

## CONTINUITY OF FUZZY PROPER FUNCTIONS ON ŠOSTAK'S $I$ -FUZZY TOPOLOGICAL SPACES

RAJAKUMAR ROOPKUMAR AND CHANDRAN KALAIVANI

ABSTRACT. The relations among various types of continuity of fuzzy proper function on a fuzzy set and at fuzzy point belonging to the fuzzy set in the context of Šostak's  $I$ -fuzzy topological spaces are discussed. The projection maps are defined as fuzzy proper functions and their properties are proved.

### 1. Introduction

From the work of Chang [3], the research on fuzzy topology has been started and hundreds of papers on various concepts like neighborhood structures [15, 16, 23], product topology [16, 25], separation axioms [14, 22, 23], compactness [10, 12, 13], connectedness [4, 15], continuity of functions [16, 24], etc., have been published. Šostak introduced  $I$  fuzzy topology in a new way. This topology is later redefined by various researchers, and is called *smooth fuzzy topology* [18]. We prefer to call this topology by Šostak's  $I$ -fuzzy topology. In the context of Šostak's  $I$ -fuzzy topological spaces, neighborhood structures [5], base and subbase [17], product topology [21], compactness [1, 6, 7, 8], separation axioms [21], gradation preserving functions [11] are also studied.

Regarding functions in fuzzy setting, the fuzzy proper function and its continuity on Chang topological spaces are first introduced by Chakraborty and Ahsanulla [2]. Chaudhuri and Das [4] proved the equivalent statements of continuity of fuzzy proper function in the context of Chang topology. Fath Allah and Mamoud [9] introduced the fuzzy graph, strongly fuzzy graph of a proper fuzzy proper function on Chang topological space. They proved the closed graph theorem under some sufficient conditions and also proved various results relating separation axioms, the continuity of fuzzy proper function and the closedness of its graph. The notions of smooth fuzzy continuity and weakly smooth fuzzy continuity of a fuzzy proper function on Šostak's  $I$ -fuzzy topological spaces and their properties were discussed in [18].

---

Received February 19, 2010.

2010 *Mathematics Subject Classification.* 54A40.

*Key words and phrases.* fuzzy proper function, Šostak's  $I$ -fuzzy topology, smooth fuzzy continuity.

This paper is organized as follows. In Section 2, we recall some basic definitions and results from the literature and also prove some preliminary results required for our discussion.

The section 3 is devoted to discuss the various equivalent statements of smooth fuzzy continuity and weak smooth fuzzy continuity in Šostak's  $I$ -fuzzy topological spaces. As in the classical case of continuity of functions between two topological spaces, we prove that fuzzy continuity on a fuzzy set is equivalent to the fuzzy continuity at every fuzzy point of the fuzzy set. When we study the weak continuity of a fuzzy proper function at a fuzzy point, obviously, we have to consider two notions, one is defined in terms of fuzzy neighborhoods and the other in terms of quasi neighborhoods. We establish that weak fuzzy continuity on a fuzzy set implies weak fuzzy continuity at every fuzzy point in the fuzzy set, in both notions, and neither of the converse is true, which is in contrast with the classical case. To get the converse of one of these results, we introduce two new notions, namely,  $\alpha$ -weakly smooth fuzzy continuous functions and positive minimum Šostak's  $I$ -fuzzy topological spaces.

The projection functions from a product of fuzzy sets into a fuzzy set are defined in Section 4, as fuzzy proper functions and prove that they are smooth fuzzy continuous,  $\alpha$ -weakly smooth fuzzy continuous and weakly smooth fuzzy continuous. We also prove that every fuzzy proper function  $F : A \rightarrow \prod B_j$  can be expressed as  $[F_j]$  for some suitable  $F_j : A \rightarrow B_j$  and study the relationship between continuity of  $F$  and that of its coordinate functions  $F_j$ .

## 2. Preliminaries

Throughout this paper  $X, Y$  denote fixed non-empty sets and  $A, B$  denote fuzzy subsets of  $X, Y$  respectively,  $I$  denotes the closed interval  $[0, 1]$  and  $I^X$  denotes the set of all fuzzy subsets of  $X$ . Let  $X = \{x_1, x_2, \dots, x_n\}$  and  $\lambda_i \in I$  for  $i = 1, 2, \dots, n$ . By  $A_{[x_1, x_2, \dots, x_n]}^{[\lambda_1, \lambda_2, \dots, \lambda_n]}$ , we shall mean the fuzzy subset  $A$  of  $X$  which maps each  $x_i$  to  $\lambda_i \forall i = 1, 2, \dots, n$ . A fuzzy point [15] in  $X$  is defined by  $P_x^\lambda(t) = \begin{cases} \lambda & \text{if } t = x \\ 0 & \text{if } t \neq x \end{cases}$ , where  $0 < \lambda \leq 1$  and by adopting [14], we say that for  $A \in I^X$ ,  $P_x^\lambda \in A$  if  $\lambda < A(x)$ .

**Definition 2.1** ([18]). A Šostak's  $I$ -fuzzy topology on a fuzzy set  $A \in I^X$  is a map  $\tau : \mathfrak{J}_A \rightarrow I$ , where  $\mathfrak{J}_A = \{U \in I^X : U \leq A\}$ , satisfying the following three axioms:

- (1)  $\tau(0) = \tau(A) = 1$ ,
- (2)  $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2), \forall A_1, A_2 \in \mathfrak{J}_A$ ,
- (3)  $\tau(\bigvee_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau(A_i)$  for every family  $(A_i)_{i \in \Gamma} \subseteq \mathfrak{J}_A$ .

The pair  $(A, \tau)$  is called a Šostak's  $I$ -fuzzy topological space or simply sfts.

**Definition 2.2** ([2]). A fuzzy subset  $F$  of  $X \times Y$  is said to be a fuzzy proper function from  $A$  to  $B$  if

- (1)  $F(x, y) \leq \min \{A(x), B(y)\}$  for each  $(x, y) \in X \times Y$ ,
- (2) for each  $x \in X$  with  $A(x) > 0$ , there exists a unique  $y_0 \in Y$  such that  $F(x, y_0) = A(x)$  and  $F(x, y) = 0$  if  $y \neq y_0$ .

**Definition 2.3** ([2]). Let  $F$  be a fuzzy proper function from  $A$  to  $B$ . If  $U \leq A$  and  $V \leq B$ , then  $F(U) \leq B$  and  $F^{-1}(V) \leq A$  are defined by

$$\begin{aligned} F(U)(y) &= \sup \{F(x, y) \wedge U(x) : x \in X\}, \forall y \in Y, \\ F^{-1}(V)(x) &= \sup \{F(x, y) \wedge V(y) : y \in Y\}, \forall x \in X. \end{aligned}$$

The following lemma is an immediate consequence of the above definition.

**Lemma 2.4.** *If  $P_x^\lambda \in A$  and  $F : A \rightarrow B$  is a fuzzy proper function, then  $F(P_x^\lambda) = P_y^\lambda$ , where  $y \in Y$  is unique such that  $F(x, y) = A(x)$ .*

Next we present an easy formula for finding inverse image of a fuzzy set under a fuzzy proper function.

**Lemma 2.5.** *Let  $F : A \rightarrow B$  be a fuzzy proper function. If  $V \in I^Y$  and  $V \leq B$ , then  $F^{-1}(V)(x) = A(x) \wedge V(y)$ , where  $y \in Y$  is unique such that  $F(x, y) = A(x)$ .*

**Lemma 2.6.** *If  $A \in I^X$ , then  $A = \bigvee \{P_x^\lambda : P_x^\lambda \in A\}$ .*

**Lemma 2.7.** *If  $P_x^\lambda \in F^{-1}(V)$ , then  $F(P_x^\lambda) \in V$ .*

*Proof.* Let  $P_x^\lambda \in F^{-1}(V)$ . Then, we have  $\lambda < F^{-1}(V)(x) = A(x) \wedge V(y)$ , where  $y \in Y$  is unique such that  $F(x, y) = A(x)$ . By Lemma 2.4, we have  $F(P_x^\lambda) = P_y^\lambda$ . Hence  $\lambda < A(x) \wedge V(y) \leq V(y)$  implies that  $F(P_x^\lambda) \in V$ .  $\square$

**Definition 2.8** ([2]). Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Chang fuzzy topological spaces. Then  $F$  is said to be fuzzy continuous on  $A$  if  $F^{-1}(V) \in \tau_1$  for each  $V \in \tau_2$ .

**Definition 2.9** ([4]). Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Chang fuzzy topological spaces. Then  $F$  is said to be fuzzy continuous at  $P_x^\lambda \in A$  if for every  $V \in \tau_2$  with  $F(P_x^\lambda) \in V$ , there exists  $U \in \tau_1$  such that  $P_x^\lambda \in U$  and  $F(U) \leq V$ .

**Definition 2.10** ([2]). Let  $P_x^\lambda \in A$  and  $U \leq A$  are said to be quasi-coincident referred to  $A$  (written as  $P_x^\lambda qU[A]$ ) if there exists  $x \in X$  such that  $\lambda + U(x) > A(x)$ . If  $P_x^\lambda$  is not quasi-coincident with  $U$  in  $A$ , we write  $P_x^\lambda \bar{q}U[A]$ .

**Definition 2.11** ([4]). Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Chang fuzzy topological spaces. Then  $F$  is said to be fuzzy continuous at  $P_x^\lambda \in A$  if for every  $V \in \tau_2$  with  $F(P_x^\lambda) qV[B]$ , there exists  $U \in \tau_1$  such that  $P_x^\lambda qU[A]$  and  $F(U) \leq V$ .

**Definition 2.12** ([2]). Let  $F : A \rightarrow B$  and  $G : B \rightarrow C$  be fuzzy proper functions, where  $A \in I^X$ ,  $B \in I^Y$  and  $C \in I^Z$ . Then  $G \circ F$  is a fuzzy proper

function from  $A$  to  $C$  defined by

$$(G \circ F)(x, z) = \bigvee_{y \in Y} \{F(x, y) \wedge G(y, z)\}.$$

**Theorem 2.13** ([2]). *Let  $F : A \rightarrow B$  and  $G : B \rightarrow C$  be fuzzy proper functions. Then  $(G \circ F)^{-1}(U) = F^{-1}(G^{-1}(U))$ ,  $\forall U \leq C$ .*

**Lemma 2.14.** *Let  $F : A \rightarrow B$  and  $G : B \rightarrow C$  be fuzzy proper functions, where  $A \in I^X$ ,  $B \in I^Y$  and  $C \in I^Z$ . Then  $(G \circ F)(U) = G(F(U))$ ,  $\forall U \leq A$ .*

*Proof.* Let  $U \leq A$  and  $z \in Z$ .

$$\begin{aligned} (G \circ F)(U)(z) &= \bigvee_{x \in X} \{(G \circ F)(x, z) \wedge U(x)\} \\ &= \bigvee_{x \in X} \left\{ \bigvee_{y \in Y} \{F(x, y) \wedge G(y, z)\} \wedge U(x) \right\} \\ &= \bigvee_{y \in Y} \left\{ \bigvee_{x \in X} \{F(x, y) \wedge G(y, z) \wedge U(x)\} \right\} \\ &= \bigvee_{y \in Y} \left\{ G(y, z) \wedge \bigvee_{x \in X} \{F(x, y) \wedge U(x)\} \right\} \\ &= \bigvee_{y \in Y} \{G(y, z) \wedge F(U)(y)\} = G(F(U))(z). \end{aligned}$$

Hence we get the desired result.  $\square$

### 3. Fuzzy continuity on sfts

We first recall the definitions of smooth fuzzy continuous and weakly smooth fuzzy continuous functions from the literature.

**Definition 3.1** ([18]). Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Šostak's  $I$ -fuzzy topological spaces. Then  $F$  is said to be smooth fuzzy continuous on  $A$  if  $\tau_1(F^{-1}(V)) \geq \tau_2(V)$ .

**Definition 3.2** ([18]). Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Šostak's  $I$ -fuzzy topological spaces. Then  $F$  is said to be weakly smooth fuzzy continuous on  $A$ , if  $\tau_1(F^{-1}(V)) > 0$  whenever  $\tau_2(V) > 0$ .

To discuss the point-wise continuity of a fuzzy proper function, in the context of sfts, we introduce smooth fuzzy continuity at a fuzzy point, weakly smooth fuzzy continuity at a fuzzy point and qn-weakly smooth fuzzy continuity at a fuzzy point as follow.

**Definition 3.3.** Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Šostak's  $I$ -fuzzy topological spaces. Then  $F$  is said to

be smooth fuzzy continuous at a fuzzy point  $P_x^\lambda \in A$  if for every  $V \leq B$  with  $F(P_x^\lambda) \in V$ , there exists  $U \leq A$  such that  $P_x^\lambda \in U$ ,  $F(U) \leq V$  and  $\tau_1(U) \geq \tau_2(V)$ .

**Definition 3.4.** Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Šostak's  $I$ -fuzzy topological spaces. Then  $F$  is said to be weakly smooth fuzzy continuous at a fuzzy point  $P_x^\lambda \in A$  if for every  $V \leq B$  with  $\tau_2(V) > 0$  and  $F(P_x^\lambda) \in V$ , there exists  $U \leq A$  such that  $\tau_1(U) > 0$ ,  $P_x^\lambda \in U$  and  $F(U) \leq V$ .

**Definition 3.5.** Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Šostak's  $I$ -fuzzy topological spaces. Then  $F$  is said to be qn-weakly smooth fuzzy continuous at a fuzzy point  $P_x^\lambda \in A$  if for every q-neighborhood  $V$  of  $F(P_x^\lambda)$ , there exists a q-neighborhood  $U$  of  $P_x^\lambda$  such that  $F(U) \leq V$ .

In the above definition, by q-neighborhood of a fuzzy point, we mean a fuzzy set which is quasi-coincident with the fuzzy point and whose gradation of openness is positive.

**Theorem 3.6.** A fuzzy proper function  $F$  is smooth fuzzy continuous on  $A$  if and only if  $F$  is weakly smooth fuzzy continuous at every fuzzy point  $P_x^\lambda \in A$ .

*Proof.* Assume  $F$  is smooth fuzzy continuous on  $A$ . Let  $P_x^\lambda \in A$ ,  $V \leq B$  with  $F(P_x^\lambda) = P_y^\lambda \in V$ , where  $y$  is such that  $F(x, y) = A(x)$ . If we take  $U = F^{-1}(V)$ , then we have  $\tau_1(U) \geq \tau_2(V)$  and  $F(U) = F(F^{-1}(V)) \leq V$ . The inequalities  $\lambda < A(x)$ ,  $\lambda < V(y)$  imply that  $U(x) = F^{-1}(V)(x) = A(x) \wedge V(y) > \lambda$  and hence  $P_x^\lambda \in U$ . Hence  $F$  is weakly smooth fuzzy continuous at every fuzzy point of  $A$ .

Conversely, assume that  $F$  is smooth fuzzy continuous at every fuzzy point  $P_x^\lambda \in A$ . Let  $V \leq B$ . For every  $P_x^\lambda \in F^{-1}(V)$ , there exists  $U_{x_\lambda} \leq A$  such that  $P_x^\lambda \in U_{x_\lambda}$  and  $F(U_{x_\lambda}) \leq V$ , and  $\tau_1(U_{x_\lambda}) \geq \tau_2(V)$ . Then clearly  $F^{-1}(V) = \bigvee U_{x_\lambda}$  and  $\tau_1(F^{-1}(V)) = \tau_1(\bigvee U_{x_\lambda}) \geq \wedge \tau_1(U_{x_\lambda}) \geq \tau_2(V)$ .

Hence  $F$  is smooth fuzzy continuous on  $A$ .  $\square$

**Theorem 3.7.** If a fuzzy proper function  $F$  is weakly smooth fuzzy continuous on  $A$ , then  $F$  is weakly smooth fuzzy continuous at every fuzzy point  $P_x^\lambda \in A$ .

*Proof.* Assume that  $F$  is weakly smooth fuzzy continuous on  $A$ . Let  $P_x^\lambda \in A$ ,  $V \leq B$  with  $\tau_2(V) > 0$  and  $F(P_x^\lambda) \in V$ . If  $U = F^{-1}(V)$ , then  $\tau_1(U) = \tau_1(F^{-1}(V)) > 0$ ,  $F(U) \leq V$  and  $P_x^\lambda \in U$ . Hence  $F$  is weakly smooth fuzzy continuous at every  $P_x^\lambda \in A$ .  $\square$

**Counter example 3.8.** There exists a weakly smooth fuzzy continuous function at every fuzzy point of a fuzzy set  $A$  which is not weakly smooth fuzzy continuous on  $A$ .

Let  $X = \{r, s\}$ ,  $Y = \{\xi, \zeta\}$  and  $A_{[r,s]}^{[0.8,0.7]}$ ,  $B_{[\xi,\zeta]}^{[0.9,0.8]}$  be fuzzy subsets of  $X$  and  $Y$  respectively. Define the fuzzy subsets  $U_n \leq A$ ,  $V_1 \leq B$  by  $U_n_{[r,s]}^{[0.7-\frac{1}{n+2}, 0.7-\frac{1}{n+1}]}$ ,  $\forall n = 1, 2, 3, \dots$ , and  $V_1_{[\xi,\zeta]}^{[0.7,0.7]}$ .

If  $\tau_1 : \mathfrak{J}_A \rightarrow I$  by

$$\tau_1(U) = \begin{cases} 1, & U = 0 \text{ or } A, \\ \frac{1}{n}, & U = U_n, \forall n = 1, 2, 3, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\tau_2 : \mathfrak{J}_B \rightarrow I$  by

$$\tau_2(V) = \begin{cases} 1, & V = 0 \text{ or } B, \\ 0.6, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then obviously  $(A, \tau_1)$ ,  $(B, \tau_2)$  are Šostak's  $I$ -fuzzy topological spaces.

Let the proper function  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be defined by

$$F(r, \xi) = 0.8, \quad F(r, \zeta) = 0, \quad F(s, \xi) = 0.7, \quad F(s, \zeta) = 0.$$

We first note that  $F$  is not weakly smooth fuzzy continuous on  $A$ , since  $\tau_2(V_1_{[\xi,\zeta]}^{[0.7,0.7]}) = 0.6 > 0$  but  $\tau_1(F^{-1}(V_1)_{[r,s]}^{[0.7,0.7]}) = 0$ .

Next we prove that  $F$  is weakly smooth fuzzy continuous at every fuzzy point in  $A$ . Let  $P_r^\lambda \in A$ . Then  $0 < \lambda < 0.8$  and  $F(P_r^\lambda) = P_\xi^\lambda$ .

**Case 1:**  $0 < \lambda < 0.7$ .

In this case, if  $V \in \mathfrak{J}_B$ , with  $\tau_2(V) > 0$  and  $F(P_r^\lambda) = P_\xi^\lambda \in V$ , then  $\lambda < V_1(\xi) = 0.7$  and  $V$  is either  $V_1$  or  $B$ .

$V = V_1$ : Now by using  $0.7 - \frac{1}{n+2} \uparrow 0.7$  as  $n \rightarrow \infty$ , we can choose a positive integer  $n$  such that  $\lambda < U_n(r) < 0.7$ . For this  $n$ , we get

$$P_r^\lambda \in U_n \text{ and } F(U_n)_{[\xi,\zeta]}^{[0.7-\frac{1}{n+2}, 0]} \leq V_1_{[\xi,\zeta]}^{[0.7,0.7]}.$$

$V = B$ : Clearly  $P_r^\lambda \in A$  and  $F(A)_{[\xi,\zeta]}^{[0.8,0.7]} \leq B_{[\xi,\zeta]}^{[0.9,0.8]}$ .

**Case 2:**  $0.7 \leq \lambda < 0.8$ .

Here  $B$  is the only fuzzy set such that  $\tau_2(B) > 0$  and  $F(P_r^\lambda) \in B$  and hence  $A$  does the required job.

Consequently,  $F$  is weakly smooth fuzzy continuous at every  $P_r^\lambda \in A$ . Similarly, we can prove that  $F$  is weakly smooth fuzzy continuous at every  $P_s^\lambda \in A$ .

The converse of Theorem 3.7 can be achieved under some sufficient conditions, for which we need to introduce the following definitions.

**Definition 3.9.** Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  and  $(B, \tau_2)$  be Šostak's  $I$ -fuzzy topological spaces. Then  $F$  is said to be  $\alpha$ -weakly smooth fuzzy continuous on  $A$ , if  $\tau_1(F^{-1}(V)) \geq \alpha$  whenever  $V \in \mathfrak{J}_B$  and  $\tau_2(V) \geq \alpha$ .

**Definition 3.10.** Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function,  $(A, \tau_1)$  and  $(B, \tau_2)$  be Šostak's  $I$ -fuzzy topological spaces and  $\alpha \in (0, 1]$ .  $F$  is said to be  $\alpha$ -weakly smooth fuzzy continuous at a fuzzy point  $P_x^\lambda \in A$  if for every  $V \leq B$  with  $\tau_2(V) \geq \alpha$  and  $F(P_x^\lambda) \in V$ , there exists  $U \leq A$  such that  $\tau_1(U) \geq \alpha$ ,  $P_x^\lambda \in U$  and  $F(U) \leq V$ .

**Definition 3.11.** Let  $(A, \tau)$  be a Šostak's  $I$ -fuzzy topological space. Then  $\tau$  is said to be positive minimum Šostak's  $I$ -topology if  $\bigwedge_{i \in \Gamma} \tau(U_i) > 0$ , whenever  $U_i \in \mathfrak{J}_A$  and  $\tau(U_i) > 0$  for all  $i \in \Gamma$ .

This is a natural expectation in a Šostak's  $I$ -fuzzy topology, because if we call a fuzzy set  $U$  with  $\tau(U) > 0$ , an open fuzzy set, then we get that arbitrary union of open fuzzy sets is open in a positive minimum Šostak's  $I$ -fuzzy topological space, whereas this does not hold in a Šostak's  $I$ -fuzzy topological space.

**Theorem 3.12.** A fuzzy proper function  $F$  is  $\alpha$ -weakly smooth fuzzy continuous on  $A$  if and only if  $F$  is  $\alpha$ -weakly smooth fuzzy continuous at every fuzzy point  $P_x^\lambda \in A$ .

*Proof.* Assume that  $F$  is  $\alpha$ -weakly smooth fuzzy continuous on  $A$ . Let  $P_x^\lambda \in A$ . If  $V \leq B$  is given with  $\tau_2(V) \geq \alpha$  and  $F(P_x^\lambda) \in V$ , then we take  $U = F^{-1}(V)$ . As in the proof of Theorem 3.7, we get  $P_x^\lambda \in U$ . Since  $\tau_2(V) \geq \alpha$ ,  $\tau_1(U) = \tau_1(F^{-1}(V)) \geq \alpha$  and  $F(U) = F(F^{-1}(V)) \leq V$ . Hence  $F$  is  $\alpha$ -weakly smooth fuzzy continuous at  $P_x^\lambda$ .

Conversely, assume that  $F$  is  $\alpha$ -weakly smooth fuzzy continuous at every fuzzy point  $P_x^\lambda \in A$ . Let  $V \leq B$  and  $\tau_2(V) \geq \alpha$ . For any  $P_x^\lambda \in F^{-1}(V)$ , by Lemma 2.7, we have  $F(P_x^\lambda) \in V$ . Therefore, by assumption, there exists  $U_{x_\lambda} \leq A$  with  $\tau_1(U_{x_\lambda}) \geq \alpha$  such that  $P_x^\lambda \in U_{x_\lambda}$  and  $F(U_{x_\lambda}) \leq V$ . Then by using Lemma 2.6, we get  $F^{-1}(V) = \bigvee U_{x_\lambda}$  and  $\tau_1(F^{-1}(V)) = \tau_1(\bigvee U_{x_\lambda}) \geq \bigwedge \tau_1(U_{x_\lambda}) \geq \alpha$ . Thus  $F$  is fuzzy  $\alpha$ -weakly continuous on  $A$ .  $\square$

Similarly, we can prove the following theorem.

**Theorem 3.13.** Let  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be a fuzzy proper function and  $(A, \tau_1)$  be a positive minimum Šostak's  $I$ -fuzzy topological spaces.  $F$  is weakly smooth fuzzy continuous on  $A$  if and only if  $F$  is weakly smooth fuzzy continuous at every fuzzy point of  $A$ .

**Theorem 3.14.** If a fuzzy proper function  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  is weakly smooth fuzzy continuous on  $A$ , then  $F$  is  $qn$ -weakly smooth fuzzy continuous at every fuzzy point  $P_x^\lambda \in A$ .

*Proof.* Assume  $F$  is weakly smooth fuzzy continuous on  $A$ . Let  $V \leq B$  be a  $q$ -neighborhood of  $F(P_x^\lambda) = P_y^\lambda$ , where  $y \in Y$  is such that  $F(x, y) = A(x)$ . Then we have  $\tau_2(V) > 0$  and  $P_y^\lambda qV[B]$ . Take  $U = F^{-1}(V)$ . By our assumption

$\tau_1(U) > 0$ . We claim that  $P_x^\lambda qU[A]$ .

$$\begin{aligned} \lambda + U(x) &= \lambda + F^{-1}(V)(x) = \lambda + (A(x) \wedge V(y)) \\ &= (\lambda + A(x)) \wedge (\lambda + V(y)) \\ &> (\lambda + A(x)) \wedge B(y) && \text{since } \lambda + V(y) > B(y) \\ &> A(x) \wedge B(y) \geq A(x) && \text{since } B(y) \geq A(x). \end{aligned}$$

Hence  $F$  is qn-weakly smooth fuzzy continuous at every fuzzy point of  $A$ .  $\square$

**Counter example 3.15.** There exists a fuzzy proper function on  $A$  such that it is qn-weakly smooth fuzzy continuous at every fuzzy point of  $A$  but it is not weakly smooth fuzzy continuous on  $A$ .

Let  $X = \{r, s\}$ ,  $Y = \{\xi, \zeta\}$ ,  $A_{[r,s]}^{[1,0.8]}$ ,  $B_{[\xi,\zeta]}^{[1,0.9]}$ ,  $U_1^{[0.8,0.7]}_{[r,s]}$  and  $V_1^{[0.8,0.8]}_{[\xi,\zeta]}$ .

We define  $\tau_1 : \mathfrak{I}_A \rightarrow I$  by

$$\tau_1(U) = \begin{cases} 1, & U = 0 \text{ or } A, \\ 0.9, & U = U_1, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\tau_2 : \mathfrak{I}_B \rightarrow I$  by

$$\tau_2(V) = \begin{cases} 1, & V = 0 \text{ or } B, \\ 0.8, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Define a fuzzy proper function

$$F : (A, \tau_1) \rightarrow (B, \tau_2) \text{ by } F(r, \xi) = 1, F(r, \zeta) = 0, F(s, \xi) = 0, F(s, \zeta) = 0.8.$$

$F$  is not weakly smooth fuzzy continuous on  $A$ , since  $\tau_2(V_1^{[0.8,0.8]}_{[\xi,\zeta]}) = 0.8$  and  $\tau_1(F^{-1}(V_1^{[0.8,0.8]}_{[\xi,\zeta]})) = 0$ .

We claim that  $F$  is qn-weakly smooth fuzzy continuous at every fuzzy point of  $A$ . We note that if  $P_r^\lambda \in A$ , then  $F(P_r^\lambda) = P_\xi^\lambda$ .

**Case 1:**  $0 < \lambda \leq 0.2$ .

In this case,  $B$  is the only q-neighborhood for  $P_\xi^\lambda$  in  $B$ , clearly  $A$  is a q-neighborhood of  $P_r^\lambda$  such that  $F(A) \leq B$ .

**Case 2:**  $0.2 < \lambda < 1$ .

In this case,  $V_1$  and  $B$  are the possible q-neighborhoods of  $P_\xi^\lambda$  in  $B$ . For  $V_1$ , we find that  $U_1^{[0.8,0.7]}_{[r,s]}$  is q-neighborhood of  $P_x^\lambda$  and  $F(U_1^{[0.8,0.7]}_{[r,s]}) \leq V_1^{[0.8,0.8]}_{[\xi,\zeta]}$ . And for  $B$ , we choose  $A$  as before.

Hence  $F$  is qn-weakly smooth fuzzy continuous at every  $P_r^\lambda$  of  $A$ . Similarly, we can prove that  $F$  is qn-weakly smooth fuzzy continuous at every  $P_s^\lambda \in A$ .

*Remark 3.16.* At this juncture, it is necessary to compare [4, Theorem 3.3], in which it is stated that “Let  $A \in I^X$ ,  $B \in I^Y$  and  $T, T'$  be the Chang topologies



on  $A$  and  $B$  respectively. (i)  $F : (A, T) \rightarrow (B, T')$  is fuzzy continuous if and only if (v)  $\forall x_p \in A$  and  $\forall y \in Y$  such that  $F(x, y) \neq 0$  and  $\forall V \in T'$  with  $y_p q V[B]$ ,  $\exists U \in T$  such that  $F(U) \leq V$ ."

Unfortunately, the proof of (v)  $\Rightarrow$  (i) is not correct, because they used the statement " $B(y) - b > p \Rightarrow x_p \bar{q} W[A]$ ", where  $x \in X$  and  $y \in Y$  such that  $F(x, y) \neq 0$  and  $W(x) = b$ , which does not hold. From  $F(x, y) \leq A(x) \wedge B(y)$  and  $F(x, y) \neq 0 \Rightarrow F(x, y) = A(x)$ , we conclude that  $B(y) \geq A(x)$ . Using  $B(y) - b > p$  and  $b = W(x)$ , it is immediate that  $B(y) > W(x) + p$ . From these two inequalities, we could not arrive at  $A(x) \geq W(x) + p$ . We deliberately justify our argument that the statement (v) does not imply the statement (i) by slightly modifying the Counter example 3.15.

**Counter example 3.17.** There exists a fuzzy proper function which satisfies the statement (v) and does not satisfy the statement (i).

Let  $X = \{r, s\}$ ,  $Y = \{\xi, \zeta\}$ ,  $A_{[r,s]}^{[1,0.8]}$ ,  $B_{[\xi,\zeta]}^{[1,0.9]}$ ,  $U_{1[r,s]}^{[0.8,0.7]}$  and  $V_{1[\xi,\zeta]}^{[0.8,0.8]}$ . If  $T = \{0, A, U_1\}$  and  $T' = \{0, B, V_1\}$ , then  $(X, T)$  and  $(Y, T')$  are Chang fuzzy topological spaces. Define a fuzzy proper function

$$F : (A, T) \rightarrow (B, T') \text{ by } F(r, \xi) = 1, F(r, \zeta) = 0, F(s, \xi) = 0, F(s, \zeta) = 0.8.$$

Here  $F$  satisfies the statement (v) but  $F$  is not fuzzy continuous on  $A$ , since  $V_1 \in T'$  but  $F^{-1}(V_1) \notin T$ . For details c.f. Counter example 3.15.

**Theorem 3.18.** If  $F$  is  $\alpha$ -weakly smooth fuzzy continuous  $\forall \alpha \in (0, 1]$  on  $A$ , then  $F$  is weakly smooth fuzzy continuous on  $A$ .

*Proof.* Let  $V \leq B$  with  $\tau_2(V) > 0$ . Choose  $\alpha$  such that  $0 < \alpha \leq \tau_2(V)$ . Since  $F$  is  $\alpha$ -weakly smooth fuzzy continuous,  $\tau_1(F^{-1}(U)) \geq \alpha > 0$  and hence  $F$  is weakly smooth fuzzy continuous on  $A$ .  $\square$

**Counter example 3.19.** There exists a fuzzy proper function  $F$  which is weakly smooth fuzzy continuous but not  $\alpha$ -weakly smooth fuzzy continuous for some  $\alpha \in (0, 1]$ .

Let  $X = \{r, s\}$ ,  $Y = \{\xi, \zeta\}$ . If  $A_{[r,s]}^{[0.7,0.5]}$ ,  $B_{[\xi,\zeta]}^{[0.8,0.6]}$ ,  $U_{1[r,s]}^{[0.6,0.5]}$ ,  $U_{2[r,s]}^{[0.5,0.4]}$ ,  $V_{1[\xi,\zeta]}^{[0.6,0.6]}$ ,  $V_{2[\xi,\zeta]}^{[0.5,0.4]}$ , then  $U_1, U_2 \leq A \in I^X$ ,  $V_1, V_2 \leq B \in I^Y$ . We define smooth fuzzy topologies  $\tau_1$  on  $A$  and  $\tau_2$  on  $B$  by

$$\tau_1(U) = \begin{cases} 1, & U = 0 \text{ or } A, \\ 0.6, & U = U_1, \\ 0.5, & U = U_2, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tau_2(V) = \begin{cases} 1, & V = 0 \text{ or } B, \\ 0.7, & V = V_1, \\ 0.3, & V = V_2, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that  $(A, \tau_1)$ ,  $(B, \tau_2)$  are two Šostak's  $I$ -fuzzy topological spaces. Let the proper function  $F : (A, \tau_1) \rightarrow (B, \tau_2)$  be defined by

$$F(r, \xi) = 0.7, F(r, \zeta) = 0, F(s, \xi) = 0, F(s, \zeta) = 0.5.$$

We note that  $F^{-1}(V_1)_{[r,s]}^{[0.6,0.5]}$ ,  $F^{-1}(V_2)_{[r,s]}^{[0.5,0.4]}$ , and  $F^{-1}(B)_{[r,s]}^{[0.7,0.5]}$ . Therefore  $F$  is weakly smooth fuzzy continuous on  $A$ . But  $F$  is not  $\alpha$ -weakly smooth fuzzy continuous for  $\alpha = 0.65$ . Because  $\tau_2(V_1) = 0.7 > \alpha$  but  $\tau_1(F^{-1}(V_1)) = \tau_1(U_1) = 0.6 \not> \alpha$ .

**Theorem 3.20.** *If  $F$  is weakly smooth fuzzy continuous at  $P_x^\lambda \in A$  and  $G$  is weakly smooth fuzzy continuous at  $F(P_x^\lambda) \in B$ , then  $G \circ F$  is weakly smooth fuzzy continuous at  $P_x^\lambda \in A$ .*

*Proof.* Let  $W$  be a neighborhood of  $(G \circ F)(P_x^\lambda) = G(F(P_x^\lambda))$ . Since  $G$  is weakly smooth fuzzy continuous at  $F(P_x^\lambda)$  and  $W$  is a neighborhood of  $G(F(P_x^\lambda))$ , then there exists a neighborhood  $V$  of  $F(P_x^\lambda)$  such that  $G(V) \leq W$ . Since  $F$  is weakly smooth fuzzy continuous at  $P_x^\lambda$  and  $V$  is a neighborhood of  $F(P_x^\lambda)$ , then there exists a neighborhood  $U$  of  $P_x^\lambda$  such that  $F(U) \leq V$ . By Lemma 2.14,  $(G \circ F)(U) = G(F(U)) \leq G(V) \leq W$ .  $\square$

**Theorem 3.21** ([18]). *If  $F : A \rightarrow B$  is smooth fuzzy continuous on  $A$  and  $G : B \rightarrow C$  is smooth fuzzy continuous on  $B$ , then  $G \circ F : A \rightarrow C$  is smooth fuzzy continuous on  $A$ .*

Similarly, we can prove the following theorems.

**Theorem 3.22.** *If  $F : A \rightarrow B$  is weakly smooth fuzzy continuous on  $A$  and  $G : B \rightarrow C$  is weakly smooth fuzzy continuous on  $B$ , then  $G \circ F : A \rightarrow C$  is weakly smooth fuzzy continuous on  $A$ .*

**Theorem 3.23.** *Let  $\alpha \in (0, 1]$ . If  $F : A \rightarrow B$  is  $\alpha$ -weakly smooth fuzzy continuous on  $A$  and  $G : B \rightarrow C$  is  $\alpha$ -weakly smooth fuzzy continuous on  $B$ , then  $G \circ F : A \rightarrow C$  is  $\alpha$ -weakly smooth fuzzy continuous on  $A$ .*

**Theorem 3.24.** *If  $F$  is  $\alpha$ -weakly smooth fuzzy continuous at  $P_x^\lambda \in A$  and  $G$  is  $\alpha$ -weakly smooth fuzzy continuous at  $F(P_x^\lambda) \in B$ , then  $G \circ F$  is  $\alpha$ -weakly smooth fuzzy continuous at  $P_x^\lambda \in A$ .*

**Theorem 3.25.** *If  $F$  is  $qn$ -weakly smooth fuzzy continuous at  $P_x^\lambda \in A$  and  $G$  is  $qn$ -weakly smooth fuzzy continuous at  $F(P_x^\lambda) \in B$ , then  $G \circ F$  is  $qn$ -weakly smooth fuzzy continuous at  $P_x^\lambda \in A$ .*

#### 4. Projection maps on product of fuzzy sets

In this section, we define projection maps on product of fuzzy subsets as fuzzy proper functions. Throughout this section, we use  $\prod$  to denote  $\prod_{j \in J}$ , where  $J$  is a fixed index set. Now we recall the product topology on product of fuzzy sets from [21].

**Definition 4.1** ([21]). Let  $\{(X_j, \tau_j) : j \in J\}$  be family of sfts and  $P_k : \prod X_j \rightarrow X_k$  denote the  $k$ -th projection map. Let  $\mathcal{S} = \{P_k^{-1}(U_k) : \tau_k(U_k) > 0, k \in J\}$  and  $\mathcal{B}_{\mathcal{S}}$  be the collection of all finite intersections of members of  $\mathcal{S}$ . Define  $\tau : \mathcal{S} \rightarrow I$  by  $\tau(P_k^{-1}(U_k)) = \tau_k(U_k)$ . Then  $\tau_{\mathcal{S}}$  is defined as follows:

$$\tau_{\mathcal{S}}(U) = \begin{cases} \inf \{\tau(E_1), \tau(E_2)\}, & U = E_1 \wedge E_2 \text{ where } E_1, E_2 \in \mathcal{S}, \\ \sup \tau(W_i), & U = \bigvee_i W_i \text{ where each } W_i \in \mathcal{B}_{\mathcal{S}}, \\ 0, & \text{otherwise,} \end{cases}$$

$\tau_{\mathcal{S}}$  is called the product of  $\tau_j$ 's and  $(\prod X_j, \tau_{\mathcal{S}})$  is called the product of fuzzy topological spaces  $\{(X_j, \tau_j) : j \in J\}$ .

**Definition 4.2.** Let  $A \in I^X$ ,  $B_j \in I^{Y_j}$ ,  $\forall j \in J$ . If  $F_j : A \rightarrow B_j$ , then we define  $[F_j] : A \rightarrow \prod B_j$  by

$$([F_j])(x, [y_j]) = \inf_{j \in J} F_j(x, y_j), \quad \forall (x, [y_j]) \in X \times \prod Y_j.$$

**Definition 4.3.** Let  $A_j \in I^{X_j}$  for every  $j \in J$ . For every  $k \in J$ , the  $k$ -th projection map  $P_k : \prod A_j \rightarrow A_k$  is defined as a fuzzy proper function such that for every  $[x_j] \in \prod X_j$ , and  $y_k \in X_k$ ,

$$P_k([x_j], y_k) = \begin{cases} \prod A_j([x_j]) & y_k = x_k \\ 0 & y_k \neq x_k. \end{cases}$$

**Lemma 4.4.** Let  $P_k : \prod A_j \rightarrow A_k$  be a smooth projection map. If  $U_k \leq A_k$ , then  $P_k^{-1}(U_k) = \prod_{j \neq k} A_j \times U_k$ .

*Proof.* For an arbitrary  $[x_j] \in \prod X_j$ ,

$$\begin{aligned} P_k^{-1}(U_k)([x_j]) &= \prod A_j([x_j]) \wedge U_k(x_k) \\ &= \left( \inf_{j \in J} A_j(x_j) \right) \wedge U_k(x_k) \\ &= \inf_{j \in J} (A_j(x_j) \wedge U_k(x_k)) \\ &= \inf_{j \neq k} (A_j(x_j) \wedge U_k(x_k)) \quad (\text{since } A_k(x_k) \geq U_k(x_k)) \\ &= \left( \inf_{j \neq k} A_j(x_j) \right) \wedge U_k(x_k) \\ &= \left( \prod_{j \neq k} A_j \times U_k \right)([x_j]). \end{aligned}$$

□

**Theorem 4.5.** Let  $P_k : \prod A_j \rightarrow A_k$  be the  $k$ -th projection, where  $k \in J$ . Then

- (1)  $P_k$  is smooth fuzzy continuous on  $\prod A_j$ ,
- (2)  $P_k$  is weakly smooth fuzzy continuous on  $\prod A_j$ ,
- (3)  $P_k$  is  $\alpha$ -weakly smooth fuzzy continuous on  $\prod A_j$ ,  $\forall \alpha \in (0, 1]$ .

*Proof.* Let  $U_k \leq A_k$ . From the previous lemma, and by using the definition of product topology  $\tau$  on sfts, we have  $\tau(P_k^{-1}(U_k)) = \tau(U_k \times \prod_{j \neq k} A_j) = \tau_k(U_k)$ . Consequently, the theorem follows.  $\square$

**Lemma 4.6.** If  $F : A \rightarrow \prod B_k$ , then  $F = [P_j \circ F]$ , where  $P_j : \prod B_k \rightarrow B_j$  is the  $j$ -th projection map for every  $j \in J$ .

*Proof.* Let  $x \in X$  be arbitrary. By definition, there exists unique  $[y_j] \in \prod Y_j$  is such that  $F(x, [y_j]) = A(x)$ . To prove this lemma, we shall show that

$$([P_j \circ F])(x, [z_j]) = \begin{cases} A(x) & [z_j] = [y_j] \\ 0 & [z_j] \neq [y_j]. \end{cases}$$

Assume that  $[z_j] = [y_j]$ . Let  $k \in J$  be arbitrary. Then we have  $z_k = y_k$  and hence

$$\begin{aligned} (P_k \circ F)(x, z_k) &= (P_k \circ F)(x, y_k) \\ &= \bigvee_{[w_j] \in \prod Y_j} \{F(x, [w_j]) \wedge P_k([w_j], y_k)\} \\ &= F(x, [y_j]) \wedge P_k([y_j], y_k) \\ &= A(x) \wedge \prod B_j([y_j]) \\ &= A(x) \text{ (since } F(x, [y_j]) = A(x) \leq \prod B_j([y_j]) \text{)}. \end{aligned}$$

Therefore  $([P_j \circ F])(x, [z_j]) = \inf_{j \in J} (P_j \circ F)(x, z_j) = \inf_{j \in J} A(x) = A(x)$ .

Next assume that  $[z_j] \neq [y_j]$ . Then there exists  $i \in J$  such that  $z_i \neq y_i$ . Now

$$\begin{aligned} (P_i \circ F)(x, z_i) &= \bigvee_{[w_j] \in \prod Y_j} \{F(x, w_j) \wedge P_i([w_j], z_i)\} \\ &= F(x, [y_j]) \wedge P_i([y_j], z_i) = 0. \end{aligned}$$

Thus  $([P_j \circ F])(x, [z_j]) = \inf_{j \in J} (P_j \circ F)(x, z_j) = 0$ , since  $(P_i \circ F)(x, z_i) = 0$  and  $i \in J$ .  $\square$

From the above lemma, it follows that if  $F : A \rightarrow \prod B_j$  and  $F = [F_j]$ , then  $F_j = P_j \circ F$ ,  $\forall j \in J$ , where  $P_j$  is the  $j$ -th projection on  $\prod B_j$ .

*Remark 4.7.* If  $x \in X$  and  $[y_j] \in \prod Y_j$  such that  $F(x, [y_j]) = A(x)$ , then for every  $k \in J$ , for the same given  $x \in X$ , the  $k$ -th coordinate  $y_k$  of  $[y_j]$  gives  $(P_k \circ F)(x, y_k) = A(x)$ .

**Theorem 4.8.** *Let  $F : (A, \sigma) \rightarrow (\prod A_j, \tau)$  be a proper function such that  $F = [F_j]$ , where  $F_j : (A, \sigma) \rightarrow (A_j, \tau_j)$  for every  $j \in J$ . If  $F$  is smooth fuzzy continuous on  $A$ , then  $F_j$  is smooth fuzzy continuous for every  $j \in J$  on  $A$ .*

*Proof.* Assume that  $F$  is smooth fuzzy continuous. By using Lemma 4.6, we have  $F_j = P_j \circ F$  for every  $j \in J$ . Since each  $P_j$  is smooth fuzzy continuous and  $F$  is smooth fuzzy continuous, applying Theorem 3.21, we get  $F_j = P_j \circ F$  is smooth fuzzy continuous on  $A$  for every  $j \in J$ .  $\square$

**Counter example 4.9.** The converse of the above theorem is not true.

Let  $X = \{r, s\}$ ,  $Y = \{\xi, \zeta\}$ ,  $A_{[r,s]}^{[0.6,0.7]} \in I^X$ ,  $B_1_{[\xi,\zeta]}^{[0.8,0.7]}$ ,  $B_2_{[\xi,\zeta]}^{[0.7,0.9]} \in I^Y$ ,  $U_n_{[r,s]}^{[0.6-\frac{1}{n+2}, 0.5-\frac{1}{n+2}]}$ ,  $\forall n = 1, 2, 3, \dots$ ,  $V_0_{[\xi,\zeta]}^{[0.5,0.4]}$ , and  $W_n_{[\xi,\zeta]}^{[0.5-\frac{1}{n+2}, 0.6-\frac{1}{n+2}]}$ ,  $\forall n = 1, 2, 3, \dots$ . If  $\sigma : \mathfrak{I}_A \rightarrow I$  is defined by

$$\sigma(U) = \begin{cases} 1, & U = 0 \text{ or } A, \\ \frac{1}{n}, & U = U_n, n = 1, 2, 3, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

$\tau_1 : \mathfrak{I}_{B_1} \rightarrow I$  by

$$\tau_1(V) = \begin{cases} 1, & V = 0 \text{ or } B_1, \\ 0.1, & V = V_0, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\tau_2 : \mathfrak{I}_{B_2} \rightarrow I$  is defined by

$$\tau_2(W) = \begin{cases} 1, & W = 0 \text{ or } B_2, \\ \frac{1}{n+1}, & W = W_n, n = 1, 2, 3, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

then  $(X, \tau_1)$  and  $(Y, \tau_2)$  are Šostak's  $I$ -fuzzy topological spaces. Define a fuzzy proper function  $F : (A, \sigma) \rightarrow (B_1 \times B_2, \tau)$  by

$$F(r, (\xi, \zeta)) = A(r), F(r, (\xi, \xi)) = 0, F(r, (\zeta, \xi)) = 0, F(r, (\zeta, \zeta)) = 0;$$

and

$$F(s, (\zeta, \xi)) = A(s), F(s, (\xi, \xi)) = 0, F(s, (\xi, \zeta)) = 0, F(s, (\zeta, \zeta)) = 0.$$

Let  $U = \bigvee_{n \in \mathbb{N}} (B_1 \times W_n)$ . Since  $\sigma\left(F^{-1}(U)_{[r,s]}^{[0.6,0.5]}\right) = 0 < \tau(U) = 0.5$ , we get  $F$  is not smooth fuzzy continuous on  $A$ . But  $F_i : (A, \sigma) \rightarrow (B_i, \tau_i)$  are weakly smooth fuzzy continuous for  $i = 1, 2$ .

**Theorem 4.10.** *Let  $F = [F_j] : (A, \sigma) \rightarrow (\prod A_j, \tau)$  be a proper function. If  $F$  is weakly smooth fuzzy continuous, then  $F_j$  is weakly smooth fuzzy continuous for every  $j \in J$ .*

*Proof.* Assume that  $F$  is weakly smooth fuzzy continuous. By Lemma 4.6, we have  $F_j = P_j \circ F$  for every  $j \in J$ . Since each  $P_j$  is weakly smooth fuzzy continuous and  $F$  is weakly smooth fuzzy continuous by using Theorem 3.22, we get that  $F_j = P_j \circ F$  is weakly smooth fuzzy continuous on  $A$  for every  $j \in J$ .  $\square$

**Counter example 4.11.** The converse of the above theorem is not true.

See the same function given in Counter example 4.9.

The converse holds when we use a positive minimum Šostak's  $I$ -fuzzy topology in the domain of  $F$  or when  $F$  is  $\alpha$ -weakly smooth fuzzy continuous.

**Theorem 4.12.** *Let  $(A, \sigma)$  be a positive minimum Šostak's  $I$ -fuzzy topological space and  $(A_j, \tau_j)$  be a fuzzy topological space. A fuzzy proper function  $F : (A, \sigma) \rightarrow (\prod A_j, \tau)$  is weakly smooth fuzzy continuous if and only if  $F_j$  is weakly smooth fuzzy continuous for every  $j \in J$ .*

*Proof.* Since one part of the proof is analogous to that of Theorem 4.10, we prove only that if  $F_j$  is weakly smooth fuzzy continuous on  $A$  for every  $j \in J$ , then  $F$  is weakly smooth fuzzy continuous on  $A$ . Let  $B \in \mathcal{B}_{\mathcal{F}}$ . Then  $B = \prod U_j$ , where  $U_j = A_j, \forall j \in J$  with  $j \neq j_1, j_2, \dots, j_n$  and  $\tau_j(U_j) > 0, \forall j = j_1, j_2, \dots, j_n$ , for some  $j_1, j_2, \dots, j_n \in J$ . We claim that  $F^{-1}(B) = \bigwedge_{j \in J} (P_j \circ F)^{-1}(U_j)$ .

$$\begin{aligned} (F^{-1}(B))(x) &= A(x) \wedge B([x_j]) \\ &\quad \left( \text{where } [x_j] \in \prod X_j \text{ is such that } F(x, [x_j]) = A(x) \right) \\ &= A(x) \wedge \prod U_j([x_j]) = A(x) \wedge \inf_{j \in J} U_j(x_j) \\ &= \inf_{j \in J} (A(x) \wedge U_j(x_j)) = \inf_{j \in J} ((P_j \circ F)^{-1}(U_j))(x) \\ &\quad \text{(by using Remark 4.7)} \\ &= \left( \bigwedge_{j \in J} F_j^{-1}(U_j) \right)(x) \quad \text{(since } P_j \circ F = F_j). \end{aligned}$$

Therefore

$$\begin{aligned} \sigma(F^{-1}(B)) &= \sigma \left( \bigwedge_{j \in J} F_j^{-1}(U_j) \right) \\ &= \sigma \left( \bigwedge_{i=1}^n F_{j_i}^{-1}(U_{j_i}) \bigwedge_{j \neq j_1, j_2, \dots, j_n} \bigwedge F_j^{-1}(A_j) \right) \\ &= \sigma \left( \bigwedge_{i=1}^n F_{j_i}^{-1}(U_{j_i}) \right) \quad \text{(since } F_j^{-1}(A_j) = A, \forall j) \\ &\geq \bigwedge_{i=1}^n \sigma(F_{j_i}^{-1}(U_{j_i})). \end{aligned}$$

Since  $F_{j_i}$  is weakly smooth fuzzy continuous, we have  $\sigma(F_{j_i}^{-1}(U_{j_i})) > 0$  for every  $i = 1, 2, \dots, n$ . Hence  $\sigma(F^{-1}(B)) > 0$ .

If  $U \in \prod A_j$  with  $\tau(U) > 0$ , then  $U = \bigvee_{k \in K} B_k$ , where each  $B_k \in \mathcal{B}_{\mathcal{F}}$  and  $K$  is any index set. Since  $\sigma(F^{-1}(B_k)) > 0, \forall k \in K$ , and  $\sigma$  is a positive minimum Šostak's  $I$ -fuzzy topology, we get  $\sigma(F^{-1}(U)) = \sigma(F^{-1}(\bigvee_{k \in K} B_k)) = \sigma(\bigvee_{k \in K} F^{-1}(B_k)) \geq \bigwedge_{k \in K} \sigma(F^{-1}(B_k)) > 0$ .  $\square$

**Theorem 4.13.** *Let  $\alpha \in (0, 1]$ .  $F : A \rightarrow \prod B_j$  is  $\alpha$ -weakly smooth fuzzy continuous if and only if  $F_j$  is  $\alpha$ -weakly smooth fuzzy continuous for every  $j \in J$ .*

## References

- [1] H. Aygün and S. E. Abbas, *Some good extensions of compactness in Šostak's  $L$ -fuzzy topology*, Hacet. J. Math. Stat. **36** (2007), no. 2, 115–125.
- [2] M. K. Chakraborty and T. M. G. Ahsanullah, *Fuzzy topology on fuzzy sets and tolerance topology*, Fuzzy Sets and Systems **45** (1992), no. 1, 103–108.
- [3] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.
- [4] A. K. Chaudhuri and P. Das, *Some results on fuzzy topology on fuzzy sets*, Fuzzy Sets and Systems **56** (1993), no. 3, 331–336.
- [5] M. Demirci, *Neighborhood structures of smooth topological spaces*, Fuzzy Sets and Systems **92** (1997), no. 1, 123–128.
- [6] ———, *Three topological structures of smooth topological spaces*, Fuzzy Sets and Systems **101** (1999), no. 1, 185–190.
- [7] ———, *On several types of compactness in smooth topological spaces*, Fuzzy Sets and Systems **90** (1997), no. 1, 83–88.
- [8] M. K. El Gayyar, E. E. Kerre, and A. A. Ramadan, *Almost compactness and near compactness in smooth topological spaces*, Fuzzy Sets and Systems **62** (1994), no. 2, 193–202.
- [9] M. A. Fath Alla and F. S. Mahmoud, *Fuzzy topology on fuzzy sets: functions with fuzzy closed graphs and strong fuzzy closed graphs*, J. Fuzzy Math. **9** (2001), no. 3, 525–533.
- [10] S. Ganguly and S. Saha, *A note on compactness in a fuzzy setting*, Fuzzy Sets and Systems **34** (1990), no. 1, 117–124.
- [11] R. N. Hazra and S. K. Samanta, and K. C. Chattopadhyay, *Fuzzy topology redefined*, Fuzzy Sets and Systems **45** (1992), no. 1, 79–82.
- [12] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56** (1976), no. 3, 621–633.
- [13] ———, *A comparison of different compactness notions in fuzzy topological spaces*, J. Math. Anal. Appl. **64** (1978), no. 2, 446–454.
- [14] H. C. Ming, *Fuzzy topological spaces*, J. Math. Anal. Appl., **110**(1) (1985), 141–178.
- [15] P. P. Ming and L. Y. Ming, *Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), no. 2, 571–599.
- [16] ———, *Fuzzy topology, II. Product and quotient spaces*, J. Math. Anal. Appl. **77** (1980), no. 1, 20–37.
- [17] W. Peeters, *Subspaces of smooth fuzzy topologies and initial smooth fuzzy structures*, Fuzzy Sets and Systems **104** (1999), no. 3, 423–433.
- [18] A. A. Ramadan, M. A. Fath Alla, and S. E. Abbas, *Smooth Fuzzy topology on fuzzy sets*, J. Fuzzy Math. **10** (2002), no. 1, 59–68.
- [19] A. P. Šostak, *On a fuzzy topological structure*, Rend. Circ. Mat. Palermo (2) Suppl. No. **11** (1985), 89–103.
- [20] ———, *On some modification of fuzzy topologies*, Mat. Vesnik **41** (1989), no. 1, 51–64.

- [21] R. Srivastava, *On separation axioms in a newly defined fuzzy topology*, Fuzzy Sets and Systems **62** (1994), no. 3, 341–346.
- [22] R. Srivastava, S. N. Lal, and A. K. Srivastava, *Fuzzy Hausdorff topological spaces*, J. Math. Anal. Appl. **81** (1981), no. 2, 497–506.
- [23] ———, *Fuzzy  $T_1$ -topological spaces*, J. Math. Anal. Appl. **102** (1984), no. 2, 442–448.
- [24] R. H. Warren, *Neighborhoods, bases and continuity in fuzzy topological spaces*, Rocky Mountain J. Math. **8** (1978), no. 3, 459–470.
- [25] C. K. Wong, *Fuzzy topology: Product and quotient theorems*, J. Math. Anal. Appl. **45** (1974), 512–521.

RAJAKUMAR ROOPKUMAR  
DEPARTMENT OF MATHEMATICS  
ALAGAPPA UNIVERSITY  
KARAIKUDI 630003, INDIA  
*E-mail address:* roopkumarr@rediffmail.com

CHANDRAN KALAIVANI  
DEPARTMENT OF MATHEMATICS  
ALAGAPPA UNIVERSITY  
KARAIKUDI 630003, INDIA  
*E-mail address:* kalai\_posa@yahoo.co.in