X$ such that $y_{2n+1} = Ax_{2n} = Dx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Cx_{2n+2}$ for $n = 0, 1, 2, \cdots$, we prove $\{y_n\}$ is a Cauchy sequence. For $x = x_{2n}, y = x_{2n+1}$ by (ii), we have

$$M^2(Ax_{2n}, Bx_{2n+1}, kt) * [M(Cx_{2n}, Ax_{2n}, kt)M(Dx_{2n+1}, Bx_{2n+1}, kt)] * M^2(Dx_{2n+1}, Bx_{2n+1}, kt) + aM(Dx_{2n+1}, Bx_{2n+1}, kt)M(Cx_{2n}, Bx_{2n+1}, 2kt) \geq [pM(Cx_{2n}, Ax_{2n}, t) + qM(Cx_{2n}, Dx_{2n+1}, t)] \times M(Cx_{2n}, Bx_{2n+1}, 2kt),$$
$$N^2(Ax_{2n}, Bx_{2n+1}, kt) \diamond [N(Cx_{2n}, Ax_{2n}, kt)N(Dx_{2n+1}, Bx_{2n+1}, kt)] \diamond N^2(Dx_{2n+1}, Bx_{2n+1}, kt) + aN(Dx_{2n+1}, Bx_{2n+1}, kt)N(Cx_{2n}, Bx_{2n+1}, 2kt) \leq [pN(Cx_{2n}, Ax_{2n}, t) + qN(Cx_{2n}, Dx_{2n+1}, t)] \times N(Cx_{2n}, Bx_{2n+1}, 2kt).$$

Hence

$$\begin{aligned}&M(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt)\\&+aM(y_{2n+1}, y_{2n+2}, kt)M(y_{2n}, y_{2n+2}, 2kt)\\&\geq (p+q)M(y_{2n}, y_{2n+1}, t)M(y_{2n}, y_{2n+2}, 2kt),\\&N(y_{2n+1}, y_{2n+2}, kt)N(y_{2n}, y_{2n+2}, 2kt)\\&+aN(y_{2n+1}, y_{2n+2}, kt)N(y_{2n}, y_{2n+2}, 2kt)\\&\leq (p+q)N(y_{2n}, y_{2n+1}, t)N(y_{2n}, y_{2n+2}, 2kt).\end{aligned}$$

So, we have

$$\begin{aligned}&M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t),\\&N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t).\end{aligned}$$

Similarly, also we have

$$\begin{aligned}&M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t),\\&N(y_{2n+2}, y_{2n+3}, kt) \leq N(y_{2n+1}, y_{2n+2}, t).\end{aligned}$$

For $k \in (0,1)$, if $k_1 = \frac{1}{k} > 1$ and $t = k_1 t_1$, then

$$\begin{aligned}&M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, k_1 t_1)\\&\geq \cdots \geq M(y_0, y_1, k_1^n t_1),\\&N(y_n, y_{n+1}, t) \leq N(y_{n-1}, y_n, k_1 t_1)\\&\leq \cdots \leq N(y_0, y_1, k_1^n t_1).\end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence and completeness of X, $\{y_n\}$ converges to $z \in X$. Hence

$$\begin{aligned}\lim_{n\to\infty} Ax_{2n} &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Dx_{2n+1}\\&= \lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Bx_{2n+1}\\&= \lim_{n\to\infty} Cx_{2n+2} = \lim_{n\to\infty} Cx_{2n} = z.\end{aligned}$$

Since A, C are weak compatible of type(γ), $Az = Cz$.

Now, taking $x = z$ and $y = x_{2n+1}$ in (ii), we have as $n \to \infty$,

$$\begin{aligned}&M^2(Az, z, kt) * [M(Cz, Az, kt)M(z, z, kt)]\\&*M^2(z, z, kt) + aM(z, z, kt)M(Cz, z, 2kt)\\&\geq [pM(Cz, Az, t) + qM(Cz, z, t)]M(Cz, z, 2kt),\\&N^2(Az, z, kt) \diamond [N(Cz, Az, kt)N(z, z, kt)]\\&\diamond N^2(z, z, kt) + aN(z, z, kt)N(Cz, z, 2kt)\\&\leq [pN(Cz, Az, t) + qN(Cz, z, t)]N(Cz, z, 2kt).\end{aligned}$$

It follows that

$$\begin{aligned}&M^2(Az, z, kt) + aM(Az, z, 2kt)\\&\geq [p + qM(Az, z, t)]M(Az, z, 2kt),\\&N^2(Az, z, kt) \leq qN(Az, z, t)N(Az, z, 2kt).\end{aligned}$$

Since $M(x, y, \cdot)$ is nondecreasing and $N(x, y, \cdot)$ is nonincreasing for all $x, y \in X$, we have

$$M(Az, z, t) \geq \frac{p-a}{1-q} = 1, \;\; N(Az, z, t) \leq \frac{0}{1-q} = 0$$

for all $t > 0$, so $Az = z$. Hence $Az = Cz = z$.

Similarly, since B, D are weak compatible of type(γ), we get $Bz = Dz$. For taking $x = x_{2n}$ and $y = z$ in (ii), we have as $n \to \infty$,

$$\begin{aligned}&M^2(z, Bz, kt) * [M(z, z, kt)M(Dz, Bz, kt)]\\&*M^2(Dz, Bz, kt) + aM(Dz, Bz, kt)M(z, Bz, 2kt)\\&\geq [pM(z, z, t) + qM(z, Dz, t)]M(z, Bz, 2kt),\\&N^2(z, Bz, kt) \diamond [N(z, z, kt)N(Dz, Bz, kt)]\\&\diamond N^2(Dz, Bz, kt) + aN(Dz, Bz, kt)N(z, Bz, 2kt)\\&\leq [pN(z, z, t) + qN(z, Dz, t)]N(z, Bz, 2kt),\end{aligned}$$

then

$$\begin{aligned}&M^2(z, Bz, kt) + aM(z, Bz, 2kt)\\&\geq [p + qM(z, Dz, t)]M(z, Bz, 2kt),\\&N^2(z, Bz, kt) \leq qN(z, Dz, t)N(z, Bz, 2kt)\end{aligned}$$

Thus it follows that

$$M(z, Bz, t) \geq \frac{p-a}{1-q} = 1, \;\; N(z, Bz, t) \leq \frac{0}{1-q} = 0$$

for all $t > 0$, so $Bz = z$. hence $Bz = Dz = z$. Therefore z is a common fixed point of A, B, C and D.

Let w be another common fixed point of A, B, C and D. Then we have

$$\begin{aligned}&M^2(Az, Bw, kt) * [M(Cz, Az, kt)M(Dw, Bw, kt)]\\&\quad *M^2(Dw, Bw, kt)\\&\quad +aM(Dw, Bw, kt)M(Cw, Bw, 2kt)\\&\geq [pM(Cz, Az, t)\\&\quad +qM(Cz, Dw, t)]M(Cz, Bw, 2kt),\\&N^2(Az, Bw, kt) \diamond [N(Cz, Az, kt)N(Dw, Bw, kt)]\\&\quad \diamond N^2(Dw, Bw, kt)\\&\quad +aN(Dw, Bw, kt)N(Cw, Bw, 2kt)\\&\leq [pN(Cz, Az, t)\\&\quad +qN(Cz, Dw, t)]N(Cz, Bw, 2kt).\end{aligned}$$

So,

$$\begin{aligned}&M^2(z, w, kt) + aM(z, w, 2kt)\\&\geq [p + qM(z, w, t)]M(z, w, 2kt),\\&N^2(z, w, kt) \leq qN(z, w, t)N(z, w, 2kt).\end{aligned}$$

Therefore

$$M(z, w, t) \geq \frac{p-a}{1-q} = 1, \;\; N(z, w, t) \leq \frac{0}{1-q} = 0$$

for all $t > 0$, so $z = w$. hence A, B, C and D have unique common fixed point on X. □

Example 4.3. Let (X, d) be the metric space with $X = [0, 1]$. Denote $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and let M_d, N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows :

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \ \ N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X and $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space. Define self mappings A, B, C and D by

$$\begin{aligned} A(X) &= 1 \\ B(X) &= 1 \\ C(X) &= \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{x is irrational} \end{cases}, \\ D(X) &= \tfrac{x+1}{2}. \end{aligned}$$

If we define $\{x_n\} \subset X$ by $x_n = 1 - \frac{1}{n}$, then we have for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$ and $A1 = 1 = C1$,

$$\lim_{n\to\infty} M(CAx_n, 1, t) \leq M(A1, 1, t) = 1,$$
$$\lim_{n\to\infty} N(CAx_n, 1, t) \geq N(A1, 1, t) = 0.$$

Also, for $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 1$ and $B1 = 1 = D1$,

$$\lim_{n\to\infty} M(DBx_n, 1, t) \leq M(B1, 1, t) = 1,$$
$$\lim_{n\to\infty} N(DBx_n, 1, t) \geq N(B1, 1, t) = 0.$$

Therefore, (A, C) and (B, D) are weak compatible of type(γ). Then all the conditions of Theorem 4.2 are satisfied and 1 is a unique common fixed point of A, B, C and D on X.

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