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GMM Estimation for Seasonal Cointegration

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Abstract

This paper considers a generalized method of moments(GMM) estimation for seasonal cointegration as the extension of Kleibergen (1999). We propose two iterative methods for the estimation according to whether parameters in the model are simultaneously estimated or not. It is shown that the GMM estimator coincides in form to a maximum likelihood estimator or a feasible two-step estimator. In addition, we derive its asymptotic distribution that takes the same form as that in Ahn and Reinsel (1994).

Keywords: Generalized method of moments estimation, vector error correction model, vector autoregressive model.

1. Introduction

Hylleberg *et al.* (1990) introduced the concept of seasonal cointegration for seasonal time series with unit roots at seasonal frequencies, *i.e.*, roots of modulus 1, whereas the concept of usual cointegration, by Engle and Granger (1987), is for nonseasonal time series with unit roots at zero frequency, *i.e.*, roots of exactly 1. Since then, many approaches for analyzing seasonal cointegration have been developed. Among others, Ahn and Reinsel (1994) (AR1994) and Ahn *et al.* (2004) used an iterative method considering all the frequencies of seasonal unit roots simultaneously. Johansen and Schaumburg (1999) considered a switching algorithm based on partial regression, to avoid the complexity generated with the simultaneous estimation. Cubadda (2001) considered the complex error correction model, which uses only the nonstationary roots existing on the upper half unit circle, in order to overcome the complicated algorithm of the previous works.

In the literature for seasonal cointegration, normality is usually adopted for constructing the likelihood function because the maximum likelihood(ML) approach has been the prevalent method for estimating the cointegrating parameters in vector error correction model(VECM). The popularity of the ML approach occurs due to its sound theoretical basis, computational simplicity and superior performance relative to some other estimators (Brüggemann and Lütkepohl, 2005). However, the potentially poor small-sample performances of the ML estimator(MLE) were pointed out by several earlier works, especially, in non-seasonal cointegration analysis. Among others, Phillips (1994) showed that finite-sample moments of the MLE do not exist and Gonzalo (1994) and Hansen *et*

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al. (1998) found that the small-sample properties of the MLE are not well approximated by its asymptotic distribution and in particular, that the MLE produces occasional outliers which are far away from the true parameter values. In this respect, Seong (2008) considered a simple feasible two-step (or generalized least squares) estimator for seasonal cointegration which does not produce the kind of outlying estimates observed for the MLE.

Since empirical distributions of financial and macroeconomic data, frequently analyzed through seasonal cointegration, are skewed and leptokurtic, the inferences, based on the normality assumption, may no longer have optimal properties and hence be misleading. Recently, econometricians and financial economists are searching for alternative estimation methods to relax the strong (parametric) assumption in data analysis. For the purpose, we may consider a generalized method of moments(GMM) estimation which needs only certain moment conditions without any parametric assumptions.

Since the GMM estimation was first introduced by Hansen (1982), its variants have been applied to numerous fields, including finance and macroeconomics. Kitamura and Philips (1997) developed the GMM for a nonstationary regression model. Quintos (1998) and Kleibergen (1999) adopted it for estimating long-run equilibriums in nonseasonal cointegration.

In this paper, we extend the work of Kleibergen to seasonal cointegration. Two iterative methods for constructing the GMM estimator are proposed according to whether parameters in the model are simultaneously estimated or not. The first (simultaneous estimation) method considers all the parameters simultaneously as in AR1994 and, then, the GMM estimator is inexplicitly given by an iterative equation which takes the same form as that for the MLE. Contrarily, the second (switching estimation) method classifies the parameters into two groups: stationary or nonstationary. If one group of parameters is provided as initial values, the other group of parameters is explicitly expressed as a function of the group. Therefore, the switching estimation can be seen as a kind of feasible two-step estimation for seasonal cointegration. These two methods are different from that of Kleibergen in that identification condition for the parameters is given before the first order moment condition of objective function is obtained. Our methods involve simpler calculations and are more efficient than that of Kleibergen.

The paper is structured as follows. In Section 2, we propose two iterative GMM estimation methods for seasonal cointegration. Their asymptotic distributions are derived in Section 3. Section 4 contains proofs of the theorems in Section 3.

2. GMM Estimation for Seasonal Cointegration

Consider the m-dimensional vector AR(VAR) model of order p as follows:

$$\Phi(L)\boldsymbol{y}_t = \left(I_m - \sum_{j=1}^{p} \Phi_j L^j\right) \boldsymbol{y}_t = \boldsymbol{\epsilon}_t, \tag{2.1}$$

where I_m denotes an $m \times m$ identity matrix and L is a lag operator, such that $L\boldsymbol{y}_t = \boldsymbol{y}_{t-1}$, and the m-dimensional random vector $\boldsymbol{\epsilon}_t$ is assumed to be independent with $\mathrm{E}(\boldsymbol{\epsilon}_t) = 0$, $\mathrm{Cov}(\boldsymbol{\epsilon}_t) = \Omega$, $\sup_t \mathrm{E}(|\epsilon_{j,t}|^{2+\delta}) < \infty$ for some $\delta > 0$ and $j = 1, 2, \ldots, m$, and $\epsilon_{j,t}$ is the j^{th} element of $\boldsymbol{\epsilon}_t$. We assume that the roots of the characteristic equation $\det(\Phi(z)) = 0$ are on or outside the unit circle.

For simplicity, it is assumed that the process $\boldsymbol{y}_t$ is observed on a quarterly basis. Models with other seasonal periods, *e.g.*, monthly, can be easily implemented as in Ahn *et al.* (2004). Then, as in

AR1994, if we expand (2.1) by Lagrange expansion at seasonal unit roots $z = 1$, -1, i and $-i$ (*i.e.*, frequencies 0, π, $\pi/2$ and $3\pi/2$, respectively), we obtain the following VECM:

$$\Phi^*(L)\left(1-L^4\right)\boldsymbol{y}_t = \alpha_1\beta_1'\boldsymbol{u}_{t-1} + \alpha_2\beta_2'\boldsymbol{v}_{t-1} + \left(\alpha_3\beta_4' + \alpha_4\beta_3'\right)\boldsymbol{w}_{t-1} + \left(-\alpha_3'\beta_3' + \alpha_4\beta_4'\right)\boldsymbol{w}_{t-2} + \boldsymbol{\epsilon}_t,$$

where

$$\boldsymbol{u}_t = (1+L)\left(1+L^2\right)\boldsymbol{y}_t, \qquad \boldsymbol{v}_t = (1-L)\left(1+L^2\right)\boldsymbol{y}_t, \qquad \boldsymbol{w}_t = \left(1-L^2\right)\boldsymbol{y}_t$$

and $\Phi^*(L)$ is a matrix polynomial of order $p-4$, and α_j and β_j are $m \times r_j$ matrices with rank equal to r_j for $j = 1, \ldots, 4$ and $r_3 = r_4$. For a unique parameterization, we need to normalize the β_j's such that

$$\beta_1' = [I_{r_1}, \beta_{10}'], \qquad \beta_2' = [I_{r_2}, \beta_{20}'], \qquad \beta_3' = [I_{r_3}, \beta_{30}'], \qquad \beta_4' = [O_{r_4}, \beta_{40}'],$$

where O_{r_j} is an $r_j \times r_j$ matrix of zeros, and β_{j0}'s are $(m-r_j)\times r_j$ matrices of unknown parameters. Note that r_1, r_2 and $r_3(r_4)$ denote the cointegrating ranks at the unit roots 1, -1 and $i(-i)$, respectively, and $\beta_1'\boldsymbol{u}_t$, $\beta_2'\boldsymbol{v}_t$, $(\beta_3' + \beta_4'L)\boldsymbol{w}_t$ and $(\beta_4' - \beta_3'L)\boldsymbol{w}_t$ are stationary processes, *i.e.*, cointegrating relationships.

The VECM can be rewritten in a compact form or regression setting to easily perform the estimation procedure:

$$\boldsymbol{z}_t = \Pi\boldsymbol{x}_{t-1} + \boldsymbol{\epsilon}_t, \tag{2.2}$$

where

$$\begin{aligned}
&\boldsymbol{z}_t = \left(1-L^4\right)\boldsymbol{y}_t, \qquad \Pi = \left[\alpha_1\beta_1', \alpha_2\beta_2', \alpha_3\beta_4' + \alpha_4\beta_3', -\alpha_3\beta_3' + \alpha_4\beta_4', \Phi_1^*, \ldots, \Phi_{p-4}^*\right], \\
&\boldsymbol{x}_{t-1} = \left[\boldsymbol{u}_{t-1}', \boldsymbol{v}_{t-1}', \boldsymbol{w}_{t-1}', \boldsymbol{w}_{t-2}', \tilde{\boldsymbol{z}}_{t-1}'\right]' \quad \text{and} \quad \tilde{\boldsymbol{z}}_{t-1} = \left[\boldsymbol{z}_{t-1}', \ldots, \boldsymbol{z}_{t-(p-4)}'\right]'.
\end{aligned}$$

We can also rewrite the equation in a matrix notation:

$$Z = \Pi X + E,$$

where

$$\begin{aligned}
Z &= [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_T], & X &= [\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{T-1}] = \left[U', V', W', W^{*\prime}, \tilde{Z}'\right]', \\
U &= [\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{T-1}] = \left[U_1', U_2'\right]', & V &= [\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{T-1}] = \left[V_1', V_2'\right]', \\
W &= [\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{T-1}] = \left[W_1', W_2'\right]', & W^* &= [\boldsymbol{w}_{-1}, \ldots, \boldsymbol{w}_{T-2}] = \left[W_1^{*\prime}, W_2^{*\prime}\right]', \\
\tilde{Z} &= [\tilde{\boldsymbol{z}}_0, \ldots, \tilde{\boldsymbol{z}}_{T-1}] \quad \text{and} & E &= [\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_T].
\end{aligned}$$

We now consider the GMM estimator for seasonal cointegration. In regression setting (2.2), we can obtain the simple orthogonal condition that the regressor of the process is orthogonal to the error. Hence, the moment condition for GMM estimation is given as

$$\mathrm{E}[m_t(\theta)] = \mathrm{E}\left[\mathrm{vec}\left(\boldsymbol{\epsilon}_t\boldsymbol{x}_{t-1}'\right)\right] = 0,$$

where $\mathrm{vec}(\cdot)$ vectorizes a matrix column-wise from left to right. By using the weighting matrix:

$$\hat{V}_T = \left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{t-1}\boldsymbol{x}_{t-1}'\right) \otimes \hat{\Omega},$$

where $\hat{\Omega}$ is a consistent estimator for Ω and $\otimes$ denotes the Kronecker product, we can obtain the objective function for the efficient GMM estimator

$$\begin{aligned} Q_T^*(\theta) &= T \cdot \text{vec}\left(\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\epsilon}_t \boldsymbol{x}_{t-1}'\right)' \hat{V}_T^{-1} \text{vec}\left(\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\epsilon}_t \boldsymbol{x}_{t-1}'\right) \\ &= \text{vec}\left(EX'\right)' \left(XX' \otimes \hat{\Omega}\right)^{-1} \text{vec}\left(EX'\right), \end{aligned} \tag{2.3}$$

where

$$\theta = \left(\theta_1', \theta_2'\right)', \quad \theta_1 = \text{vec}\left(\beta_{10}', \beta_{20}', \beta_{30}', \beta_{40}'\right) \quad \text{and} \quad \theta_2 = \text{vec}\left(\alpha_1, \ldots, \alpha_4, \Phi_1^*, \ldots, \Phi_{p-4}^*\right)'.$$

Note that θ_1 comprises nonstationary parameters including cointegrating vectors, and θ_2 stationary parameters including adjustment vectors and coefficient matrices related to short run dynamics. Now we propose two iterative GMM estimation methods for seasonal cointegration.

2.1. Simultaneous estimation method

Due to the nonlinearity of the parameters occurred by the reduced rank structure for $\Phi(1)$, $\Phi(-1)$ and $\Phi(i)$, we apply a Gauss-Newton method to minimize the objective function $Q_T^*(\theta)$. The general form of the updating equation by the Gauss-Newton method is given by

$$\begin{aligned} \theta^{(k+1)} &= \theta^{(k)} - \left[\left(\frac{\partial \bar{m}(\theta)'}{\partial \theta}\right) \hat{V}_T^{-1} \left(\frac{\partial \bar{m}(\theta)}{\partial \theta'}\right)\right]^{-1} \left[\left(\frac{\partial \bar{m}(\theta)'}{\partial \theta}\right) \hat{V}_T^{-1} \bar{m}(\theta)\right]\Bigg|_{\theta^{(k)}} \\ &\equiv \theta^{(k)} - \left[H_t(\theta)\right]^{-1} \left[G_t(\theta)\right]|_{\theta^{(k)}}, \end{aligned} \tag{2.4}$$

where $\bar{m}(\theta) = 1/T\,\text{vec}(\boldsymbol{\epsilon}X')$ and $\theta^{(k)}$ is an estimate at the previous iteration. In order to obtain the Hessian and gradient matrices, $H_t(\theta)$ and $G_t(\theta)$, respectively, in Equation (2.4), we derive $\partial\boldsymbol{\epsilon}_t/\partial\theta'$ and $\partial\bar{m}(\theta)'/\partial\theta$ as follow:

$$\frac{\partial \boldsymbol{\epsilon}_t}{\partial \theta'} = -\begin{pmatrix} \boldsymbol{u}_{2,t-1} \otimes \alpha_1' \\ \boldsymbol{v}_{2,t-1} \otimes \alpha_2' \\ \boldsymbol{w}_{2,t-1} \otimes \alpha_4' - \boldsymbol{w}_{2,t-2} \otimes \alpha_3' \\ \boldsymbol{w}_{2,t-1} \otimes \alpha_3' + \boldsymbol{w}_{2,t-2} \otimes \alpha_4' \\ \beta_1 \boldsymbol{u}_{t-1} \otimes I_m \\ \beta_2' \boldsymbol{v}_{t-1} \otimes I_m \\ (\beta_4' \boldsymbol{w}_{t-1} - \beta_3' \boldsymbol{w}_{t-2}) \otimes I_m \\ (\beta_3' \boldsymbol{w}_{t-1} + \beta_4' \boldsymbol{w}_{t-2}) \otimes I_m \\ \tilde{\boldsymbol{z}}_{t-1} \otimes I_m \end{pmatrix}',$$

$$\frac{\partial \bar{m}(\theta)'}{\partial \theta} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\partial \epsilon_t}{\partial \theta'}\right)' (\boldsymbol{x}_{t-1}' \otimes I_m) = -\frac{1}{T}\begin{pmatrix} U_2X' \otimes \alpha_1' \\ V_2X' \otimes \alpha_2' \\ W_2X' \otimes \alpha_4' - W_2^*X' \otimes \alpha_3' \\ W_2X' \otimes \alpha_3' + W_2^*X' \otimes \alpha_4' \\ \beta_1'UX' \otimes I_m \\ \beta_2'VX' \otimes I_m \\ (\beta_4'WX' - \beta_3'W^*X') \otimes I_m \\ (\beta_3'WX' + \beta_4'W^*X') \otimes I_m \\ \tilde{Z}X' \otimes I_m \end{pmatrix} \equiv -\frac{1}{T}F_1(\theta).$$

Therefore, we obtain the following updating equation

$$\begin{aligned}\theta^{(k+1)} &= \theta^{(k)} - \left[\left(\frac{\partial\bar{m}(\theta')}{\partial\theta}\right)\hat{V}_T^{-1}\left(\frac{\partial\bar{m}(\theta)}{\partial\theta'}\right)\right]^{-1}\left[\left(\frac{\partial\bar{m}(\theta')}{\partial\theta}\right)\hat{V}_T^{-1}\bar{m}(\theta)\right]\Bigg|_{\theta^{(k)}} \\ &= \theta^{(k)} + \left(F_1(\theta)\left(XX'\otimes\hat{\Omega}_T\right)^{-1}F_1(\theta)'\right)^{-1}\left(F_1(\theta)\left(XX'\otimes\hat{\Omega}_T\right)^{-1}\bar{m}(\theta)\right)\Bigg|_{\theta^{(k)}} \end{aligned} \tag{2.5}$$

since $\hat{V}_T = XX'/T\otimes\hat{\Omega}$. This equation coincides in form to the Newton-Raphson equation for the MLE in AR1994.

2.2. Switching estimation method

Switching method first classifies the unknown parameters into two parameter sets, θ_1 and θ_2, and estimates them alternately by using a given parameter set. Specifically, the method is done by the two following steps. In the first step, give the initial estimators of θ_2 and Ω. Then, the VECM can be represented as follows:

$$\dot{\boldsymbol{z}}_t = \alpha_1\beta_{10}'\boldsymbol{u}_{2,t-1} + \alpha_2\beta_{20}'\boldsymbol{v}_{2,t-1} + \alpha_{30}\beta_{30}'\boldsymbol{w}_{2,t-1} + \alpha_{40}\beta_{40}'\boldsymbol{w}_{2,t-2} + \boldsymbol{\epsilon}_t, \tag{2.6}$$

where

$$\begin{aligned}\dot{\boldsymbol{z}} &= \boldsymbol{z}_t - \alpha_1\boldsymbol{u}_{1,t-1} - \alpha_2\boldsymbol{v}_{1,t-1} - \alpha_4\boldsymbol{w}_{1,t-1} + \alpha_3\boldsymbol{w}_{1,t-2} - \Phi_1^*\boldsymbol{z}_{t-1} - \cdots - \Phi_{(p-4)}^*\boldsymbol{z}_{t-p+4} \\ &= \boldsymbol{z}_t - \alpha_1\boldsymbol{u}_{1,t-1} - \alpha_2\boldsymbol{v}_{1,t-1} - \alpha_4\boldsymbol{w}_{1,t-1} + \alpha_3\boldsymbol{w}_{1,t-2} - \Phi^*\tilde{\boldsymbol{z}}_{t-1}.\end{aligned}$$

Therefore, we can construct the moment condition function as

$$\bar{m}(\theta_1) = \frac{1}{T}\sum_{t=1}^{T}\mathrm{vec}(\boldsymbol{\epsilon}_t\boldsymbol{x}_{t-1}') = \frac{1}{T}\sum_{t=1}^{T}\mathrm{vec}(\dot{\boldsymbol{z}}_t\boldsymbol{x}_{t-1}') - \frac{1}{T}F_2\theta_1,$$

where

$$F_2 = \begin{bmatrix}\sum_{t=1}^{T}(\boldsymbol{x}_{t-1}\boldsymbol{u}_{2,t-1}'\otimes\alpha_1)' \\ \sum_{t=1}^{T}(\boldsymbol{x}_{t-1}\boldsymbol{v}_{2,t-1}'\otimes\alpha_2)' \\ \sum_{t=1}^{T}(\boldsymbol{x}_{t-1}\boldsymbol{w}_{2,t-1}'\otimes\alpha_4 - \boldsymbol{x}_{t-1}\boldsymbol{w}_{2,t-2}'\otimes\alpha_3)' \\ \sum_{t=1}^{T}(\boldsymbol{x}_{t-1}\boldsymbol{w}_{2,t-1}'\otimes\alpha_3 + \boldsymbol{x}_{t-1}\boldsymbol{w}_{2,t-2}'\otimes\alpha_4)'\end{bmatrix}'.$$

Since $\bar{m}(\theta_1)$ is linear in θ_1, the objective function is quadratic in θ_1,

$$Q_T^*(\theta_1) = \left(\frac{1}{T}\sum_{t=1}^{T}\mathrm{vec}(\dot{\boldsymbol{z}}_t\boldsymbol{x}_{t-1}') - \frac{1}{T}F_2\theta_1\right)'\hat{V}_T^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\mathrm{vec}(\dot{\boldsymbol{z}}_t\boldsymbol{x}_{t-1}) - \frac{1}{T}F_2\theta_1\right).$$

The first order condition for minimizing the function with respect to θ_1 is

$$F_2'\hat{V}_T^{-1}\sum_{t=1}^{T}\mathrm{vec}(\dot{\boldsymbol{z}}_t\boldsymbol{x}_{t-1}') = F_2'\hat{V}_T^{-1}F_2\theta_1.$$

Then, the efficient GMM estimator for θ_1 is

$$\tilde{\theta}_1 = \left(F_2'\left(XX' \otimes \hat{\Omega}\right)^{-1} F_2\right)^{-1} \left(F_2'\left(XX' \otimes \hat{\Omega}\right)^{-1} \sum_{t=1}^{T} \mathrm{vec}\left(\dot{\boldsymbol{z}}_t \boldsymbol{x}_{t-1}\right)\right). \tag{2.7}$$

In the second step, by using the estimator of θ_1 obtained in the first step, we construct the moment condition for θ_2:

$$\bar{m}(\theta_2) = \frac{1}{T}\sum_{t=1}^{T} \mathrm{vec}\left(\boldsymbol{\epsilon}_t \boldsymbol{x}_{t-1}'\right) = \frac{1}{T}\sum_{t=1}^{T} \mathrm{vec}(\boldsymbol{z}_t \boldsymbol{x}_{t-1}') - \frac{1}{T} F_3 \theta_2,$$

where

$$F_3 = \begin{bmatrix} \sum_{t=1}^{T} (\boldsymbol{x}_{t-1}\boldsymbol{u}_{t-1}'\beta_1 \otimes \alpha_1)' \\ \sum_{t=1}^{T} (\boldsymbol{x}_{t-1}\boldsymbol{v}_{t-1}'\beta_2 \otimes \alpha_2)' \\ \sum_{t=1}^{T} ((\boldsymbol{x}_{t-1}\boldsymbol{w}_{t-1}'\beta_4 - \boldsymbol{x}_{t-1}\boldsymbol{w}_{t-2}'\beta_3) \otimes I_m)' \\ \sum_{t=1}^{T} ((\boldsymbol{x}_{t-1}\boldsymbol{w}_{t-1}'\beta_3 + \boldsymbol{x}_{t-1}\boldsymbol{w}_{t-2}'\beta_4) \otimes I_m)' \\ \sum_{t=1}^{T} (\boldsymbol{x}_{t-1}\tilde{\boldsymbol{z}}_{t-1}) \otimes I_m \end{bmatrix}'.$$

Similarly to the first step, since $\bar{m}(\theta_2)$ is linear in θ_2, the GMM estimator for θ_2 is given by

$$\theta_2 = \left(F_3'\left(XX' \otimes \hat{\Omega}\right)^{-1} F_3\right)^{-1} \left(F_3'\left(XX' \otimes \hat{\Omega}\right)^{-1} \sum_{t=1}^{T} \mathrm{vec}\left(\boldsymbol{z}_t \boldsymbol{x}_{t-1}'\right)\right). \tag{2.8}$$

These two steps are repeated until the parameter estimates converge.

The switching method is noticeable in that in each step it proposes a closed form of estimators. In addition it can be seen to be a feasible two-step estimator for seasonal cointegration. Therefore, as mentioned in Seong (2008), it can be an alternative of the MLE because it does not produce outlying estimates observed for the MLE.

We remark that when the data dimension(m) or VAR order(p) is large, the switching method is preferred since the Gauss-Newton iteration of the simultaneous method, in Section 2.1, is more likely to fail.

3. Asymptotic Distribution of GMM Estimator

In this section we derive the asymptotic distribution for the GMM estimator in the following theorems with their proofs given in Section 4.

Theorem 3.1. *Let $\hat{\theta}$ denote the simultaneous GMM estimator for θ obtained from the updating Equation (2.4). If we use initial consistent estimators, then*

$$T(\hat{\beta}_{10}' - \beta_{10}') \xrightarrow{d} (\alpha_1'\Omega^{-1}\alpha_1)^{-1}\alpha_1'\Omega^{-1}\tilde{G}_1' F_{11}^{*\,-1},$$

$$T(\hat{\beta}_{20}' - \beta_{20}') \xrightarrow{d} (\alpha_2'\Omega^{-1}\alpha_2)^{-1}\alpha_2'\Omega^{-1}\tilde{G}_2' F_{22}^{*\,-1},$$

$$T\begin{pmatrix} \hat{\beta}_{30}' - \beta_{30}' \\ \hat{\beta}_{40}' - \beta_{40}' \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \tilde{F_{33}} & \tilde{F_{34}} \\ \tilde{F_{43}} & \tilde{F_{44}} \end{pmatrix}^{-1} \begin{pmatrix} vec(\tilde{G}_3\Omega^{-1}\alpha_4 - \tilde{G}_4\Omega^{-1}\alpha_3) \\ vec(\tilde{G}_3\Omega^{-1}\alpha_3 - \tilde{G}_4\Omega^{-1}\alpha_4) \end{pmatrix}$$

and

$$\sqrt{T}(\hat{\theta}_2 - \theta_2) \xrightarrow{d} N^*,$$

where $\xrightarrow{d}$ *denotes the convergence in distribution and vec(N^*) is a normal random vector with mean 0 and covariance matrix* $\Omega \otimes \Gamma_{\tilde{x}}^{-1}$ *for* $\Gamma_{\tilde{x}} = Cov(\tilde{x}_{t-1})$ *and* $\tilde{x}_{t-1} = [\boldsymbol{u}'_{t-1}\beta_1, \boldsymbol{v}'_{t-1}\beta_2, \boldsymbol{w}'_{t-1}\beta_4 - \boldsymbol{w}'_{t-2}\beta_3, \boldsymbol{w}'_{t-1}\beta_3 + \boldsymbol{w}'_{t-2}\beta_4, \tilde{\boldsymbol{z}}'_{t-1}]'$.

Theorem 3.2. *Let* $\tilde{\theta}$ *denote the switching GMM estimator for* θ *obtained from (2.7) and (2.8). If initial consistent estimators are given, the asymptotic distribution of* $\tilde{\theta}$ *is the same as that in Theorem 3.1.*

We remark that the GMM estimators are asymptotically equivalent to the MLE; however, their the finite-sample properties may be very different (Seong, 2008).

4. Proof for Theorems

4.1. Proof of Theorem 3.1

Since $M = I - X'(XX')^{-1}X$ and $P = X'(XX')^{-1}X$ are idempotent and symmetric and $XP = X$, $UP = U$, $VP = V$, $WP = W$ and $W^*P = W^*$ are satisfied. Using these properties, the first part of updating Equation (2.4) can be represented as

$$F_1\left(XX' \otimes \hat{\Omega}\right)^{-1} F_1' = \begin{pmatrix} U_2 \otimes \alpha_1'\Omega^{-1} \\ V_2 \otimes \alpha_2'\Omega^{-1} \\ W_2 \otimes \alpha_4'\Omega^{-1} - W_2^* \otimes \alpha_3'\Omega^{-1} \\ W_2 \otimes \alpha_3'\Omega^{-1} + W_2^* \otimes \alpha_4'\Omega^{-1} \\ \beta_1'U \otimes \Omega^{-1} \\ \beta_2'V \otimes \Omega^{-1} \\ (\beta_4'W - \beta_3'W^*) \otimes \Omega^{-1} \\ (\beta_3'W + \beta_4'W^*) \otimes \Omega^{-1} \\ \tilde{Z}X'(XX')^{-1} \otimes \Omega^{-1} \end{pmatrix} \begin{pmatrix} U_2 \otimes \alpha_1' \\ V_2 \otimes \alpha_2' \\ W_2 \otimes \alpha_4' - W_2^* \otimes \alpha_3' \\ W_2 \otimes \alpha_3' + W_2^* \otimes \alpha_4' \\ \beta_1'U \otimes I_m \\ \beta_2'V \otimes I_m \\ (\beta_4'W - \beta_3'W^*) \otimes I_m \\ (\beta_3'W + \beta_4'W^*) \otimes I_m \\ X\tilde{Z}' \otimes I_m \end{pmatrix}'.$$

The second part of updating Equation (2.4) can be represented as

$$F_1\left(XX' \otimes \hat{\Omega}\right)^{-1} \mathrm{vec}\left(\sum_{t=1}^{T} \boldsymbol{\epsilon}_t \boldsymbol{x}'_{t-1}\right) = \begin{pmatrix} U_2X'(XX')^{-1} \otimes \alpha_1'\Omega^{-1} \\ V_2X'(XX')^{-1} \otimes \alpha_2'\Omega^{-1} \\ W_2X'(XX')^{-1} \otimes \alpha_4'\Omega^{-1} - W_2^*X'(XX')^{-1} \otimes \alpha_3'\Omega^{-1} \\ W_2X'(XX')^{-1} \otimes \alpha_3'\Omega^{-1} + W_2^*X'(XX')^{-1} \otimes \alpha_4'\Omega^{-1} \\ \beta_1'UX'(XX')^{-1} \otimes \Omega^{-1} \\ \beta_2'VX'(XX')^{-1} \otimes \Omega^{-1} \\ \left(\beta_4'WX'(XX')^{-1} - \beta_3'W^*X'(XX')^{-1}\right) \otimes \Omega^{-1} \\ \left(\beta_3'WX'(XX')^{-1} + \beta_4'W^*X'(XX')^{-1}\right) \otimes \Omega^{-1} \\ \left(\tilde{Z}X' \otimes I_m\right) \end{pmatrix} \mathrm{vec}(EX')$$

$$
= \begin{pmatrix}
\mathrm{vec}\left(\alpha_1'\Omega^{-1}EX'(XX')^{-1}XU_2'\right) \\
\mathrm{vec}\left(\alpha_2'\Omega^{-1}EX'(XX')^{-1}XV_2'\right) \\
\mathrm{vec}\left(\alpha_4'\Omega^{-1}EX'(XX')^{-1}XW_2' - \alpha_3'\Omega^{-1}EX'(XX')^{-1}XW_2^{*'}\right) \\
\mathrm{vec}\left(\alpha_3'\Omega^{-1}EX'(XX')^{-1}XW_2' + \alpha_4'\Omega^{-1}EX'(XX')^{-1}XW_2^{*'}\right) \\
\mathrm{vec}\left(\Omega^{-1}EX'(XX')^{-1}XU'\beta_1\right) \\
\mathrm{vec}\left(\Omega^{-1}EX'(XX')^{-1}XV'\beta_2\right) \\
\mathrm{vec}\left(\Omega^{-1}EX'(XX')^{-1}X\left(W'\beta_4 - W^{*'}\beta_3\right)\right) \\
\mathrm{vec}\left(\Omega^{-1}EX'(XX')^{-1}X\left(W'\beta_3 + W^{*'}\beta_4\right)\right) \\
\mathrm{vec}\left(EX'(XX')^{-1}X\tilde{Z}\right)
\end{pmatrix}.
$$

Therefore, we obtain the expression:

$$
F_1\left(XX' \otimes \hat{\Omega}\right)^{-1}\mathrm{vec}\left(\sum_{t=1}^{T}\boldsymbol{\epsilon}_t\boldsymbol{x}_{t-1}'\right) = \begin{pmatrix}
\mathrm{vec}\left(\alpha_1'\Omega^{-1}EU_2'\right) \\
\mathrm{vec}\left(\alpha_2'\Omega^{-1}EV_2'\right) \\
\mathrm{vec}\left(\alpha_4'\Omega^{-1}EW_2' - \alpha_3'\Omega^{-1}EW_2^{*'}\right) \\
\mathrm{vec}\left(\alpha_3'\Omega^{-1}EW_2' + \alpha_4'\Omega^{-1}EW_2^{*'}\right) \\
\mathrm{vec}\left(\Omega^{-1}EU'\beta_1\right) \\
\mathrm{vec}\left(\Omega^{-1}EV'\beta_2\right) \\
\mathrm{vec}\left(\Omega^{-1}E\left(W'\beta_4 - W^{*'}\beta_3\right)\right) \\
\mathrm{vec}\left(\Omega^{-1}E\left(W'\beta_3 + W^{*'}\beta_4\right)\right) \\
\mathrm{vec}\left(E\tilde{Z}'\right)
\end{pmatrix}.
$$

The asymptotic distributions can be derived by Lemma 1 in AR1994 (p.327). Let $D = \mathrm{diag}(D_1, D_2)$ where $D_1 = \mathrm{diag}(TI_{r_1(m-r_1)}, TI_{r_2(m-r_2)}, TI_{2r_3(m-r_3)})$, $D_2 = \mathrm{diag}(T^{1/2}I_b)$ and b is the dimension of θ_2. By using the expressions of $\partial\boldsymbol{\epsilon}_t/\partial\theta'$ and $\partial\bar{m}(\theta)'/\partial\theta$ in Section 2.1, we obtain the following asymptotic properties:

$$
D^{-1}F_1\left(XX' \otimes \hat{\Omega}\right)^{-1}F_1'D^{-1} \xrightarrow{d} \begin{pmatrix}
\tilde{F}_{11} & 0 & 0 & 0 & 0 \\
0 & \tilde{F}_{22} & 0 & 0 & 0 \\
0 & 0 & \tilde{F}_{33} & \tilde{F}_{34} & 0 \\
0 & 0 & \tilde{F}_{43} & \tilde{F}_{44} & 0 \\
0 & 0 & 0 & 0 & \Gamma_{\tilde{X}} \otimes \Omega^{-1}
\end{pmatrix}
$$

and

$$
D^{-1}F_1\left(XX' \otimes \hat{\Omega}\right)^{-1}\mathrm{vec}\left(\sum_{t=1}^{T}\boldsymbol{\epsilon}_t\boldsymbol{x}_{t-1}'\right) \xrightarrow{d} \begin{pmatrix}
\mathrm{vec}\left(\alpha_1'\Omega^{-1}G_1'J_1\right) \\
\mathrm{vec}\left(\alpha_2'\Omega^{-1}G_2'J_2\right) \\
\mathrm{vec}\left(\alpha_4'\Omega^{-1}G_3'J_i - \alpha_3'\Omega^{-1}G_4'J_i\right) \\
\mathrm{vec}\left(\alpha_3'\Omega^{-1}G_3'J_i + \alpha_4'\Omega^{-1}G_4'J_i\right) \\
\mathrm{vec}\left(N^*\right)
\end{pmatrix},
$$

where

$$\tilde{F}_{11} = J_1'F_{11}J_1 \otimes \alpha_1'\Omega^{-1}\alpha_1,$$

$$\tilde{F}_{22} = J_{-1}' F_{22} J_{-1} \otimes \alpha_2' \Omega^{-1} \alpha_2,$$
$$\tilde{F}_{33} = J_i' F_{33} J_i \otimes \alpha_4' \Omega^{-1} \alpha_4 - J_i' F_{34} J_i \otimes \alpha_4' \Omega^{-1} \alpha_3 - J_i' F_{43} J_i \otimes \alpha_3' \Omega^{-1} \alpha_4 + J_i' F_{44} J_i \otimes \alpha_3' \Omega^{-1} \alpha_3,$$
$$\tilde{F}_{34} = J_i' F_{33} J_i \otimes \alpha_4' \Omega^{-1} \alpha_3 + J_i' F_{34} J_i \otimes \alpha_4' \Omega^{-1} \alpha_4 - J_i' F_{43} J_i \otimes \alpha_3' \Omega^{-1} \alpha_3 - J_i' F_{44} J_i \otimes \alpha_3' \Omega^{-1} \alpha_4,$$
$$\tilde{F}_{43} = \tilde{F}_{34}'$$

and

$$\tilde{F}_{44} = J_i' F_{33} J_i \otimes \alpha_3' \Omega^{-1} \alpha_3 + J_i' F_{34} J_i \otimes \alpha_3' \Omega^{-1} \alpha_4 + J_i' F_{43} J_i \otimes \alpha_4' \Omega^{-1} \alpha_3 + J_i' F_{44} J_i \otimes \alpha_4' \Omega^{-1} \alpha_4.$$

4.2. Proof of Theorem 2.2

When consistent estimators for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, Φ_j^*'s and Ω are given, we obtain

$$D_1\left(\tilde{\theta}_1 - \theta_1\right) = \left(D_1^{-1} F_2' \left(XX' \otimes \hat{\Omega}\right)^{-1} F_2 D_1^{-1}\right)^{-1} \left(D_1^{-1} F_2' \left(XX' \otimes \hat{\Omega}\right)^{-1}\right) \sum_{t=1}^{T} \operatorname{vec}\left(\boldsymbol{\epsilon}_t \boldsymbol{x}_{t-1}'\right). \quad (4.1)$$

Then, by using the lemma in AR1994, asymptotic property of the first part of Equation (4.1) can be derived as follows:

$$\left(D_1^{-1} F_2' \left(XX' \otimes \hat{\Omega}\right)^{-1} F_2 D^{-1}\right)^{-1}$$
$$= \left(D_1^{-1} \begin{bmatrix} U_2 X' \otimes \alpha_1' \\ V_2 X' \otimes \alpha_2' \\ W_2 X' \otimes \alpha_4' - W_2^* X' \otimes \alpha_3 \\ W_2 X' \otimes \alpha_3' + W_2^* X' \otimes \alpha_4' \end{bmatrix} \left(XX \otimes \hat{\Omega}\right)^{-1} \begin{bmatrix} U_2 X' \otimes \alpha_1' \\ V_2 X' \otimes \alpha_2' \\ W_2 X' \otimes \alpha_4' - W_2^* X' \otimes \alpha_3 \\ W_2 X' \otimes \alpha_3' + W_2^* X' \otimes \alpha_4' \end{bmatrix}' D^{-1} \right)^{-1}$$
$$\xrightarrow{d} \begin{bmatrix} \tilde{F}_{11} & & & \\ & \tilde{F}_{22} & & \\ & & \tilde{F}_{33} & \tilde{F}_{34} \\ & & \tilde{F}_{43} & \tilde{F}_{44} \end{bmatrix}^{-1}.$$

Similarly, asymptotic distribution of the second part of Equation (4.1) is given as follows:

$$D_1^{-1} F_2' \left(XX' \otimes \hat{\Omega}\right)^{-1} \sum_{t=1}^{T} \operatorname{vec}\left(\boldsymbol{\epsilon}_t \boldsymbol{x}_{t-1}'\right) = D_1^{-1} \begin{pmatrix} \operatorname{vec}\left(\alpha_1' \hat{\Omega}^{-1} E U_2'\right) \\ \operatorname{vec}\left(\alpha_2' \hat{\Omega}^{-1} E V_2'\right) \\ \operatorname{vec}\left(\alpha_4' \hat{\Omega}^{-1} E W_2' - \alpha_3' \hat{\Omega}^{-1} E W_2^{*\prime}\right) \\ \operatorname{vec}\left(\alpha_3' \hat{\Omega}^{-1} E W_2' + \alpha_4' \hat{\Omega}^{-1} E W_2^{*\prime}\right) \end{pmatrix}$$
$$\xrightarrow{d} \begin{pmatrix} \operatorname{vec}\left(\alpha_1' \hat{\Omega}^{-1} G_1' J_1\right) \\ \operatorname{vec}\left(\alpha_2' \hat{\Omega}^{-1} G_2' J_2\right) \\ \operatorname{vec}\left(\alpha_4' \hat{\Omega}^{-1} G_3' J_i - \alpha_3' \hat{\Omega}^{-1} G_4' J_i\right) \\ \operatorname{vec}\left(\alpha_3' \hat{\Omega}^{-1} G_3' J_i + \alpha_4' \hat{\Omega}^{-1} G_4' J_i\right) \end{pmatrix}.$$

Alternately, when the consistent estimators for $\beta_1, \beta_2, \beta_3, \beta_4$ and Ω are given,

$$D_2\left(\tilde{\theta}_2 - \theta_2\right) = \left(D_2^{-1} F_3' \left(XX' \otimes \hat{\Omega}\right)^{-1} F_3 D_2^{-1}\right)^{-1} \left(D_2^{-1} F_3' \left(XX' \otimes \hat{\Omega}\right)^{-1} \sum_{t=1}^{T} \operatorname{vec}\left(\boldsymbol{\epsilon}_t \boldsymbol{x}_{t-1}'\right)\right). \quad (4.2)$$

Then, asymptotic distribution of the first part of Equation (4.2) is given as follows:

$$\left(D_2^{-1} F_3' \left(XX' \otimes \hat{\Omega}\right)^{-1} F_3 D_2^{-1}\right)^{-1}$$

$$= \left(D_2^{-1} \begin{bmatrix} \beta_1' U X' \otimes I_m \\ \beta_2' V X' \otimes I_m \\ (\beta_4' W X' - \beta_3' W^* X') \otimes I_m \\ (\beta_3' W X' + \beta_4' W^* X') \otimes I_m \\ \tilde{Z} X \otimes I_m \end{bmatrix} \left(XX' \otimes \hat{\Omega}\right)^{-1} \begin{bmatrix} \beta_1' U X' \otimes I_m \\ \beta_2' V X' \otimes I_m \\ (\beta_4' W X' - \beta_3' W^* X') \otimes I_m \\ (\beta_3' W X' + \beta_4' W^* X') \otimes I_m \\ \tilde{Z} X' \otimes I_m \end{bmatrix}' D_2^{-1} \right)^{-1}$$

$$\xrightarrow{d} \Gamma_{\tilde{X}} \otimes \Omega^{-1}$$

and that of the second part is given as follows:

$$D_2^{-1} F_3' \left(XX' \otimes \hat{\Omega}\right)^{-1} \sum_{t=1}^{T} \text{vec}\left(\boldsymbol{\epsilon}_t \boldsymbol{x}_{t-1}'\right) = D_2^{-1} \begin{bmatrix} \text{vec}\left(\hat{\Omega}^{-1} E U' \beta_1\right) \\ \text{vec}\left(\hat{\Omega}^{-1} E V' \beta_2\right) \\ \text{vec}\left(\hat{\Omega}^{-1} E (W' \beta_4 - W^{*\prime} \beta_3)\right) \\ \text{vec}\left(\hat{\Omega}^{-1} E (W' \beta_3 + W^{*\prime} \beta_4)\right) \\ \text{vec}\left(E \tilde{Z}'\right) \end{bmatrix}$$

$$\xrightarrow{d} \text{vec}(N^*).$$

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