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REMARKS ON γ-OPERATIONS INDUCED BY A TOPOLOGY

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Abstract. Császár [3] introduced the notions of γ-compact and γ-operation on a topological space. In this paper, we introduce the notions of almost Γ-compact, (γ,τ)-continuous function and (γ,τ)-open function defined by γ-operation on a topological space and investigate some properties for such notions.

1. Introduction and preliminaries

Let X be a non-empty set with the power set $\exp X$. A function $\gamma : \exp X \to \exp X$ is said to be *monotonic* [2] if and only if $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$. The collection of all monotonic functions is denoted by $\Gamma(X)$ and the elements of $\Gamma(X)$ are said to be *operations*. If $\gamma \in \Gamma(X)$, then a set $A \subseteq X$ is said to be γ-*open* [2] if $A \subseteq \gamma A$. For $A \subseteq X$, we denote by $i_\gamma A$ the union of all γ-open sets contained in A, i.e., the largest γ-open set contained in A. The complement of a γ-open set is said to be γ-*closed*. Any intersection of γ-closed sets is γ-closed, and for $A \subseteq X$, we denote by $c_\gamma A$ the intersection of all γ-closed sets containing A, i.e. the smallest γ-closed set containing A. If τ is a topology on X and we write c for *closure*, i for *interior*, $s = ci$, $p = ic$, $\alpha = ici$, $\beta = cic$, then $c, i, s, p, \alpha, \beta$ are all all elements of $\Gamma(X)$ and s-open, p-open, α-open, β-open sets are said to be *semiopen* [5], *preopen* [6], α-*open* [7], β-*open* [1], respectively. Let $\gamma \in \Gamma(X)$ and $\gamma' \in \Gamma(Y)$. Then a function $f : X \to Y$ is said to be (γ,γ')-*continuous* [4] if for each γ'-open set V in Y, $f^{-1}(V)$ is γ-open in X. And f is said to be (γ,γ')-*open* [4] if for each γ-open set A in X, $f(A)$ is γ'-open in Y. In this paper, we introduce the notions of (γ,τ)-continuous function and (γ,τ)-open function and investigate characterizations for such functions. We also introduce the notion of almost Γ-compact defined by γ-operation on a topological space.

Theorem 1.1 ([2]). *Let $\gamma \in \Gamma(X)$ and $A \subseteq X$. Then the statements are hold*:

(1) *Any union of γ-open sets is γ-open.*

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(2) $i_\gamma A = X - c_\gamma(X - A)$, $c_\gamma A = X - i_\gamma(X - A)$.

2. (γ, τ)-continuous function and (γ, τ)-open function

We recall the notion of γ-operation introduced in [3]: Let (X, μ) be a topological space, and $\gamma : \exp X \to \exp X$ a mapping such that
(1) $A \subseteq B \Rightarrow \gamma A \subseteq \gamma B$.
(2) $\gamma\emptyset = \emptyset$, $\gamma X = X$.
(3) For $A \subseteq X$ and an open set $G \subseteq X$, $G \cap \gamma A \subseteq \gamma(G \cap A)$.

Now we call the mapping γ an *associated operation* with μ on X. We will denote an associated operation γ with μ by γ_μ (simply γ).

According to [2], a set $A \subseteq X$ is said to be γ-*open* if and only if $A \subseteq \gamma A$. Let (X, μ) be a topological space and γ an associated operation with μ. Then an open set is always γ-open [3].

Definition 2.1. Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. Then a function $f : (X, \mu) \to (Y, \tau)$ is (γ, τ)-continuous if for every open set F in Y, $f^{-1}(F)$ is γ-open in X.

Remark 2.2. Let (X, μ) and (Y, τ) be topological spaces and let γ and γ' associated operations with μ and τ, respectively. Then the (γ, γ')-continuous function $f : X \to Y$ is (γ, τ)-continuous if the associated operation $\gamma' : \exp Y \to \exp Y$ is defined by $\gamma' = int$, where int is the *interior* in Y.

Theorem 2.3. *Let $f : (X, \mu) \to (Y, \tau)$ be a function on topological spaces and γ an associated operation with μ. Then f is (γ, τ)-continuous if and only if for each $x \in X$ and each open set V containing $f(x)$, there exists a γ-open set U containing x such that $f(U) \subseteq V$.*

Proof. Suppose f is (γ, τ)-continuous. Then for each $x \in X$ and each open set V containing $f(x)$, $f^{-1}(V)$ is γ-open. Now put $U = f^{-1}(V)$, then the γ-open U satisfies that $x \in U$ and $f(U) \subseteq V$.

For the converse, let V be an open set in Y. Then for each $x \in f^{-1}(V)$, by hypothesis, there exists a γ-open set U containing x such that $U \subseteq f^{-1}(V)$, and so $f^{-1}(V) = \cup U$. By Theorem 1.1(1), $f^{-1}(V)$ is γ-open. □

Theorem 2.4. *Let $f : (X, \mu) \to (Y, \tau)$ be a function on topological spaces and γ an associated operation with μ. Then a function f is (γ, τ)-continuous if and only if $f^{-1}(int(B)) \subseteq \gamma f^{-1}(B)$ for $B \subseteq Y$.*

Proof. Let f be (γ, τ)-continuous and $B \subseteq Y$. Since $f^{-1}(int(B))$ is γ-open in X and γ is monotonic,

$$f^{-1}(int(B)) \subseteq \gamma f^{-1}(int(B)) \subseteq \gamma f^{-1}(B).$$

For the converse, let B be open in Y. Then by hypothesis, $f^{-1}(B) = f^{-1}(int(B)) \subseteq \gamma f^{-1}(B)$, so that $f^{-1}(B)$ is γ-open in X. Hence f is (γ, τ)-continuous. □

Theorem 2.5. *Let $f : (X, \mu) \to (Y, \tau)$ be a function on topological spaces and γ an associated operation with μ. Then the following are equivalent*:

(1) *f is (γ, τ)-continuous.*
(2) *For every closed set F in Y, $f^{-1}(F)$ is γ-closed in X.*
(3) *$f^{-1}(int(B)) \subseteq i_\gamma f^{-1}(B)$ for $B \subseteq Y$.*
(4) *$c_\gamma f^{-1}(B) \subseteq f^{-1}(cl(B))$ for $B \subseteq Y$.*
(5) *$f(c_\gamma A) \subseteq cl(f(A))$ for $A \subseteq X$.*

Proof. Straightforward. □

Definition 2.6. Let (X, τ) and (Y, ν) be topological spaces and γ an associated operation with ν. Then a function $f : (X, \tau) \to (Y, \nu)$ is said to be (τ, γ)-open if for every open set G in X, $f(G)$ is γ-open in Y.

Remark 2.7. Let (X, τ) and (Y, ν) be topological spaces and let γ and γ' associated operations with τ and ν, respectively. Then the (γ, γ')-open function $f : X \to Y$ is (τ, γ')-open if $\gamma : \exp X \to \exp X$ is a mapping defined by $\gamma = int$, where *int* is the *interior* in X.

Theorem 2.8. *Let $f : (X, \tau) \to (Y, \nu)$ be a function on topological spaces and γ an associated operation with ν. Then the following equivalent*:

(1) *f is (τ, γ)-open.*
(2) *$int(f^{-1}(B)) \subseteq f^{-1}(i_\gamma B)$ for $B \subseteq Y$.*
(3) *$f^{-1}(c_\gamma B) \subseteq cl(f^{-1}(B))$ for $B \subseteq Y$.*
(4) *$f(int(A)) \subseteq i_\gamma f(A)$ for $A \subseteq X$.*

Proof. (1) ⇒ (2) For $B \subseteq Y$, by (1), $f(int(f^{-1}(B)))$ is a γ-open set such that $f(int(f^{-1}(B))) \subseteq B$. It implies $f(int(f^{-1}(B))) \subseteq i_\gamma B$, and hence $int(f^{-1}(B)) \subseteq f^{-1}(i_\gamma B)$.

(2) ⇔ (3) It follows from Theorem 1.1.

(2) ⇒ (4) Obvious.

(4) ⇒ (1) Let A be open in X; then by (4) $f(A) = f(int(A)) \subseteq i_\gamma f(A)$, and so $f(A) = i_\gamma f(A)$. Thus $f(A)$ is γ-open. □

Theorem 2.9. *Let a function $f : (X, \tau) \to (Y, \nu)$ be topological spaces and γ an associated operation with ν. Then f is (τ, γ)-open if and only if for $A \subseteq X$, $f(int(A)) \subseteq \gamma f(A)$.*

Proof. Suppose that f is (τ, γ)-open. Then for $A \subseteq X$, by Theorem 2.8, $f(int(A)) \subseteq i_\gamma f(A)$ and since $i_\gamma f(A) \subseteq \gamma i_\gamma f(A)$ and γ is monotonic, we have

$$f(int(A)) \subseteq i_\gamma f(A) \subseteq \gamma i_\gamma f(A) \subseteq \gamma f(A).$$

The converse is obvious. □

Definition 2.10. Let (X, μ) be a topological space and γ an associated operation with μ. Then X is Γ-T_2 if for every two distinct points x and y in X, there exist two γ-open sets U and V containing x and y, respectively, such that $U \cap V = \emptyset$.

Definition 2.11. Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. A function $f : X \to Y$ has a Γ_τ-*closed graph* (resp. *strongly* Γ_τ-*closed graph*) if for each $(x, y) \in (X \times Y) - G(f)$, there exist a γ-open set U and an open set V containing x and y, respectively, such that $(U \times V) \cap G(f) = \emptyset$ (resp. $(U \times cl(V)) \cap G(f) = \emptyset$), where $G(f) = \{(x, f(x)) : x \in X\}$.

Lemma 2.12. *Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. A function $f : X \to Y$ has a Γ_τ-closed graph (resp. strongly Γ_τ-closed graph) if for each $(x, y) \notin G(f)$, there exist a γ-open set U and an open set V containing x and y, respectively, such that $f(U) \cap V = \emptyset$ (resp. $f(U) \cap cl(V) = \emptyset$).*

Theorem 2.13. *Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. If $f : X \to Y$ is (γ, τ)-continuous and Y is a T_2 space, then f has a strongly Γ_τ-closed graph.*

Proof. Let $x, y \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and since Y is T_2, there exist two open sets U and V such that $f(x) \in U$, $y \in V$ and $U \cap V = \emptyset$ and so $U \cap cl(V) = \emptyset$. Since f is (γ, τ)-continuous, by Theorem 2.3, there exists a γ-open set W of x such that $f(W) \subset U$, and so $f(W) \cap cl(V) = \emptyset$. Thus by Lemma 2.12, f has a strongly Γ_τ-closed graph. □

Theorem 2.14. *Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. If $f : X \to Y$ is a surjective function with a strongly Γ_τ-closed graph, then Y is T_2-space.*

Proof. Let y and z be distinct points in Y. Then there exists an $x \in X$ such that $f(x) = y$. Thus $(x, z) \notin G(f)$, since f has a strongly Γ_τ-closed graph, there exist a γ-open set U and an open set V of x and z, respectively, such that $f(U) \cap cl(V) = \emptyset$. This implies $y \notin cl(V)$, so there exists an open set W such that $W \cap V = \emptyset$. Consequently, Y is T_2. □

Theorem 2.15. *Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. If $f : X \to Y$ is a (γ, τ)-continuous injection with a Γ_τ-closed graph, then X is Γ-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X. Then $f(x_1) \neq f(x_2)$. This implies that $(x_1, f(x_2)) \in (X \times Y) - G(f)$. By hypothesis, there exist a γ-open set U and an open set V of x_1 and $f(x_2)$, respectively, such that $(U \times V) \cap G(f) = \emptyset$. Since f is (γ, τ)-continuous, there exists a γ-open set W containing x_2 such that $f(W) \subset V$. Hence $f(W \cap U) \subseteq f(W) \cap f(U) = \emptyset$. Therefore $W \cap U = \emptyset$ and so X is Γ-T_2. □

Let (X, μ) be a topological space and γ an associated operation with μ. A collection $\mathbf{S} = \{S_i \subseteq X : S_i$ is γ-open, $i \in I\}$ is called a γ-*open cover* for X if $X = \cup_{i \in I} S_i$. The space (X, μ) is said to be γ-*compact* (resp. *almost* γ-*compact*) [3] if for each γ-open cover $\mathbf{S} = \{S_i \subseteq X : S_i$ is γ-open, $i \in I\}$, there exists a finite index set $F \subseteq I$ such that $X = \cup_{i\in F} S_i$ (resp. $X = \cup_{i\in F} cl(S_i)$). And we recall that a topological space (X, μ) is said to be *quasi H-closed* [8] if for each open cover $\mathbf{C} = \{G_i \subseteq X : G_i \text{ is open}, i \in I\}$, there exists a finite index set $F \subseteq I$ such that $X = \cup_{i\in F} cl(G_i)$.

Definition 2.16. Let (X, μ) be a topological space and γ an associated operation with μ. The space (X, μ) is said to be *almost* Γ*-compact* if for each γ-open cover $\mathbf{S} = \{S_i \subseteq X : S_i \text{ is } \gamma\text{-open}, i \in I\}$, there exists a finite index set $F \subseteq I$ such that $X = \cup_{i\in F} c_\gamma S_i$.

Remark 2.17. Let (X, μ) be a topological space and γ an associated operation with μ. Since every open set is γ-open, in general, $c_\gamma(A) \subseteq cl(A)$. So obviously the following implications are obtained but the converses may not be true as shown in the next example.

$$\gamma\text{-compact} \Rightarrow \text{almost } \Gamma\text{-compact} \Rightarrow \text{almost } \gamma\text{-compact}$$

Example 2.18. (1) Let X be a topological space and the associated operation $\gamma = int$ in X. Then since γ-compactness is compactness and almost Γ-compactness (almost γ-compactness) is quasi H-closedness, generally the converse is not true.

(2) Let N denote the set of natural numbers. Consider a topology $\mu = \{\emptyset, N_o, N_e, N\}$ where $N_o = \{2n - 1 : n \in N\}$ and $N_e = \{2n : n \in N\}$. Define $\gamma : \exp N \to \exp N$ by $\gamma(A) = int(cl(A))$ for $A \in \exp(N)$. Then the mapping γ is an associated operation with μ on N. Obviously the set of all γ-open sets is $\exp N$, and so we get the conclusion that the space N is almost γ-compact but not almost Γ-compact.

Theorem 2.19. *Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. Let $f : X \to Y$ be a surjective (γ, τ)-continuous function. Then if X is γ-compact, then Y is compact.*

Proof. Obvious. □

Theorem 2.20. *Let (X, μ) and (Y, τ) be topological spaces and γ an associated operation with μ. Let $f : X \to Y$ be a (γ, τ)-continuous and surjective function. If X is almost Γ-compact, then Y is quasi H-closed.*

Proof. Let $\mathcal{S} = \{S_i : i \in J\}$ be an open cover of Y. Then $\{f^{-1}(S_i) : S_i \in \mathcal{S}, i \in J\}$ is a γ-open cover of X and by almost Γ-compactness, there is a finite subcover $\{c_\gamma f^{-1}(S_{j_1}), c_\gamma f^{-1}(S_{j_2}), \ldots, c_\gamma f^{-1}(S_{j_n}) : S_j \in \mathcal{S}, j = j_1, j_2, \ldots, j_n\}$ such that $X \subseteq \cup c_\gamma f^{-1}(S_j)$. Then from Theorem 2.5(4),

$$Y = f(X) \subseteq f(\cup c_\gamma f^{-1}(S_j)) \subseteq \cup f(f^{-1}(cl(S_j))) \subseteq \cup cl(S_j).$$

Hence Y is quasi H-closed. □

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