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Existence of Solutions for the Impulsive Semilinear Fuzzy Intergrodifferential Equations with Nonlocal Conditions and Forcing Term with Memory in n-dimensional Fuzzy Vector Space (E_N^n, d_ε)

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Abstract

In this paper, we study the existence and uniqueness of solutions for the impulsive semilinear fuzzy integrodifferential equations with nonlocal conditions and forcing term with memory in n-dimensional fuzzy vector space (E_N^n, d_ε) by using Banach fixed point theorem. That is an extension of the result of Kwun et al. [9] to impulsive system.

Key Words: Existence and uniqueness of solutions, impulsive, semilinear, integrodifferential equations, nonlocal,memory

1. Introduction

Many authors have studied several concepts of fuzzy systems. Diamond and Kloeden [4] proved the fuzzy optimal control for the following system:

$$\dot{x}(t) = a(t)x(t) + u(t), \;\; x(0) = x_0,$$

where $x(\cdot)$ and $u(\cdot)$ are nonempty compact interval-valued functions on E^1. Kwun and Park [6] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E_N^1 by using Kuhn-Tucker theorems. The theory of differential equations with discontinuous trajectories during the last twenty years has been to a great extent stimulated by their numerous applications to problem arsing in mechanics, electrical engineerin, the theory of automatic control, medicine and biology. For the monographs of the theory of impulsive differential equations, see Bainov and Simenov [1], Lakshimikantham et al. [10], Samoileuko and Perestyuk [15], where numerous properties of their solutions are studied and detailed bibliographies are given. Rogovchenko [14] follows the ideas of the theory of impulsive differential equations which treats the changes of the state of the evolution process due to a short-term perturbations whose duration can be negligible in comparison with the duration of the process as an instant impulses. In 2007, Park et al. [12] studied the existence and uniqueness of fuzzy solutions and controllability for the impulsive semilinear fuzzy integrodifferential equations in one-dimensional fuzzy vector space E_N^1. R. Rodríguez-López [13] studied periodic boundary value problems for impulsive fuzzy differential equations. Fuzzy integrodifferential equations are a field of interest, due to their applicability to the analysis of phenomena with memory where imprecision is inherent. Balasubramaniam and Muralisankar [2] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. They considered the semilinear one-dimensional heat equation on a connected domain $(0, 1)$ for material with memory. In one-dimensional fuzzy vector space E_N^1, Park et al. [11] proved the existence and uniqueness of fuzzy solutions and presented the sufficient condition of nonlocal controllability for the following semilinear fuzzy integrodifferential equation with nonlocal initial condition. In [7], Kwun et al. proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equations by using successive iteration. In [8], Kwun et al. investigated the continuously initial observability for the semilinear fuzzy integrodifferential equations. Bed and Gal [3] studied almost periodic fuzzy-number-valued functions. Gal and N'Guerekata [5] studied almost automorphic fuzzy-number-valued functions. Recently, Kwun et al.[9] study the the existence and uniqueness of solutions and nonlocal controllability for the semilinear fuzzy integrodifferential equations in n-dimensional fuzzy vector space.

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In this paper, we study the existence and uniqueness of solutions for the following impulsive semilinear fuzzy integrodifferential equations with nonlocal conditions and forcing term with memory in n-dimensional fuzzy vector space:

$$\frac{dx_i(t)}{dt} = A_i\left[x_i(t) + \int_0^t G(t-s)x_i(s)ds\right] \quad (1)$$
$$+f_i(t, x_i(t), \int_0^t q_i(t,s,x_i(s))ds) \text{ on } E_N^i,$$
$$x_i(0) + g_i(x_i) = x_{0_i} \in E_N^i, \quad (2)$$
$$\Delta x_i(t_k) = I_k(x_i(t_k)), \ \ t \neq t_k, \quad (3)$$
$$k = 1, 2, \cdots, m, \ i = 1, 2, \cdots, n,$$

where $A_i : [0,T] \to E_N^i$ is fuzzy coefficient, E_N^i is the set of all upper semi-continuously convex fuzzy numbers on R with $E_N^i \neq E_N^j$ $(i \neq j)$, $f_i : [0,T] \times E_N^i \times E_N^i \to E_N^i$ and $q_i : [0,T] \times [0,T] \times E_N^i \to E_N^i$ are nonlinear regular fuzzy function, $g_i : E_N^i \to E_N^i$ is a nonlinear continuous function, $G(t)$ is $n \times n$ continuous matrix such that $\frac{dG(t)x_i}{dt}$ is continuous for $x_i \in E_N^i$ and $t \in [0,T]$ with $\|G(t)\| \leq k$, $k > 0$, $x_{0_i} \in E_N^i$ is initial value and $I_k \in C(E_N^i, E_N^i)$ are bounded functions, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, where $x_i(t_k^-)$ and $x_i(t_k^+)$ represent the left and right limits of $x_i(t)$ at $t = t_k$, respectively. Given nonlinear regular fuzzy function f_i and q_i satisfy global Lipschitz conditions, i.e. there exist finite constants $k_{1i}, k_{2i}, m_i > 0$ $(i = 1, 2, \cdots, n)$ such that

$$d_H\Big([f_i(s, \xi_{1i}(s), \eta_{1i}(s))]^\alpha, \quad (4)$$
$$[f_i(s, \xi_{2i}(s), \eta_{2i}(s))]^\alpha\Big)$$
$$\leq k_{1i} d_H([\xi_{1i}(s)]^\alpha, [\xi_{2i}(s)]^\alpha)$$
$$+k_{2i} d_H([\eta_{1i}(s)]^\alpha, [\eta_{2i}(s)]^\alpha),$$

$$d_H([q_i(t,s,\varphi_{1i}(s))]^\alpha, [q_i(t,s,\varphi_{2i}(s))]^\alpha \quad (5)$$
$$\leq m_i d_H([\varphi_{1i}(s)]^\alpha, [\varphi_{2i}(s)]^\alpha)$$

for all $\xi_{ji}(s), \eta_{ji}(s), \varphi_{ji}(s) \in (E_N^i)^n (j = 1, 2)$. The nonlinear function g_i is a continuous function and satisfies the Lipschitz condition

$$d_H([g_i(x_i(\cdot))]^\alpha, [g_i(y_i(\cdot))]^\alpha) \quad (6)$$
$$\leq h_i d_H([x_i(\cdot)]^\alpha, [y_i(\cdot)]^\alpha)$$

for all $x_i(\cdot), y_i(\cdot) \in (E_N^i)^n$, h_i is a finite positive constant.

2. Preliminaries

A fuzzy set of R^n is a function $u : R^n \to [0,1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$ for any $\alpha \in [0,1]$, its α-level set.

We call $u \in E^n$ a n-dimension fuzzy number.

Wang, Li and Wen [16] defined n-dimensional fuzzy vector space and investigated its properties.

For any $u_i \in E$, $i = 1, 2, \cdots, n$, we call the ordered one-dimension fuzzy number class $u_1, u_2, \cdots, u_n$ (i.e., the Cartesian product of one-dimension fuzzy number $u_1, u_2, \cdots, u_n$) a n-dimension fuzzy vector, denote it as $(u_1, u_2, \cdots, u_n)$, and call the collection of all n-dimension fuzzy vectors (i.e., the Cartesian product $\overbrace{E \times E \times \cdots \times E}$) n-dimensional fuzzy vector space, and denote it as $(E)^n$.

Definition 2.1. [16] If $u \in E^n$, and $[u]^\alpha$ is a hyperrectangle, i.e., $[u]^\alpha$ can be represented by $\prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$, i.e., $[u_{1l}^\alpha, u_{1r}^\alpha] \times [u_{2l}^\alpha, u_{2r}^\alpha] \times \cdots \times [u_{nl}^\alpha, u_{nr}^\alpha]$ for every $\alpha \in [0,1]$, where $u_{il}^\alpha, u_{ir}^\alpha \in R$ with $u_{il}^\alpha \leq u_{ir}^\alpha$ when $\alpha \in (0,1]$, $i = 1, 2, \cdots, n$, then we call u a fuzzy n-cell number. We denote the collection of all fuzzy n-cell numbers by $L(E^n)$.

Theorem 2.1. [16] For any $u \in L(E^n)$ with $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$, there exists a unique $(u_1, u_2, \cdots, u_n) \in (E)^n$ such that $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ $(i = 1, 2, \cdots, n$ and $\alpha \in [0,1])$.

Conversely, for any $(u_1, u_2, \cdots, u_n) \in (E)^n$ with $[u_i]^\alpha = [u_{il}^\alpha, u_{ir}^\alpha]$ $(i = 1, 2, \cdots, n$ and $\alpha \in [0,1])$, there exists a unique $u \in L(E^n)$ such that $[u]^\alpha = \prod_{i=1}^n [u_{il}^\alpha, u_{ir}^\alpha]$ $(\alpha \in [0,1])$.

Note 2.1. [16] Theorem 2.1 indicates that fuzzy n-cell numbers and n-dimension fuzzy vectors can represent each other, so $L(E^n)$ and $(E)^n$ may be regarded as identity. If $(u_1, u_2, \cdots, u_n) \in (E)^n$ is the unique n-dimension fuzzy vector determined by $u \in L(E^n)$, then we denote $u = (u_1, u_2, \cdots, u_n)$.

Let $(E_N^i)^n = E_N^1 \times E_N^2 \times \cdots \times E_N^n$, E_N^i $(i = 1, 2, \times, n)$ is fuzzy subset of R. Then $(E_N^i)^n \subseteq (E)^n$.

Definition 2.2. Let $u, v \in (E_N^i)^n (i = 1, 2, \cdots, n)$

$$d_\epsilon([u]^\alpha, [v]^\alpha) = d_H\Big(\prod_{i=1}^n [u_i]^\alpha, \prod_{i=1}^n [v_i]^\alpha\Big)$$
$$= \Big(\sum_{i=1}^n (d_H([u_i]^\alpha, [v_i]^\alpha))^2\Big)^{1/2}$$

where d_H is the Hausdorff distance.

Definition 2.3. The complete metric d_∞ on $(E_N^i)^n$ is defined by

$$d_\infty(u(t), v(t)) = \sup_{0 < \alpha \leq 1} d_\epsilon([u(t)]^\alpha, [v(t)]^\alpha)$$

for any $u, v \in (E_N^i)^n$.

Definition 2.4. Let $u, v \in C([0,T] : (E_N^i)^n)$

$$H_1(u, v) = \sup_{0 \leq t \leq T} d_\infty(u(t), v(t))$$

Definition 2.5. [16] The derivative $x'(t)$ of a fuzzy process $x \in (E_N^i)^n$ is defined by

$$[x'(t)]^\alpha = \prod_{i=1}^{n} [(x_{il}^\alpha)'(t), (x_{ir}^\alpha)'(t)]$$

provided that equation defines a fuzzy $x'(t) \in (E_N^i)^n$.

Definition 2.6. [16] The fuzzy integral $\int_b^a x(t)dt$, $a, b \in [0, T]$ is defined by

$$\Big[\int_b^a x(t)dt\Big]^\alpha = \prod_{i=1}^{n} \Big[\int_b^a x_{il}^\alpha(t)dt, \int_b^a x_{ir}^\alpha(t)dt\Big]$$

provided that the Lebesgue integrals on the right hand side exist.

3. Existence and Uniqueness

In this section we consider the existence and uniqueness of the fuzzy solution for the equations (1)-(3).

We define

$$A = (A_1, A_2, \cdots, A_n),$$

$$x = (x_1, x_2, \cdots, x_n),$$

$$f = (f_1, f_2, \cdots, f_n),$$

$$q = (q_1, q_2, \cdots, q_n),$$

$$g = (g_1, g_2, \cdots, g_n),$$

and

$$x_0 = (x_{0_1}, x_{0_2}, \cdots, x_{0_n}).$$

Then

$$A,\ x,\ f,\ q,\ x_0\ \in\ (E_N^i)^n.$$

Instead of the equations(1)-(3), we consider the following fuzzy integrodifferential equations in $(E_N^i)^n$.

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] \quad (7)$$
$$+f(t, x(t), \int_0^t q(t, s, x(s))ds) \text{ on } (E_N^i)^n$$
$$x(0) + g(x) = x_0 \in (E_N^i)^n \quad (8)$$
$$\Delta x(t_k) = I_k(x(t_k)),\ \ t \neq t_k, \quad (9)$$
$$k = 1, 2, \cdots, m,\ i = 1, 2, \cdots, n,$$

with fuzzy coefficient $A : [0, T] \to (E_N^i)^n$, initial value $x_0 \in (E_N^i)^n$.

Definition 3.1. The fuzzy process $x : I = [0, T] \to (E_N^i)^n$ with α-level set $[x(t)]^\alpha = \Pi_{i=1}^n [x_i]^\alpha = \Pi_{i=1}^n [x_{il}^\alpha, x_{ir}^\alpha]$ is a fuzzy solution of the equations (7)-(9) without nonhomogeneous term if and only if

$$(x_{il}^\alpha)'(t) = \min\Big\{A_{ij}^\alpha(t)\Big[x_{ik}^\alpha(t) + \int_0^t G(t-s)x_{ik}^\alpha(s)ds\Big] : j, k = l, r\Big\},$$
$$(x_{ir}^\alpha)'(t) = \max\Big\{A_{ij}^\alpha(t)\Big[x_{ik}^\alpha(t) + \int_0^t G(t-s)x_{ik}^\alpha(s)ds\Big] : j, k = l, r\Big\},$$
$$x_{il}^\alpha(0) = x_{0_{il}}^\alpha,\ \ x_{ir}^\alpha(0) = x_{0_{ir}}^\alpha,\ i = 1, 2, \cdots, n.$$

For the sequel, we need the following assumption:

(H1) $S_i(t)$ is a fuzzy number satisfying, for $y_i \in (E_N^i)^n$, $\frac{d}{dt}S_i(t)y_i \in C^1(I : (E_N^i)^n) \bigcap C(I : (E_N^i)^n)$, the equation

$$\frac{d}{dt}S_i(t)y_i = A\left[S_i(t)y_i + \int_0^t G(t-s)S_i(s)y_i ds\right],$$

where

$$[S(t)]^\alpha = \prod_{i=1}^{n}\Big[S_i(t)\Big]^\alpha = \prod_{i=1}^{n}\Big[(S_i(t))_l^\alpha, (S_i(t))_r^\alpha\Big],$$
$$S_i(0) = I$$

and $\Big(S_i(t)\Big)_k^\alpha$ is continuous with $\Big|(S_i(t))_k^\alpha\Big| \le c_i$, $c_i > 0$, $(k = l, r)$ for all $t \in I = [0, T]$.

In order to define the solution of (7)-(9), we shall consider the space

$\Omega_i = \Big\{x_i : J \to E_N^i : (x_i)_k \in C(J_k, E_N^i), J_k = (t_k, t_{k+1}], k = 0, 1, \cdots, m,$ and there exist $(x_i)(t_k^-)$ and $(x_i)(t_k^+)(k = 1, 2, \cdots, m)$, with $(x_i)(t_k^-) = (x_i)(t_k)\Big\}$, $i = 1, 2, \cdots, n$.

Let $\Omega' = \Pi_{i=1}^n \Omega_i'$, $\Omega_i' = \Omega_i \bigcap C([0, T] : E_N^i)$, $i = 1, 2, \cdots, n$.

Lemma 3.1. If x is an integral solution of (7)-(9) , then x is given by

$$x(t) = S(t)(x_0 - g(x)) \quad (10)$$
$$+ \int_0^t S(t-s)f(s, x(s), \int_0^s q(s, \tau, x(\tau))d\tau)ds$$
$$+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)),\ \ for\ \ t \in J.$$

Proof. Let x be a solution of (4)-(6). Define $\omega(s) = S(t-s)x(s)$. Then we have that

$$\frac{d\omega(s)}{ds} = -\frac{dS(t-s)}{ds}x(s) + S(t-s)\frac{dx(s)}{ds}$$

$$= -A\Big[S(t-s)x(s) + \int_0^t G(t-s)S(s)x(s)ds\Big] + S(t-s)\frac{dx(s)}{ds}$$
$$= S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau).$$

Consider $t_k < t, k = 1,2,\cdots,m$. Then integrating the previous equation, we have

$$\int_0^t \frac{d\omega(s)}{ds}ds = \int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds$$

For $k=1$,

$$\omega(t)-\omega(0) = \int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds$$

or

$$x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds.$$

Now for $k = 2,\cdots,m$, we have that

$$\int_0^{t_1}\frac{d\omega(s)}{ds}ds + \int_{t_1}^{t_2}\frac{d\omega(s)}{ds}ds + \cdots + \int_{t_k}^{t}\frac{d\omega(s)}{ds}ds = \int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds.$$

Then

$$\omega(t_1^-)-\omega(0)+\omega(t_2^-)-\omega(t_1^+)+\cdots-\omega(t_k^+)+\omega(t) = \int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds$$

if and only if

$$\omega(t) = \omega(0) + \int_0^t S(t-s) \times f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds + \sum_{0<t_k<t}\Big[\omega(t_k^+)-\omega(t_k^-)\Big].$$

Hence

$$x(t) = S(t)(x_0-g(x)) + \int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)),$$

which proves the lemma.

Assume the following:

(H2) There exists d_{ik}, $k=1,2,\cdots,m$, $i=1,2,\cdots,n$, such that

$$d_H\Big(\Big[I_{ki}(x_i(t_k^-))\Big]^\alpha, \Big[I_{ki}(y_i(t_k^-))\Big]^\alpha\Big) \le d_{ik}d_H\Big([x_i(t)]^\alpha,[y_i(t)]^\alpha\Big),$$

where $\sum_{k=1}^m d_{ik} = \overline{d_i}$

(H3) $K_1 = max\{k_{1i}\}$, $K_2 = max\{k_{2i}\}$, $H = max\{h_i\}$, $M = max\{m_i\}$, $C = max\{c_i\}$, $D = max\{\overline{d_i}\}$.

(H4) $C\Big(H + D + \Big(K_1 + K_2 M\frac{T}{2}\Big)T\Big) < 1$.

Theorem 3.1. Let $T>0$. If hypotheses (H2)-(H4) are hold. Then, for every $x_0 \in (E_N^i)^n$, equation (10) have a unique fuzzy solution $x \in \Omega'$.

Proof. For each $x(t) \in \Omega'$ and $t \in [0,T]$, define $(Gx)(t) \in \Omega'$ by

$$(Gx)(t) = S(t)(x_0-g(x)) + \int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)),.$$

Thus, $Gx : [0,T] \to \Omega'$is continuous, so G is a mapping from Ω' into itself. There exist $G_i (i=1,2,\cdots,n)$ which is a continuous function from Ω_i'. By Definition 2.2, Definition 2.3, some properties of d_H and inequalities (4),(5) and (6), we have following inequalities. For $x_i, y_i \in \Omega_i'$,

$$\begin{aligned}
& d_H\Big(\Big[G_i(x_i(t))\Big]^\alpha, \Big[G_i(y_i(t))\Big]^\alpha\Big) \\
& \le d_H\Big(\Big[S_i(t)(x_{0_i}-g_i(x_i)) + \int_0^t S_i(t-s)f_i\Big(s,x_i(s),\int_0^t q_i(s,\tau,x_i(\tau)d\tau\Big)ds\Big]^\alpha, \\
& \quad \Big[S_i(t)(y_{0_i}-g_i(y_i)) + \int_0^t S_i(t-s)f_i\Big(s,y_i(s),\int_0^t q_i(s,\tau,y_i(\tau)d\tau\Big)ds\Big]^\alpha\Big) \\
& \quad + d_H\Big(\Big[\sum_{0<t_k<t} S_i(t-t_k)I_k(x_i(t_k^-))\Big]^\alpha, \Big[\sum_{0<t_k<t} S_i(t-t_k)I_k(y_i(t_k^-))\Big]^\alpha\Big)
\end{aligned}$$

$$\le d_H([S_i(t)g_i(x_i)]^\alpha, [S_i(t)g_i(y_i)]^\alpha) + \int_0^t d_H\Big(\Big[S_i(t-s)f_i\Big(s,x_i(s),\int_0^t q_i(s,\tau,x_i(\tau)d\tau\Big)\Big]^\alpha,$$

$$
\begin{aligned}
&\Big[S_i(t-s)f_i\Big(s,y_i(s),\int_0^t q_i(s,\tau,y_i(\tau)d\tau\Big)\Big]^\alpha\Big)\\
&+d_H\Big(\Big[\sum_{0<t_k<t}S_i(t-t_k)I_k(x_i(t_k^-))\Big]^\alpha,\\
&\qquad\Big[\sum_{0<t_k<t}S_i(t-t_k)I_k(y_i(t_k^-))\Big]^\alpha\Big)\\
&\le c_ih_id_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+c_i\int_0^t\Big(k_{1i}d_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+k_{2i}m_i\int_0^s d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)d\tau\Big)ds\\
&\quad+c_i\overline{d_i}d_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&d_\infty\Big(Gx(t),Gy(t)\Big)\\
&=\sup_{0<\alpha\le1}d_\epsilon\Big([G(x(t))]^\alpha,[G(y(t))]^\alpha\Big)\\
&=\sup_{0<\alpha\le1}\Big(\sum_{i=1}^n\Big(d_H([G_i(x_i(t))]^\alpha,[G_i(y_i(t))]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\sup_{0<\alpha\le1}\Big(\sum_{i=1}^n\Big(c_ih_id_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+c_i\int_0^t\Big(k_{1i}d_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+k_{2i}m_i\int_0^s d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)d\tau\Big)ds\\
&\quad+c_i\overline{d_i}d_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\Big(\sum_{i=1}^n\Big(\sup_{0<\alpha\le1}c_ih_id_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+\sup_{0<\alpha\le1}c_i\int_0^t\Big(k_{1i}d_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+k_{2i}m_i\int_0^s d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)d\tau\Big)ds\\
&\quad+\sup_{0<\alpha\le1}c_i\overline{d_i}d_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\Big(\sum_{i=1}^n\Big(\sup_{0<\alpha\le1}CHd_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+\sup_{0<\alpha\le1}C\int_0^t\Big(K_1d_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+K_2M\int_0^s d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)d\tau\Big)ds\\
&\quad+\sup_{0<\alpha\le1}CDd_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\Big(\sum_{i=1}^n\Big(CH\sup_{0<\alpha\le1}d_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+C\sup_{0<\alpha\le1}\int_0^t\Big(K_1d_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+K_2M\int_0^s d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)d\tau\Big)ds\\
&\quad+CD\sup_{0<\alpha\le1}d_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
&H_1(Gx,Gy)\\
&=\sup_{0\le t\le T}d_\infty\Big(Gx(t),Gy(t)\Big)\\
&=\sup_{0\le t\le T}\sup_{0<\alpha\le1}d_\epsilon\Big([G(x(t))]^\alpha,[G(y(t))]^\alpha\Big)\\
&\le\sup_{0\le t\le T}\Big(\sum_{i=1}^n\Big(CH\sup_{0<\alpha\le1}d_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+C\sup_{0<\alpha\le1}\int_0^t\Big(K_1d_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+K_2M\int_0^s d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)d\tau\Big)ds\\
&\quad+CD\sup_{0<\alpha\le1}d_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\Big(\sum_{i=1}^n\Big(CH\sup_{0\le t\le T}\sup_{0<\alpha\le1}d_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+C\sup_{0\le t\le T}\sup_{0<\alpha\le1}\int_0^t\Big(K_1d_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+K_2M\int_0^s d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)d\tau\Big)ds\\
&\quad+CD\sup_{0\le t\le T}\sup_{0<\alpha\le1}d_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\Big(\sum_{i=1}^n\Big(CH\sup_{0\le t\le T}\sup_{0<\alpha\le1}d_H([x_i(\cdot)]^\alpha,[y_i(\cdot)]^\alpha)\\
&\quad+C\sup_{0\le t\le T}\sup_{0<\alpha\le1}\Big(K_1Td_H([x_i(s)]^\alpha,[y_i(s)]^\alpha)\\
&\qquad+K_2M\frac{T^2}{2}d_H([x_i(\tau)]^\alpha,[y_i(\tau)]^\alpha)\Big)\\
&\quad+CD\sup_{0\le t\le T}\sup_{0<\alpha\le1}d_H([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\Big(C(H+D+\Big(K_1+K_2M\frac{T}{2}\Big)T\Big)\\
&\times\sup_{0\le t\le T}\sup_{0<\alpha\le1}\Big(\sum_{i=1}^n\Big(d_H(([x_i(t)]^\alpha,[y_i(t)]^\alpha)\Big)^2\Big)^{1/2}\\
&\le\Big(C(H+D+\Big(K_1+K_2M\frac{T}{2}\Big)T\Big)
\end{aligned}
$$

$$\times \sup_{0\le t\le T}\sup_{0<\alpha\le 1} d_\epsilon\Big([G(x(t))]^\alpha, [G(y(t))]^\alpha\Big)$$

$$\le \Big(C(H+D+\Big(K_1+K_2M\frac{T}{2}\Big)T\Big)$$
$$\times \sup_{0\le t\le T} d_\infty\Big(Gx(t), Gy(t)\Big)$$
$$\le \Big(C(H+D+\Big(K_1+K_2M\frac{T}{2}\Big)T\Big)H_1(x,y).$$

By hypotheses (H4), Φ is a contraction mapping. By the Banach fixed point theorem, (10) has an unique fixed point $x\in\Omega'$.

4. Example

Consider the two semilinear one-dimensional heat equations on a connected domain (0,1) for a material with memory on $E_N^i(i = 1,2)$, boundary condition $x_i(t,0) = x_i(t,1) = 0(i = 1,2)$ and with initial conditions $x_i(0,z_i)+\sum_{k=1}^p (c_k)_i x_i(t_k,z_i) = x_{0_i}(z_i)$, where $x_{0_i}(z_i)\in E_N^i(i=1,2)$, $\sum_{k=1}^p (c_k)_i x_i(t_k,z_i) = g_i(x_i), i=1,2$. Let $x_i(t,z_i)(i=1,2)$ be the internal energy and

$$f_i(t,x_i(t,z_i),\int_0^t q_i(,s,x_i(s,z_i))ds$$
$$=\widetilde{2}tx_i(t,z_i)^2+\int_0^t (t-s)x_i(s,z_i)ds, i=1,2$$

be the external heat with memory. $\Delta x_i(t_k,z_i) = x_i(t_k^+,z_i)-x_i(t_k^-,z_i)$ is impulsive effect at $t=t_k(k=1,2,\cdots,m, i=1,2)$.

Let $A=(A_1,A_2)=\Big(\tilde{2}\frac{\partial^2}{\partial z_1^2},\tilde{2}\frac{\partial^2}{\partial z_2^2}\Big)$,

$$f\Big(t,x(t),\int_0^t q(t,s,x(s))ds\Big)$$
$$=\Big(f_1(t,x_1(t),\int_0^t q_1(t,s,x_1(s))ds),$$
$$f_2(t,x_2(t),\int_0^t q_2(t,s,x_2(s))ds)\Big)$$
$$=\Big(\widetilde{2}tx_1(t,z_1)^2+\int_0^t (t-s)x_1(s,z_1)ds,$$
$$\widetilde{2}tx_2(t,z_2)^2+\int_0^t (t-s)x_2(s,z_2)ds\Big),$$

$$g(x)=(g_1(x_1),g_2(x_2))$$
$$=\Big(\sum_{k=1}^p (c_k)_1x_1(t_k,z_1),\sum_{k=1}^p (c_k)_2x_2(t_k,z_2)\Big),$$
$$x(0)+g(x)=(x_1(0)+g_1(x),x_2(0)+g_2(x)),$$

$$x_0=(x_{0_1},x_{0_2})=(\widetilde{0},\widetilde{0}),$$

$$\Delta x(t_k)=(\Delta x_1(t_k),\Delta x_2(t_k)),$$
$$t\ne t_k,\ k=1,2,\cdots,m,$$

$$\begin{aligned} I_k(x(t_k)) &= (I_k(x_1(t_k)),I_k(x_2(t_k)))\\ &= (x_1(t_k^+)-x_1(t_k^-),x_2(t_k^+)-x_2(t_k^-))\\ &= (x_1(t_k^+,z_1)-x_1(t_k^-,z_1),\\ &\quad x_2(t_k^+,z_2)-x_2(t_k^-,z_2)),\end{aligned}$$

and $G(t-s)=(e^{-(t-s)},e^{-(t-s)})$, then the balance equation becomes

$$\frac{dx(t)}{dt}=A\left[x(t)+\int_0^t G(t-s)x(s)ds\right] \quad (11)$$
$$+f(t,x(t),\int_0^t q(t,s,x(s))ds) \text{ on } (E_N^i)^n,$$
$$x(0)+g(x)=x_0\in(E_N^i)^n, \quad (12)$$
$$\Delta x(t_k)=I_k(x(t_k)),\ \ t\ne t_k, \quad (13)$$
$$k=1,2,\cdots,m,\ i=1,2,\cdots,n.$$

The α-level set of fuzzy numbers are the following:
$[\widetilde{0}]^\alpha=[\alpha-1,1-\alpha]$, $[\widetilde{2}]^\alpha=[\alpha+1,3-\alpha]$ for all $\alpha\in[0,1]$. Then α-level sets of $f(t,x(t),\int_0^t q(t,s,x(s))ds)$ is

$$\Big[f(t,x(t),\int_0^t q(t,s,x(s))ds)\Big]^\alpha$$
$$=\Big[\widetilde{2}tx_1(t)^2+\int_0^t (t-s)x_1(s)ds\Big]^\alpha$$
$$\times\Big[\widetilde{2}tx_2(t)^2+\int_0^t (t-s)x_2(s)ds\Big]^\alpha$$
$$=\Big([\widetilde{2}]^\alpha\cdot t[x_1(t)^2]^\alpha+\int_0^t (t-s)[x_1(s)]^\alpha ds\Big)$$
$$\times\Big([\widetilde{2}]^\alpha\cdot t[x_2(t)^2]^\alpha+\int_0^t (t-s)[x_2(s)]^\alpha ds\Big)$$
$$=\Big[(\alpha+1)t(x_{1l}^\alpha(t))^2+\int_0^t (t-s)x_{1l}^\alpha(s)ds,$$
$$(3-\alpha)t(x_{1r}^\alpha(t))^2+\int_0^t (t-s)x_{1r}^\alpha(s)ds\Big]$$
$$\times\Big[(\alpha+1)t(x_{2l}^\alpha(t))^2+\int_0^t (t-s)x_{2l}^\alpha(s)ds,$$
$$(3-\alpha)t(x_{2r}^\alpha(t))^2+\int_0^t (t-s)x_{2r}^\alpha(s)ds\Big].$$

Further, we have

$$d_H\Big(\Big[f_i\Big(t,x_i(t),\int_0^t q_i(t,s,x_i(s))ds\Big)\Big]^\alpha,$$

$$
\begin{aligned}
&\Big[f_i\Big(t, y_i(t), \int_0^t q_i(t,s,y_i(s))ds\Big)\Big]^\alpha\Big) \\
&= t[\alpha+1, 3-\alpha]\Big(d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha) \\
&\qquad + \int_0^t (t-s) d_H([x_i(s)]^\alpha, [y_i(s)]^\alpha) ds\Big) \\
&\le t[\alpha+1, 3-\alpha]\Big(d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha) \\
&\qquad + \frac{T^2}{2} d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha)\Big) \\
&\le k_{1i} d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha) \\
&\qquad + k_{2i} d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha),
\end{aligned}
$$

$$
\begin{aligned}
&d_H\big([g_i(x_i(\cdot))]^\alpha, [g_i(y_i(\cdot))]^\alpha\big) \\
&= d_H\Big(\Big[\sum_{k=1}^p c_k(x_i(t_k))\Big]^\alpha, \Big[\sum_{k=1}^p c_k(y_i(t_k))\Big]^\alpha\Big) \\
&\le \Big|\sum_{k=1}^p c_k\Big| d_H([x_i(t_k)]^\alpha, [y_i(t_k)]^\alpha) \\
&= h_i d_H([x_i(\cdot)]^\alpha, [y_i(\cdot)]^\alpha),
\end{aligned}
$$

where k_{1i}, k_{2i} and h_i satisfy the inequality (4), (5) and (6) respectively. Then all the conditions stated in Theorem 3.1 are satisfied, so the problem (11)-(13) has a unique fuzzy solution.

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