

BIFURCATION ANALYSIS OF A DELAYED EPIDEMIC MODEL WITH DIFFUSION

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ABSTRACT. In this paper, a class of delayed epidemic model with diffusion is investigated. By analyzing the associated characteristic transcendental equation, its linear stability is investigated and Hopf bifurcation is demonstrated. Some explicit formulae determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory. Some numerical simulation are also carried out to support our analytical findings. Finally, biological explanations and main conclusions are given.

1. Introduction

In recent years, there has been a growing interest in the study of the dynamical behavior (including stable, unstable, persistent and oscillatory behavior) of epidemic models which plays an important role in the study of mathematical epidemiology. Great attention has been paid to the dynamics properties of the epidemic models which have significant biological background. Many excellent and interesting results have been obtained [1–5, 7, 9–11, 13–14, 16–21, 25–26]. In 1997, Zhang and Fang [24] investigated dispersal properties of the following epidemic model:

$$(1) \quad \begin{cases} \dot{s}_1 = -k_1 s_1 I_1 + s_1(a_1 - s_1) + \varepsilon(s_2 - s_1), \\ \dot{s}_2 = s_2(a_2 - s_2) + \varepsilon(s_1 - s_2), \\ \dot{I}_1 = k_1 s_1 I_1 - bI_1 - cI_1, \end{cases}$$

where s_1, s_2 represent the susceptible numbers in place 1 at time t and the susceptible numbers in place 2 at time t , respectively, I_1 denotes the infective numbers in place 1 at time t , k is infective coefficient, $k s_1$ is infective rate, a_1, a_2 are the carrying capacities in place 1 and in place 2, respectively, b is

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death rate of infective numbers, c is eliminative rate of the infective numbers, c is diffusion rate between the susceptible numbers. All the parameters are positive constants.

Considering the factor that the infective numbers in place 1 will stay latent before becoming infectious, we think that time delay has important biologic meaning in epidemic models and so it is more plausible to incorporate the time delay into the model. Based on the point of view, we revise model (1) into the following delayed epidemic model with diffusion:

$$(2) \quad \begin{cases} \dot{s}_1 = -k_1 s_1 I_1(t - \tau) + s_1(a_1 - s_1) + \varepsilon(s_2 - s_1), \\ \dot{s}_2 = s_2(a_2 - s_2) + \varepsilon(s_1 - s_2), \\ \dot{I}_1 = k_1 s_1 I_1(t - \tau) - b I_1(t - \tau) - c I_1(t - \tau), \end{cases}$$

where all the parameters are positive constants. The more detail biological meaning of the coefficients of the system (2) are same as those in (1) and one can see [24].

In this paper, we study the stability, the local Hopf bifurcation for system (2). To the best of our knowledge, it is the first time to deal with the research of Hopf bifurcation for the model (2).

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the boundary equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Biological explanations and some main conclusions are drawn in Section 5.

2. Stability of the boundary equilibrium and local Hopf bifurcations

In this section, we shall study the stability of the boundary equilibrium and the existence of local Hopf bifurcations. Throughout the paper, we assume that (H1). The equation

$$(3) \quad \begin{cases} s_1(a_1 - s_1) + \varepsilon(s_2 - s_1) = 0, \\ s_2(a_2 - s_2) + \varepsilon(s_1 - s_2) = 0 \end{cases}$$

has an unique positive root.

It is easy to see that if the condition (H1) holds, then Eq.(2) has an unique boundary equilibrium $E_0(s_1^*, s_2^*, I_1^*)$, where $I_1^* = 0$ and s_1^*, s_2^* are the positive roots of Eq.(3).

Let $\bar{s}_1(t) = s_1(t) - s_1^*, \bar{s}_2(t) = s_2(t) - s_2^*, \bar{I}_1(t) = I_1(t)$ and still denote $\bar{s}_i(t) (i = 1, 2), \bar{I}_1(t)$ by $s_i(t) (i = 1, 2), I_1(t)$, respectively. Then (2) becomes

$$(4) \quad \begin{cases} \dot{s}_1(t) = (a_1 - 2s_1^* - \varepsilon)s_1 + \varepsilon s_2 - k s_1^* I_1(t - \tau) - s_1^2 - k s_1 I_1(t - \tau), \\ \dot{s}_2(t) = \varepsilon s_1 + (a_2 - 2s_2^* - \varepsilon)s_2 - s_2^2, \\ \dot{I}_1(t) = (k s_1^* - b - c)I_1(t - \tau) + k s_1 I_1(t - \tau), \end{cases}$$

The linearization of Eq.(4) at $(0, 0, 0)$ is

$$(5) \quad \begin{cases} \dot{s}_1(t) = (a_1 - 2s_1^* - \varepsilon)s_1 + \varepsilon s_2 - ks_1^* I_1(t - \tau), \\ \dot{s}_2(t) = \varepsilon s_1 + (a_2 - 2s_2^* - \varepsilon)s_2, \\ \dot{I}_1(t) = (ks_1^* - b - c)I_1(t - \tau) \end{cases}$$

whose characteristic equation is

$$(6) \quad [\lambda - (ks_1^* - b - c)e^{-\lambda\tau}]\{[\lambda - (a_1 - 2s_1^* - \varepsilon)][\lambda - (a_2 - 2s_2^* - \varepsilon)] - \varepsilon^2\} = 0.$$

Obviously, we have

$$(7) \quad \lambda - (ks_1^* - b - c)e^{-\lambda\tau} = 0$$

or

$$(8) \quad [\lambda - (a_1 - 2s_1^* - \varepsilon)][\lambda - (a_2 - 2s_2^* - \varepsilon)] - \varepsilon^2 = 0.$$

In the following, we need to investigate the distribution of roots of Eq.(7) and Eq.(8). In order to investigate the distribution of roots of the transcendental equation Eq.(7) and Eq.(8), the following lemma is useful.

Lemma 2.1 ([5]). *For the transcendental equation*

$$\begin{aligned} P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \dots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ [p_1^{(1)}\lambda^{n-1} + \dots + p_{n-1}^{(1)}\lambda + p_n^{(1)}] e^{-\lambda\tau_1} + \dots \\ &+ [p_1^{(m)}\lambda^{n-1} + \dots + p_{n-1}^{(m)}\lambda + p_n^{(m)}] e^{-\lambda\tau_m} = 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \dots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

For $\tau = 0$, (7) deduces to

$$(9) \quad \lambda = ks_1^* - b - c.$$

It is easy to see that $\lambda = ks_1^* - b - c < 0$ if the condition:

$$(H2) \quad ks_1^* < b + c$$

holds.

For $\omega_0 > 0$, $i\omega_0$ is a root of (7) if and only if

$$i\omega_0 = (ks_1^* - b - c)(\cos \omega_0\tau - i \sin \omega_0\tau).$$

Separating the real and imaginary parts, we get

$$(10) \quad \begin{cases} (ks_1^* - b - c) \cos \omega_0\tau = 0, \\ (ks_1^* - b - c) \sin \omega_0\tau = -\omega_0. \end{cases}$$

Then we obtain

$$(11) \quad \omega_0 = -(ks_1^* - b - c), \quad \tau_k = -\frac{1}{ks_1^* - b - c} \left(2k\pi + \frac{\pi}{2}\right) \quad (k = 0, 1, 2, \dots).$$

In the sequel, we consider the roots of Eq.(8). Obviously, (8) can be translated into the following form:

$$(12) \quad \lambda^2 - [a_1 + a_2 - 2(s_1^* + s_2^*) - 2\varepsilon]\lambda + (a_1 - 2s_1^* - \varepsilon)(a_2 - 2s_2^* - \varepsilon) - \varepsilon^2 = 0.$$

A set of necessary and sufficient conditions for all roots of (12) to have a negative real part is given in the following form:

$$(H3) \quad a_1 + a_2 - 2(s_1^* + s_2^*) - 2\varepsilon < 0, \quad (a_1 - 2s_1^* - \varepsilon)(a_2 - 2s_2^* - \varepsilon) - \varepsilon^2 > 0.$$

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (7) near $\tau = \tau_k$, and $\alpha(\tau_k) = 0$, and $\omega(\tau_k) = \omega_0$. Due to functional differential equation theory, for every $\tau_k, k = 0, 1, 2, 3, \dots$, there exists $\epsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k| < \epsilon$. Substituting $\lambda(\tau)$ into the left hand of (7) and taking derivative with respect to τ , we have

$$(13) \quad \left[\frac{d\lambda}{d\tau} \right]^{-1} = -\frac{e^{\lambda\tau}}{\lambda(ks_1^* - b - c)} - \frac{\tau}{\lambda}.$$

Then we obtain

$$(14) \quad \left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau} \right]_{\tau=\tau_k}^{-1} = -\frac{\sin \omega_0 \tau_k}{\omega_0(ks_1^* - b - c)} > 0.$$

The above analysis leads to the following results on the stability and Hopf bifurcation.

Theorem 2.2. *If (H1)-(H3) hold, then the equilibrium $E_0(s_1^*, s_2^*, 0)$ of system (2) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Moreover, the system (2) undergoes a Hopf bifurcation at the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ when $\tau = \tau_k, k = 0, 1, 2, 3, \dots$*

3. Direction and stability of the Hopf bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k, k = 0, 1, 2, \dots$. In this section, we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ at these critical value of τ , by using techniques from normal form and center manifold theory [8]. Throughout this section, we always assume that the system (2) undergoes Hopf bifurcation at the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ for $\tau = \tau_k, k = 0, 1, 2, \dots$, and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation at the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$.

For convenience, let $\bar{s}_i(t) = s_i(\tau t) (i = 1, 2)$ and $\bar{I}_1(t) = I_1(\tau t)$, $\tau = \tau_k + \mu$, where τ_k is defined by (11) and $\mu \in R$, drop the bar for the simplification of notations, then the system (4) can be written as an FDE in $C = C([-1, 0], R^3)$ as

$$(15) \quad \dot{u}(t) = L_\mu(u_t) + F(\mu, u_t),$$

where $u(t) = (s_1(\tau t), s_2(\tau t), I_1(\tau t))^T \in C$ and $u_t(\theta) = u(t + \theta) = (s_1(t + \theta), s_2(t + \theta), I_1(t + \theta))^T \in C$, and $L_\mu : C \rightarrow R, F : R \times C \rightarrow R$ are given by

$$\begin{aligned}
 (16) \quad L_\mu \phi &= (\tau_k + \mu) \begin{pmatrix} a_1 - 2s_1^* - \varepsilon & \varepsilon & 0 \\ \varepsilon & a_2 - 2s_2^* - \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\
 &+ (\tau_k + \mu) \begin{pmatrix} 0 & 0 & -ks_1^* \\ 0 & 0 & 0 \\ 0 & 0 & ks_1^* - b - c \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}
 \end{aligned}$$

and

$$(17) \quad F(\mu, \phi) = (\tau_k + \mu) \begin{pmatrix} -\phi_1^2(0) - k\phi_1(0)\phi_3(-1) \\ -\phi_2^2(0) \\ k\phi_1(0)\phi_3(-1) \end{pmatrix},$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$.

From the discussion in Section 2, we know that if $\mu = 0$, then the system (3.1) undergoes a Hopf bifurcation at the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ and the associated characteristic equation of the system (15) has a pair of simple imaginary roots $\pm i\omega_0\tau_k$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu), \theta \in [-1, 0]$ such that

$$(18) \quad L_\mu \phi = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta) \text{ for } \phi \in C.$$

In fact, we can choose

$$\begin{aligned}
 (19) \quad \eta(\theta, \mu) &= (\tau_k + \mu) \begin{pmatrix} a_1 - 2s_1^* - \varepsilon & \varepsilon & 0 \\ \varepsilon & a_2 - 2s_2^* - \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta) \\
 &- (\tau_k + \mu) \begin{pmatrix} 0 & 0 & -ks_1^* \\ 0 & 0 & 0 \\ 0 & 0 & ks_1^* - b - c \end{pmatrix} \delta(\theta + 1),
 \end{aligned}$$

where δ is the Dirac delta function.

For $\phi \in C([-1, 0], R^3)$, define

$$(20) \quad A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 d\eta(s, \mu)\phi(s), & \theta = 0 \end{cases}$$

and

$$(21) \quad R\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

Then (15) is equivalent to the abstract differential equation

$$(22) \quad \dot{u}_t = A(\mu)u_t + R(\mu)u_t,$$

where $u_t(\theta) = u(t + \theta), \theta \in [-1, 0]$.

For $\psi \in C([-1, 0], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C([-1, 0], R^3)$ and $\psi \in C([0, 1], (R^3)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \psi^T(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$, the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_0\tau_k$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$ respectively. By direct computation, we can obtain

$$q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0\tau_k\theta}, q^*(s) = M(1, \alpha^*, \beta^*)e^{i\omega_0\tau_k s}, M = \frac{1}{B},$$

where

$$\begin{aligned} \alpha &= \frac{\varepsilon}{i\omega_0 - (a_2 - 2s_2^* - \varepsilon)}, \quad \beta = \frac{\varepsilon\alpha - i\omega_0 + a_1 - 2s_1^* - \varepsilon}{ks_1^*e^{-i\omega_0\tau_k}}, \\ \alpha^* &= -\frac{i\omega_0 + a_1 - 2s_1^* - \varepsilon}{\varepsilon}, \quad \beta^* = \frac{ks_1^*}{i\omega_0 + ks_1^* - b - c}, \\ B &= 1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + \tau_k[\bar{\alpha}\beta^*(ks_1^* - b - c) - ks_1^*\bar{\alpha}]. \end{aligned}$$

Furthermore, $\langle q^*(s), q(\theta) \rangle = 1$ and $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

Next, we use the same notations as those in Hassard [8] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq.(15) when $\mu = 0$.

Define

$$(23) \quad z(t) = \langle q^*, u_t \rangle, W(t, \theta) = u_t(\theta) - 2\text{Re}\{z(t)q(\theta)\}.$$

on the center manifold C_0 , and we have

$$(24) \quad W(t, \theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$(25) \quad W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \dots,$$

and z and \bar{z} are local coordinates for center manifold C_0 in the direction of q^* and \bar{q}^* . Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (15),

$$\dot{z}(t) = i\omega_0\tau_k z + \bar{q}^*(\theta)f(0, W(z, \bar{z}, \theta) + 2\text{Re}\{zq(\theta)\} \stackrel{\text{def}}{=} i\omega_0\tau_k z + \bar{q}^*(0)f_0.$$

That is

$$\dot{z}(t) = i\omega_0\tau_k z + g(z, \bar{z}),$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \dots.$$

Hence, we have

$$g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = f(0, y_t) = K_{20}z^2 + K_{11}z\bar{z} + K_{02}\bar{z}^2 + K_{21}z^2\bar{z} + h.o.t.,$$

where

$$\begin{aligned} K_{20} &= -\bar{D}\tau_k (1 + k\beta e^{-i\omega_0\tau_k} + \alpha^2\alpha^* - k\beta\beta^* e^{-i\omega_0\tau_k}), \\ K_{11} &= -2\bar{D}\tau_k (1 + k\text{Re}\{\beta e^{-i\omega_0\tau_k}\} + |\alpha|^2\alpha^* - k\beta^*\text{Re}\{\beta e^{-i\omega_0\tau_k}\}), \\ K_{02} &= -\bar{D}\tau_k (1 + k\bar{\beta}e^{i\omega_0\tau_k} + \bar{\alpha}^2\alpha^* - k\bar{\beta}\beta^* e^{i\omega_0\tau_k}), \\ K_{21} &= -\bar{D}\tau_k \left\{ W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0) + k \left[\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)e^{i\omega_0\tau_k} + \frac{1}{2}W_{20}^{(3)}(-1) \right. \right. \\ &\quad \left. \left. + \beta W_{11}^{(1)}(0)e^{-i\omega_0\tau_k} + W_{11}^{(3)}(-1) \right] + \alpha^* \left[\bar{\alpha}W_{20}^{(1)}(0) + 2\alpha W_{11}^{(1)}(0) \right] \right. \\ &\quad \left. + k\beta^* \left[\frac{1}{2}\bar{\beta}W_{20}^{(1)}(0)e^{i\omega_0\tau_k} + \frac{1}{2}W_{20}^{(3)}(-1) + \beta W_{11}^{(1)}(0)e^{-i\omega_0\tau_k} + W_{11}^{(3)}(-1) \right] \right\}. \end{aligned}$$

Then we obtain

$$g_{20} = 2K_{20}, \quad g_{11} = K_{11}, \quad g_{02} = 2K_{02}, \quad g_{21} = 2K_{21}.$$

For unknown $W_{20}^{(1)}(0), W_{20}^{(3)}(-1), W_{11}^{(1)}(0), W_{11}^{(3)}(-1)$ in g_{21} , we still need to compute them.

From (22), (23), we have

$$(26) \quad W' = \begin{cases} AW - 2\text{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -1 \leq \theta < 0, \\ AW - 2\text{Re}\{\bar{q}^*(0)\bar{f}q(\theta)\} + \bar{f}, & \theta = 0 \end{cases} \stackrel{\text{def}}{=} AW + H(z, \bar{z}, \theta),$$

where

$$(27) \quad H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \dots$$

Comparing the coefficients, we obtain

$$(28) \quad (AW - 2i\tau_k\omega_0)W_{20} = -H_{20}(\theta),$$

$$(29) \quad AW_{11}(\theta) = -H_{11}(\theta),$$

.....

And we know that for $\theta \in [-1, 0)$,

$$(30) \quad H(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta).$$

Comparing the coefficients of (30) with (27) gives that

$$(31) \quad H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta),$$

$$(32) \quad H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).$$

From (28), (31) and the definition of A , we get

$$(33) \quad \dot{W}_{20}(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Noting that $q(\theta) = q(0)e^{i\omega_0\tau_k\theta}$, we have

$$(34) \quad W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_1e^{2i\omega_0\tau_k\theta},$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in R^3$ is a constant vector.

Similarly, from (29), (32) and the definition of A , we have

$$(35) \quad \dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta),$$

$$(36) \quad W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_2,$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in R^3$ is a constant vector.

In what follows, we shall seek appropriate E_1, E_2 in (34), (36), respectively. It follows from the definition of A and (31), (32) that

$$(37) \quad \int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0)$$

and

$$(38) \quad \int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0),$$

where $\eta(\theta) = \eta(0, \theta)$.

From (28), we have

$$(39) \quad H_{20}(0) = -g_{20}q(0) - g_{02}\bar{q}(0) + 2\tau_k(H_1, H_2, H_3)^T,$$

$$(40) \quad H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}(0)\bar{q}(0) + 2\tau_k(P_1, P_2, P_3)^T,$$

where

$$H_1 = -1 - k\beta e^{-i\omega_0\tau_k}, \quad H_2 = \alpha^2, \quad H_3 = k\beta e^{-i\omega_0\tau_k},$$

$$P_1 = -1 - k\operatorname{Re}\{\beta e^{-i\omega_0\tau_k}\}, \quad P_2 = |\alpha|^2, \quad P_3 = k\operatorname{Re}\{\bar{\beta} e^{-i\omega_0\tau_k}\}.$$

Noting that

$$\left(i\omega_0\tau_k I - \int_{-1}^0 e^{i\omega_0\tau_k\theta} d\eta(\theta) \right) q(0) = 0,$$

$$\left(-i\omega_0\tau_k I - \int_{-1}^0 e^{-i\omega_0\tau_k\theta} d\eta(\theta) \right) \bar{q}(0) = 0$$

and substituting (34) and (39) into (37), we have

$$\left(2i\omega_0\tau_k I - \int_{-1}^0 e^{2i\omega_0\tau_k\theta} d\eta(\theta) \right) E_1 = 2\tau_k(H_1, H_2, H_3)^T.$$

That is,

$$\begin{pmatrix} 2i\omega_0 - (a_1 - 2s_1^* - \varepsilon) & -\varepsilon & -ks_1^*e^{-2i\omega_0\tau_k} \\ -\varepsilon & 2i\omega_0 - (a_2 - 2s_2^* - \varepsilon) & 0 \\ 0 & 0 & 2i\omega_0 - (ks_1^* - b - c)e^{-2i\omega_0\tau_k} \end{pmatrix} \times E_1 = 2(H_1, H_2, H_3)^T.$$

It follows that

$$(41) \quad E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \quad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \quad E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1},$$

where

$$\begin{aligned} \Delta_1 &= \det \begin{pmatrix} 2i\omega_0 - (a_1 - 2s_1^* - \varepsilon) & -\varepsilon & -ks_1^*e^{-2i\omega_0\tau_k} \\ -\varepsilon & 2i\omega_0 - (a_2 - 2s_2^* - \varepsilon) & 0 \\ 0 & 0 & 2i\omega_0 - (ks_1^* - b - c)e^{-2i\omega_0\tau_k} \end{pmatrix}, \\ \Delta_{11} &= 2 \det \begin{pmatrix} H_1 & -\varepsilon & -ks_1^*e^{-2i\omega_0\tau_k} \\ H_2 & 2i\omega_0 - (a_2 - 2s_2^* - \varepsilon) & 0 \\ H_3 & 0 & 2i\omega_0 - (ks_1^* - b - c)e^{-2i\omega_0\tau_k} \end{pmatrix}, \\ \Delta_{12} &= 2 \det \begin{pmatrix} 2i\omega_0 - (a_1 - 2s_1^* - \varepsilon) & H_1 & -ks_1^*e^{-2i\omega_0\tau_k} \\ -\varepsilon & H_2 & 0 \\ 0 & H_3 & 2i\omega_0 - (ks_1^* - b - c)e^{-2i\omega_0\tau_k} \end{pmatrix}, \\ \Delta_{13} &= 2 \det \begin{pmatrix} 2i\omega_0 - (a_1 - 2s_1^* - \varepsilon) & -\varepsilon & H_1 \\ -\varepsilon & 2i\omega_0 - (a_2 - 2s_2^* - \varepsilon) & H_2 \\ 0 & 0 & H_3 \end{pmatrix}. \end{aligned}$$

Similarly, substituting (35) and (40) into (38), we have

$$\left(\int_{-1}^0 d\eta(\theta) \right) E_2 = 2\tau_k(P_1, P_2, P_3)^T.$$

That is

$$\begin{pmatrix} a_1 - 2s_1^* - \varepsilon & \varepsilon & -ks_1^* \\ \varepsilon & a_2 - 2s_2^* - \varepsilon & 0 \\ 0 & 0 & ks_1^* - b - c \end{pmatrix} E_2 = 2(-P_1, -P_2, -P_3)^T.$$

It follows that

$$(42) \quad E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \quad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \quad E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},$$

where

$$\begin{aligned} \Delta_2 &= \det \begin{pmatrix} a_1 - 2s_1^* - \varepsilon & \varepsilon & -ks_1^* \\ \varepsilon & a_2 - 2s_2^* - \varepsilon & 0 \\ 0 & 0 & ks_1^* - b - c \end{pmatrix}, \\ \Delta_{21} &= 2 \det \begin{pmatrix} -P_1 & \varepsilon & -ks_1^* \\ -P_2 & a_2 - 2s_2^* - \varepsilon & 0 \\ -P_3 & 0 & ks_1^* - b - c \end{pmatrix}, \\ \Delta_{22} &= 2 \det \begin{pmatrix} a_1 - 2s_1^* - \varepsilon & -P_1 & -ks_1^* \\ \varepsilon & -P_2 & 0 \\ 0 & -P_3 & ks_1^* - b - c \end{pmatrix}, \\ \Delta_{23} &= 2 \det \begin{pmatrix} a_1 - 2s_1^* - \varepsilon & \varepsilon & -P_1 \\ \varepsilon & a_2 - 2s_2^* - \varepsilon & -P_2 \\ 0 & 0 & -P_3 \end{pmatrix}. \end{aligned}$$

From (34), (36), (41), (42), we can calculate g_{21} and derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_k} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_k)\}}, \\ \beta_2 &= 2\operatorname{Re}(c_1(0)), \\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_k)\}}{\omega_0\tau_k}. \end{aligned}$$

These formulae give a description of the Hopf bifurcation periodic solutions of (15) at $\tau = \tau_k$, ($k = 0, 1, 2, 3, \dots$) on the center manifold. From the discussion above, we have the following result:

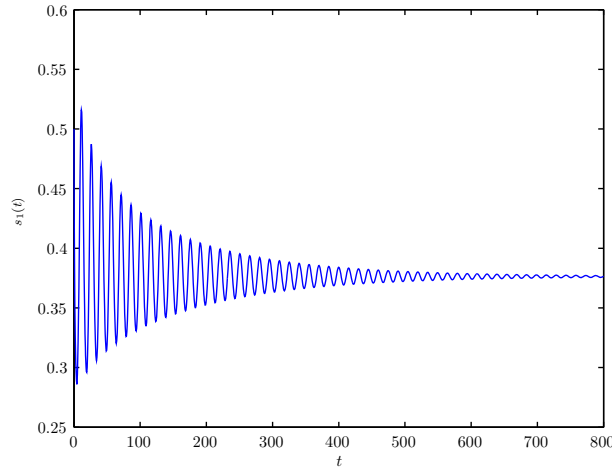


FIGURE 1

Theorem 3.1. *The periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); the periodic of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

Remark 3.2. A τT -periodic solution of (15) is a T -periodic solution of (4).

4. Numerical examples

In this section, we present some numerical results of the system (2) to verify the analytical predictions obtained in the previous section. From Section 3,

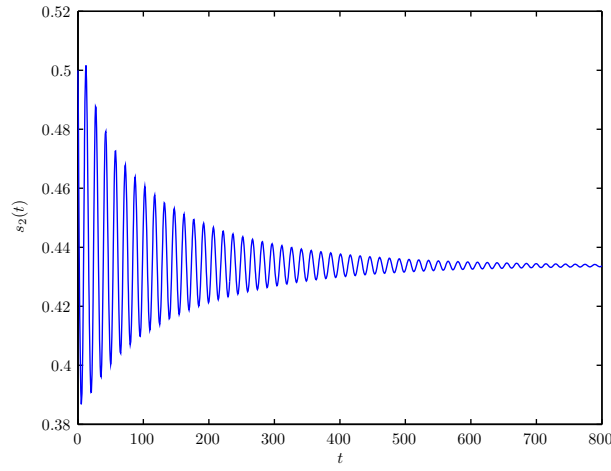


FIGURE 2

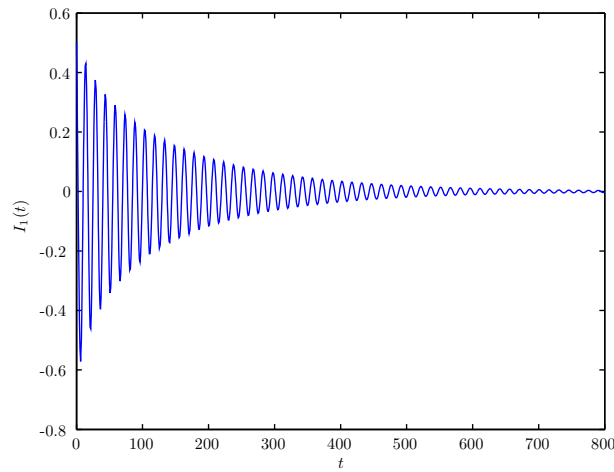


FIGURE 3

we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following system:

$$(43) \quad \begin{cases} \dot{s}_1 = -0.5s_1I_1(t - \tau) + s_1(0.3 - s_1) + 0.5(s_2 - s_1), \\ \dot{s}_2 = s_2(0.5 - s_2) + 0.5(s_1 - s_2), \\ \dot{I}_1 = 0.5s_1I_1(t - \tau) - 0.2I_1(t - \tau) - 0.4I_1(t - \tau) \end{cases}$$

which has a boundary equilibrium $E_0(s_1^*, s_2^*, I_1^*) \approx (0.3632, 0.4262, 0)$ and satisfies the conditions indicated in Theorem 2.2. When $\tau = 0$, the boundary

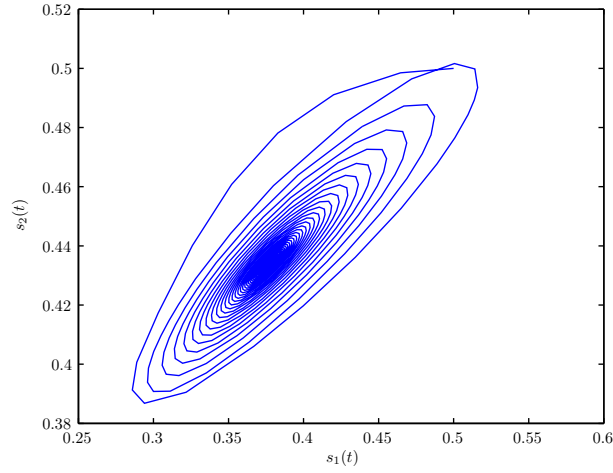


FIGURE 4

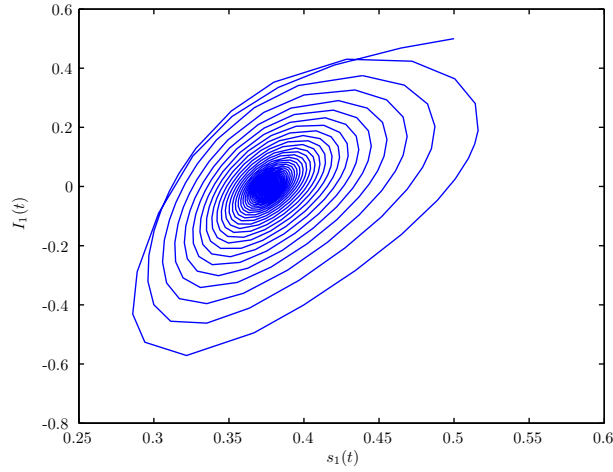


FIGURE 5

equilibrium $E_0 \approx (0.3632, 0.4262, 0)$ is asymptotically stable. Take $k = 0$ for example, by some complicated computation by means of Matlab 7.0, we get $\omega_0 \approx 1.0328$, $\tau_0 \approx 3.7544$, $\lambda'(\tau_0) \approx 0.3345 - 6.6612i$. Thus we can calculate the following values: $c_1(0) \approx -1.0531 - 3.4312i$, $\mu_2 \approx 0.4355$, $\beta_2 \approx -4.0564$, $T_2 \approx 5.4005$. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the boundary equilibrium $E_0 \approx (0.3632, 0.4262, 0)$ is stable when $\tau < \tau_0$ as is illustrated by the computer simulations (see Fig.1-7). When τ passes through the critical value τ_0 , the boundary equilibrium $E_0 \approx (0.3632, 0.4262, 0)$ loses its stability

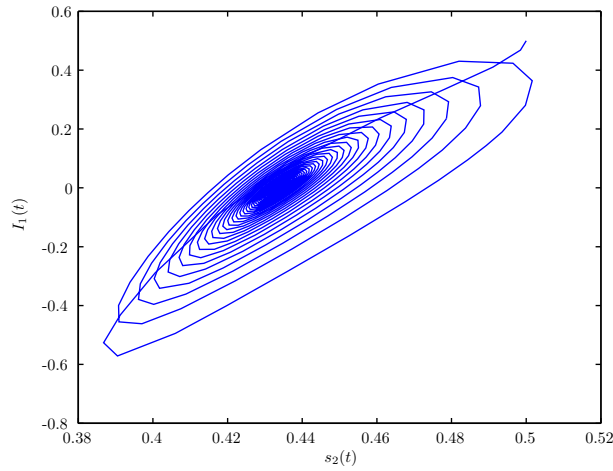


FIGURE 6

and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the boundary equilibrium $E_0 \approx (0.3632, 0.4262, 0)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from $E_0 \approx (0.3632, 0.4262, 0)$ at τ_0 are stable, which are depicted in Fig.8-14.

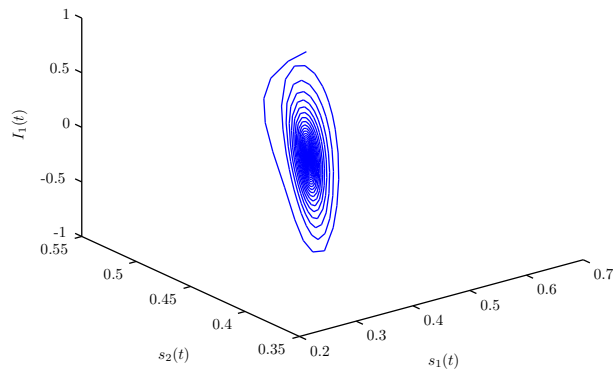


FIGURE 7

Fig.1-Fig.7 Behavior and phase portrait of the system (43) with $\tau = 3.7 < \tau_0 \approx 3.7544$. The boundary equilibrium $E_0 \approx (0.3632, 0.4262, 0)$ is asymptotically stable. The initial value is $(0.5, 0.5, 0.5)$.

Fig.8-Fig.14 Behavior and phase portrait of the system (43) with $\tau = 4 > \tau_0 \approx 3.7544$. Hopf bifurcation occurs from the boundary equilibrium $E_0 \approx (0.3632, 0.4262, 0)$. The initial value is $(0.5, 0.5, 0.5)$.

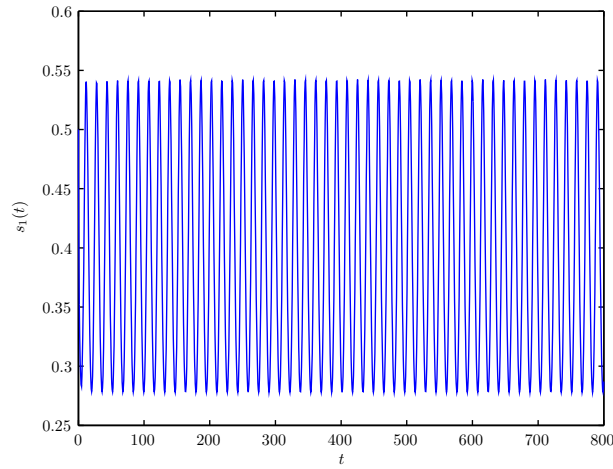


FIGURE 8

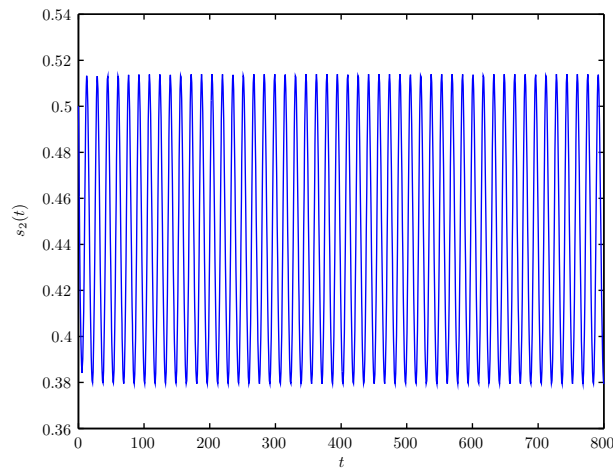


FIGURE 9

5. Biological explanations and conclusions

5.1. Biological explanations

From the analysis in Section 2, we know that under the conditions (H1)-(H3), then the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ of the system (2) is asymptotically stable for all $\tau \in [0, \tau_0)$. This shows that, in this case, the susceptible numbers in place 1, the susceptible numbers in place 2 and the infective numbers in place 2 will tend to stabilization, that is, the susceptible numbers in place 1, the susceptible numbers in place 2 and the infective numbers in place 1 will

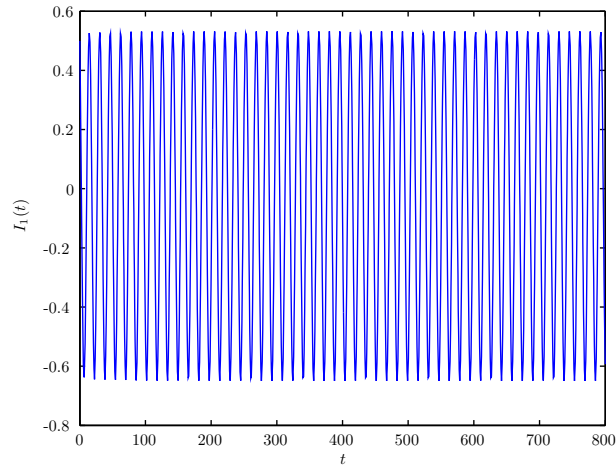


FIGURE 10

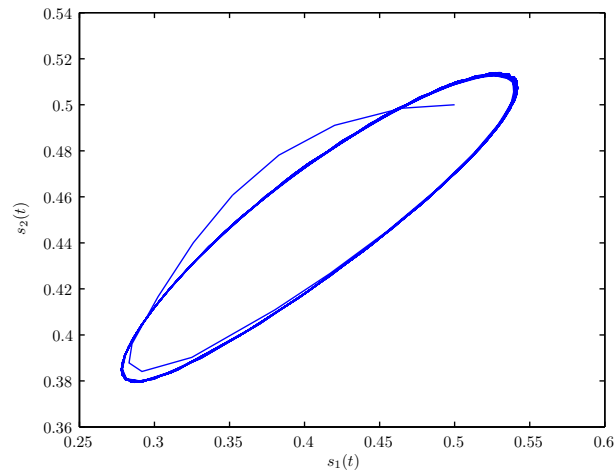


FIGURE 11

tend to $s_1^*, s_2^*, 0$, respectively, and this fact is not influenced by the delay $\tau \in [0, \tau_0)$. When τ crosses through the critical value τ_0 , the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ of the system (2) loses stability and a Hopf bifurcation occurs. This shows that the susceptible numbers in place 1, the susceptible numbers in place 2 and the infective numbers in place 1 keep in an oscillatory mode near the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$.

5.2. Conclusions

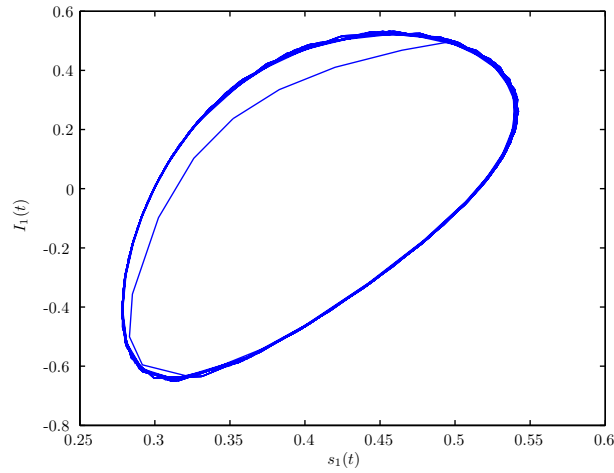


FIGURE 12

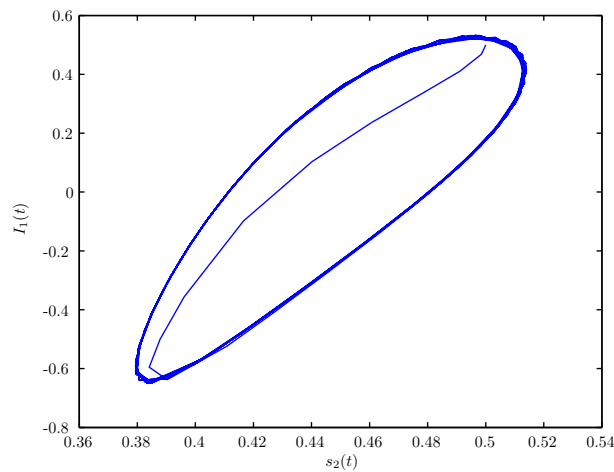


FIGURE 13

In this paper, we have investigated local stability of the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ and local Hopf bifurcation in delayed epidemic model with diffusion. We have showed that if the conditions (H1)-(H3) hold, the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$ of the system (2) is asymptotically stable for all $\tau \in [0, \tau_0)$. As the delay τ increases, the boundary equilibrium loses its stability and a sequence of Hopf bifurcations occur at the boundary equilibrium $E_0(s_1^*, s_2^*, 0)$, i.e., a family of periodic orbits bifurcates from the the positive

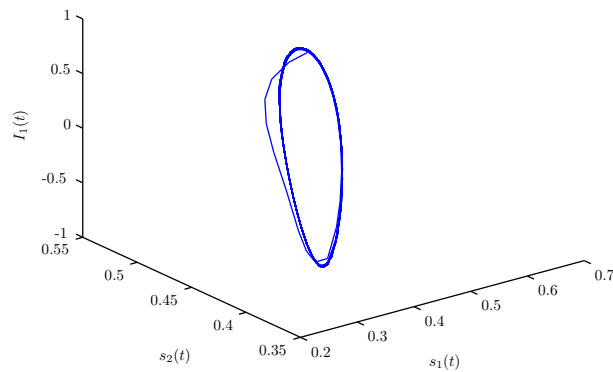


FIGURE 14

equilibrium $E_0(s_1^*, s_2^*, 0)$. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. A numerical example verifying our theoretical results is also correct.

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