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CONDITIONAL GENERALIZED FOURIER-FEYNMAN TRANSFORM OF FUNCTIONALS IN A FRESNEL TYPE CLASS

Seung Jun Chang

Abstract. In this paper we define the concept of a conditional generalized Fourier-Feynman transform on very general function space $C_{a,b}[0,T]$. We then establish the existence of the conditional generalized Fourier-Feynman transform for functionals in a Fresnel type class. We also obtain several results involving the conditional transform. Finally we present functionals to apply our results. The functionals arise naturally in Feynman integration theories and quantum mechanics.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space, that is the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. In [21], Yeh introduced the concept of a conditional Wiener integral and derived a Fourier inversion formula for changing conditional expectations into nonconditional expectations. In [17], Park and Skoug obtained a very simple formula for expressing conditional Wiener integrals in terms of ordinary Wiener integrals. The authors, in [21] and [17], derived the Kac-Feynman integral equation for time independent potential functions using their own result, respectively. In [2], Chang and Chung studied the conditional function space integrals and related topics on a very general function space $C_{a,b}[0,T]$ using the vector-valued conditioning function

$$X(x) = (x(t_1), \ldots, x(t_n)), \quad 0 = t_0 < t_1 < \cdots < t_n = T. \tag{1.1}$$

In [10], Chang, Choi and Skoug established a very simple formula for expressing conditional function space integrals in terms of nonconditional function space integrals using a very general conditioning function $X(x)$ on $C_{a,b}[0,T]$ that need not depend upon the values of X at only finitely many points of $(0,T]$

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like in the equation (1.1) above; see the equation (4.2) below. The function space $C_{a,b}[0,T]$ induced by generalized Brownian motion was introduced by J. Yeh in [19] and was used extensively by Chang and Chung [2], Chang and Skoug [4], Chang, Choi and Skoug [7–9] and Chang, Chung and Skoug [11].

In [14], Chung and Skoug introduced the concept of a conditional Feynman integral and applied their results to obtain a fundamental solution of Schrödinger equation, whereas in [18], Park and Skoug introduced the concept of a conditional Fourier-Feynman transform on Wiener space. Other work involving conditional Feynman integrals and conditional Fourier-Feynman transforms on Wiener space include [3, 13]. In [8], Chang, Choi and Skoug established various integration by parts formulas for conditional generalized Feynman integrals and conditional generalized Fourier-Feynman transforms(CGFFT) using the conditioning function $X(x) = x(T)$, $x \in C_{a,b}[0,T]$.

In this paper, working in the setting of general function space $C_{a,b}[0,T]$ and using the conditioning function X given by the equation (4.2) below, we introduce a concept of a CGFFT and obtain several results for the CGFFT of functionals in a Fresnel type class.

The Wiener process used in [3, 13, 14, 17, 18, 21] is stationary in time and is free of drift while the stochastic process used in [2, 4, 7–10, 11, 19] and in this paper is nonstationary in time and is subject to a drift $a(t)$. However when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $C_{a,b}[0,T]$ reduces to Wiener space $C_0[0,T]$.

2. Preliminaries

In this section, we briefly list some of the preliminaries from [4, 7, 9] that we need to establish our results in next sections.

Let $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ denote the function space induced by the generalized Brownian motion Y determined by continuous functions $a(t)$ and $b(t)$ where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra induced by the sup-norm, see [19, 20]. We assume in this paper that $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. Then we can consider the coordinate process $Z : [0,T] \times C_{a,b}[0,T] \to \mathbb{R}$ given by $Z(t,x) = x(t)$ which is a continuous version of Y [20]. The generalized Brownian motion Z is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.). If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., then we write $F \approx G$.

Let $L^2_{a,b}[0,T]$ be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(t)$ and $b(t)$: i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\},$$

where $|a|(t)$ is the total variation function of $a(t)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)].$$

In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ for m_L-a.e. on $[0,T]$ where m_L is the Lebesgue measure on $[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$\left(L^2_{a,b}[0,T], \|\cdot\|_{a,b}\right) \subset \left(L^2_{0,b}[0,T], \|\cdot\|_{0,b}\right) = \left(L^2[0,T], \|\cdot\|_2\right)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

For each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral

$$\langle v,x\rangle = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (v,\phi_j)_{a,b}\phi_j(t)dx(t)$$

exists for μ-a.e. $x \in C_{a,b}[0,T]$, where $\{\phi_j\}_{j=1}^\infty$ is a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that $(\phi_j,\phi_k)_{a,b} = \delta_{jk}$ (the Kronecker delta). If v is of bounded variation on $[0,T]$, then the PWZ stochastic integral $\langle v,x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_{a,b}[0,T]$.

Remark 2.1. For each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v,x\rangle$ is a Gaussian random variable on $C_{a,b}[0,T]$ with mean $\int_0^T v(s)da(s)$ and variance $\int_0^T v^2(s)db(s)$. Note that for all $u,v \in L^2_{a,b}[0,T]$,

$$\int_{C_{a,b}[0,T]} \langle u,x\rangle\langle v,x\rangle d\mu(x) = \int_0^T u(s)v(s)db(s) + \int_0^T u(s)da(s)\int_0^T v(s)da(s).$$

Hence we see that for all $u,v \in L^2_{a,b}[0,T]$, $\int_0^T u(s)v(s)db(s) = 0$ if and only if $\langle u,x\rangle$ and $\langle v,x\rangle$ are independent random variables.

Let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$D_t w = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T D_t w_1 D_t w_2 db(t) = \int_0^T z_1(t) z_2(t) db(t)$$

is a separable Hilbert space. For more details, see [9, 12].

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are homeomorphic under the linear operator given by the equation (2.1).

Throughout this paper, we assume $a \in C'_{a,b}[0,T]$ and for notational convenience we will use the notation $(w,x)^{\sim}$ instead of $\langle D_t w, x\rangle$. Then we have the following assertions.

(1) For each $w \in C'_{a,b}[0,T]$, the random variable $x \mapsto (w,x)^{\sim}$ is Gaussian with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$.
(2) $(w,\alpha x)^{\sim} = \alpha(w,x)^{\sim} = (\alpha w, x)^{\sim}$ for any real number α, $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$.
(3) If $\{w_1, \ldots, w_n\}$ is an orthonormal set in $C'_{a,b}[0,T]$, then the random variables $(w_i, x)^{\sim}$'s are independent.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] \equiv E_x[F(x)] = \int_{C_{a,b}[0,T]} F(x) d\mu(x)$$

whenever the integral exists.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\tilde{\mathbb{C}}_+$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively. For each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{-1/2}$(or $\lambda^{1/2}$) is always chosen to have positive real part.

3. A Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$

We first introduce a Banach algebra of functionals on $C_{a,b}[0,T]$ like a Fresnel class of an abstract Wiener space.

Let $\mathcal{M}(C'_{a,b}[0,T])$ be the space of $\mathbb{C}$-valued, countably additive (and hence finite) Borel measures on $C'_{a,b}[0,T]$. $\mathcal{M}(C'_{a,b}[0,T])$ is a Banach algebra under the total variation norm and with convolution as multiplication.

The Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$ of functionals on $C_{a,b}[0,T]$ is defined as the space of all stochastic Fourier transforms of elements of $\mathcal{M}(C'_{a,b}[0,T])$; that is, $F \in \mathcal{F}(C_{a,b}[0,T])$ if and only if there exists a measure f in $\mathcal{M}(C'_{a,b}[0,T])$ such that

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\} df(w) \tag{3.1}$$

for s-a.e. $x \in C_{a,b}[0,T]$. More precisely, since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0,T]$, $\mathcal{F}(C_{a,b}[0,T])$ can be regarded as the space of all s-equivalence classes of functionals of the form (3.1).

The Fresnel type class $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0,T]} d|f|(w).$$

In fact, the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication and is a Banach algebra isomorphism where f and F are related by (3.1).

We adopt the definitions and notations of [4, 7] for the concept of the generalized Feynman integral and the generalized Fourier-Feynman transform(GFFT) of functionals on $C_{a,b}[0,T]$.

Remark 3.1. In [4] Chang and Skoug introduced a Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ of functionals on $C_{a,b}[0,T]$ given by

$$\mathcal{S}(L^2_{a,b}[0,T]) = \left\{ F : F(x) \approx \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x\rangle\} d\sigma(v), \sigma \in \mathcal{M}\left(L^2_{a,b}[0,T]\right) \right\}$$

and then showed that the generalized Feynman integral and the GFFT exist for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$ under appropriate conditions.

When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $\mathcal{S}(L^2_{a,b}[0,T])$ reduces to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick [1]. For more detailed studies of Banach algebras of functionals on classical and abstract Wiener spaces, see [16, pp. 609–629]. Also, for a detailed study of functionals in $\mathcal{F}(C_{a,b}[0,T])$, see [12].

For a positive real number q_0 and $w \in C'_{a,b}[0,T]$, let

$$k(q_0; w) = \exp\left\{ (2q_0)^{-1/2} \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}} \right\}$$

and let

$$\Gamma_{q_0} = \left\{ \lambda \in \tilde{\mathbb{C}}_+ : \left| \mathrm{Im}\left(\lambda^{-1/2}\right) \right| < (2q_0)^{-1/2} \right\}. \tag{3.2}$$

Also, for $\lambda \in \tilde{\mathbb{C}}$ and $w \in C'_{a,b}[0,T]$, let

$$\psi(\lambda; w) = \exp\left\{ -\frac{1}{2\lambda} \|w\|^2_{C'_{a,b}} + \lambda^{-1/2}(w, a)_{C'_{a,b}} \right\}. \tag{3.3}$$

Then for all $\lambda \in \Gamma_{q_0}$,

$$|\psi(\lambda; w)| \le \exp\left\{ \left| \mathrm{Im}\left(\lambda^{-1/2}\right) \right| \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}} \right\} < k(q_0; w).$$

We note that for all real q with $|q| > q_0$, $(-iq)^{-1/2} = 1/\sqrt{2|q|} + i\mathrm{sign}(q)/\sqrt{2|q|}$ and $-iq \in \Gamma_{q_0}$.

For a positive real number q_0, we define a subclass $\mathcal{F}^{q_0}$ of $\mathcal{F}(C_{a,b}[0,T])$ by

$$\mathcal{F}^{q_0} = \left\{ F \in \mathcal{F}(C_{a,b}[0,T]) : \int_{C'_{a,b}[0,T]} k(q_0; w) d|f|(w) < \infty \right\},$$

where f and F are related by the equation (3.1).

Remark 3.2. Note that in case $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the function space $C_{a,b}[0, T]$ reduces to the classical Wiener space $C_0[0, T]$ and $(w, a)_{C'_{a,b}} = 0$ for all $w \in C'_{a,b}[0, T] = C'_0[0, T]$. Hence for all $\lambda \in \tilde{\mathbb{C}}_+$, $|\psi(\lambda; w)| \le 1$ and for any positive real number q_0, $\mathcal{F}^{q_0} = \mathcal{F}(C_0[0, T])$.

We now state a theorem for the GFFT of functionals in $\mathcal{F}(C_{a,b}[0, T])$ without proof. One can see similar results in [4, 5, 6].

Theorem 3.3. *Let q_0 be a positive real number. Let $F \in \mathcal{F}^{q_0}$ be given by the equation (3.1). Then for all $p \in [1, 2]$ and all real q with $|q| > q_0$, the L_p analytic GFFT of F, $T_q^{(p)}(F)$ exists and is given by the formula*

$$T_q^{(p)}(F)(y) = \int_{C'_{a,b}[0,T]} \exp\{i(w, y)^\sim\}\psi(-iq; w)df(w) \tag{3.4}$$

for s-a.e. $y \in C_{a,b}[0, T]$, where ψ is given by (3.3). Furthermore, $T_q^{(p)}(F)$ belongs to $\mathcal{F}(C_{a,b}[0, T])$.

Corollary 3.4. *Let q_0 and F be as in Theorem 3.3. Then for all real q with $|q| > q_0$, the generalized analytic Feynman integral of F, $E^{\mathrm{anf}_q}[F]$ exists and is given by the formula*

$$E^{\mathrm{anf}_q}[F] = T_q^{(1)}(F)(0) = \int_{C'_{a,b}[0,T]} \psi(-iq; w)df(w).$$

4. Conditional generalized Fourier-Feynman transform

In this section for $F : C_{a,b}[0, T] \to \mathbb{C}$ and $X : C_{a,b}[0, T] \to \mathbb{R}^n$, we first define the conditional function space integral of F given X which we denote by $E(F|X)$. Then, using the conditioning function X given by the equation (4.2) below, we define the conditional generalized Feynman integral $E^{\mathrm{anf}_q}(F|X)$ and the CGFFT $T_q^{(p)}(F|X)$.

Let $X : C_{a,b}[0, T] \to \mathbb{R}^n$ be a $\mathcal{B}(C_{a,b}[0, T])$-measurable functional whose probability distribution μ_X is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. Let F be a $\mathbb{C}$-valued μ-integrable functional on $C_{a,b}[0, T]$. Then, the conditional function space integral of F given X, denoted by $E(F|X)$ $(\vec{\eta})$, is a Lebesgue measurable function of $\vec{\eta}$, unique up to null sets in $\mathbb{R}^n$, satisfying the equation

$$\int_{X^{-1}(B)} F(x)d\mu(x) = \int_B E(F|X)(\vec{\eta})d\mu_X(\vec{\eta}) \tag{4.1}$$

for all Borel sets B in $\mathbb{R}^n$.

Let $\{g_1, \ldots, g_n\}$ be any orthonormal set in $C'_{a,b}[0, T]$. We note that the corresponding PWZ stochastic integrals $(g_j, x)^\sim$, $j = 1, \ldots, n$, form a set of independent Gaussian random variables on $C_{a,b}[0, T]$. Let $X : C_{a,b}[0, T] \to \mathbb{R}^n$ be the Gaussian random vector defined by

$$X(x) = ((g_1, x)^\sim, \ldots, (g_n, x)^\sim). \tag{4.2}$$

Remark 4.1. We note that the conditioning function X given by the equation (1.1) is the special case of X given by the equation (4.2) with

$$g_j(t) = [b(t_j) - b(t_{j-1})]^{-1/2} \int_0^t \chi_{[t_{j-1},t_j]}(s)db(s)$$

for $j = 1, \dots, n$.

Definition 4.2. Let $X : C_{a,b}[0,T] \to \mathbb{R}^n$ be given by the equation (4.2) and let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a scale-invariant measurable functional such that the function space integral $E[F(\lambda^{-1/2}\cdot)]$ exists as a finite number for all $\lambda > 0$. For $\lambda > 0$ let

$$J_\lambda(\vec{\eta}) = E(F(\lambda^{-1/2}\cdot)|X(\lambda^{-1/2}\cdot))(\vec{\eta})$$

denote the conditional function space integral of $F(\lambda^{-1/2}\cdot)$ given $X(\lambda^{-1/2}\cdot)$. If for almost all $\vec{\eta} \in \mathbb{R}^n$, there exists a function $J^*_\lambda(\vec{\eta})$, analytic in λ on $\mathbb{C}_+$ such that $J^*_\lambda(\vec{\eta}) = J_\lambda(\vec{\eta})$ for all $\lambda > 0$, then $J^*_\lambda(\cdot)$ is defined to be the conditional analytic function space integral of F given X with parameter λ and for $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}(F|X)(\vec{\eta}) = J^*_\lambda(\vec{\eta}).$$

If for fixed real $q \neq 0$, the limit

$$\lim_{\lambda \to -iq} E^{\mathrm{an}_\lambda}(F|X)(\vec{\eta})$$

exists for almost every $\vec{\eta} \in \mathbb{R}^n$, where $\lambda \to -iq$ through $\mathbb{C}_+$, we denote the value of this limit by $E^{\mathrm{anf}_q}(F|X)(\vec{\eta})$ and we call it the conditional generalized analytic Feynman integral of F given X with parameter q.

Next we define $[\cdot] : \mathbb{R}^n \to C'_{a,b}[0,T]$ by $[\vec{\eta}] = \sum_{j=1}^n \eta_j g_j$ for $\vec{\eta} = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ and we write

$$[x] \equiv [X(x)] = \sum_{j=1}^n (g_j, x)^\sim g_j$$

for $x \in C_{a,b}[0,T]$.

We quote the following theorem from [10] which plays an important role in this paper.

Theorem 4.3. *Let X be given by the equation* (4.2) *and let F be a μ-integrable functional on $C_{a,b}[0,T]$. Then*

$$E(F|X)(\vec{\eta}) = E_x[F(x - [x] + [\vec{\eta}])]. \tag{4.3}$$

In view of Theorem 4.3, we can define the CGFFT of functionals on function space $C_{a,b}[0,T]$.

Definition 4.4. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let $T_\lambda(F|X)(y,\vec{\eta})$ denote the conditional analytic function space integral of $F(y + \cdot)$ given $X(\cdot)$ with parameter λ; that is to say

$$\begin{aligned} T_\lambda(F|X)(y,\vec{\eta}) &= E^{\mathrm{an}_\lambda}(F(y+\cdot)|X)(\vec{\eta}) \\ &= E^{\mathrm{an}_\lambda}_x[F(y + x - [x] + [\vec{\eta}])]. \end{aligned} \tag{4.4}$$

Then for $p \in [1,2]$ we define the CGFFT of F given X by the formula ($\lambda \in \mathbb{C}_+$),

$$T_q^{(p)}(F|X)(y,\vec{\eta}) = \begin{cases} \underset{\lambda\to -iq}{\mathrm{l.\,i.\,m.}}\, T_\lambda(F|X)(y,\vec{\eta}), & 1 < p \le 2 \\ \lim\limits_{\lambda\to -iq} T_\lambda(F|X)(y,\vec{\eta}), & p = 1 \end{cases} \tag{4.5}$$

if it exists. In the left hand side of (4.5), the notation $\mathrm{l.\,i.\,m.}_{\lambda\to -iq}$ means that for each $\rho > 0$,

$$\lim_{\lambda\to -iq} \int_{C_{a,b}[0,T]} \big|T_\lambda(F|X)(\rho y,\vec{\eta}) - T_q^{(p)}(F|X)(\rho y,\vec{\eta})\big|^{p'} d\mu(y) = 0,$$

where $1/p + 1/p' = 1$. Note that in the case $p = 1$,

$$\begin{aligned} T_q^{(1)}(F|X)(y,\vec{\eta}) &= E^{\mathrm{anf}_q}(F(y+\cdot)|X)(\vec{\eta}) \\ &= E_x^{\mathrm{anf}_q}[F(y + x - [x] + [\vec{\eta}])]. \end{aligned} \tag{4.6}$$

5. Conditional transforms of functionals in $\mathcal{F}(C_{a,b}[0,T])$

In this section we establish the existence of the CGFFT of functionals in $\mathcal{F}(C_{a,b}[0,T])$. We then use the result for the CGFFT to obtain an expression of the GFFT of functionals in $\mathcal{F}(C_{a,b}[0,T])$. We also establish a translation theorem for the CGFFT of functionals in $\mathcal{F}(C_{a,b}[0,T])$.

Let $\{g_1,\dots,g_n\}$ be an orthonormal set in $C'_{a,b}[0,T]$. For each $w \in C'_{a,b}[0,T]$, let

$$p(w) = w - [w].$$

Then the PWZ stochastic integral $(p(w),x)^\sim$ is a Gaussian random variable with mean

$$A(p(w)) = (w - [w], a)_{C'_{a,b}} = (w,a)_{C'_{a,b}} - \sum_{j=1}^n (g_j,w)_{C'_{a,b}}(g_j,a)_{C'_{a,b}} \tag{5.1}$$

and variance

$$B(p(w)) \equiv \|w-[w]\|^2_{C'_{a,b}} = \|w\|^2_{C'_{a,b}} - \sum_{j=1}^n (g_j,w)^2_{C'_{a,b}}. \tag{5.2}$$

The following lemma is useful in establishing our main theorem for the CGFFT of functionals F in $\mathcal{F}(C_{a,b}[0,T])$. The proof follows from (4.3), (5.1), (5.2) and the change of variable formula.

Lemma 5.1. *Let X be given by the equation* (4.2) *above. For $w \in C'_{a,b}[0,T]$, let $G : C_{a,b}[0,T] \to \mathbb{C}$ be defined by $G(x) = \exp\{i(w,x)^\sim\}$. Then for $\rho > 0$ and $\vec{\eta} \in \mathbb{R}^n$,*

$$E(G(\rho\,\cdot)|X(\rho\,\cdot))(\vec{\eta}) = \exp\left\{ i(w,[\vec{\eta}])_{C'_{a,b}} - \frac{\rho^2}{2}B(p(w)) + i\rho A(p(w)) \right\}, \tag{5.3}$$

where $A(p(w))$ and $B(p(w))$ are given by the equations (5.1) *and* (5.2)*.*

For a positive real number q_0 and $w \in C'_{a,b}[0,T]$, let

$$k_n(q_0;w) = \exp\left\{(1+n)(2q_0)^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\right\} \tag{5.4}$$

and for $\lambda \in \tilde{\mathbb{C}}_+$ let

$$\Psi(\lambda;p(w)) = \exp\left\{-\frac{1}{2\lambda}B(p(w)) + i\lambda^{-1/2}A(p(w))\right\}. \tag{5.5}$$

Let Γ_{q_0} be given by (3.2). Then for all $\lambda \in \Gamma_{q_0}$,

$$\begin{aligned}
&|\Psi(\lambda;p(w))| \\
&\le \exp\left\{\left|\mathrm{Im}\left(\lambda^{-1/2}\right)\right| |A(p(w))|\right\} \\
&< \exp\left\{(2q_0)^{-1/2}\left(\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}} + \sum_{j=1}^{n}\|w\|_{C'_{a,b}}\|g_j\|^2_{C'_{a,b}}\|a\|_{C'_{a,b}}\right)\right\} \\
&= k_n(q_0;w).
\end{aligned}$$

To obtain our results for the CGFFT, for positive real number q_0, we define a subclass $\mathcal{F}_n^{q_0}$ of $\mathcal{F}(C_{a,b}[0,T])$ by

$$\mathcal{F}_n^{q_0} = \left\{F \in \mathcal{F}(C_{a,b}[0,T]) : \int_{C'_{a,b}[0,T]} k_n(q_0;w)d|f|(w) < \infty\right\},$$

where f and F are related by the equation (3.1).

Remark 5.2. (i) $\mathcal{F}_n^{q_0}$ is a subclass of $\mathcal{F}^{q_0}$ for each positive real number q_0 and every $n \in \mathbb{N}$.

(ii) When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $A(p(w)) = 0$ for all $w \in C'_{a,b}[0,T] = C'_0[0,T]$. Hence for all $\lambda \in \tilde{\mathbb{C}}_+$, $|\Psi(\lambda;p(w))| \le 1$ and for any positive real number q_0, $\mathcal{F}_n^{q_0} = \mathcal{F}(C_0[0,T])$.

In next theorem, we establish the existence of the CGFFT of functionals in $\mathcal{F}(C_{a,b}[0,T])$.

Theorem 5.3. *Let X be given by the equation* (4.2) *above and let q_0 be a positive real number. Let $F \in \mathcal{F}_n^{q_0}$ be given by the equation* (3.1)*. Then for all $p \in [1,2]$ and all real q with $|q| > q_0$, the CGFFT of F, $T_q^{(p)}(F|X)$ exists and is given by the formula*

$$\begin{aligned}
&T_q^{(p)}(F|X)(y,\vec{\eta}) \\
&= \int_{C'_{a,b}[0,T]} \exp\left\{i(w,y)^{\sim} + i(w,[\vec{\eta}])_{C'_{a,b}}\right\}\Psi(-iq;p(w))df(w)
\end{aligned} \tag{5.6}$$

for s-a.e. $y \in C_{a,b}[0,T]$, where Ψ is given by the equation (5.5)*.*

Proof. Using (3.1), (4.4), the Fubini theorem, (5.3) and (5.5), we obtain that for $\lambda > 0$ and $\vec{\eta} \in \mathbb{R}^n$,

$$
\begin{aligned}
&E\left(F\left(y+\lambda^{-1/2}\cdot\right)\Big|X\left(\lambda^{-1/2}\cdot\right)\right)(\vec{\eta})\\
&= E\left(\int_{C'_{a,b}[0,T]} \exp\left\{i\left(w, y+\lambda^{-1/2}\cdot\right)^{\sim}\right\} df(w)\Big|X\left(\lambda^{-1/2}\cdot\right)\right)(\vec{\eta})\\
&= \int_{C'_{a,b}[0,T]} \exp\{i(w,y)^{\sim}\} E\left(\exp\left\{i\left(w, \lambda^{-1/2}\cdot\right)^{\sim}\right\}\Big|X\left(\lambda^{-1/2}\cdot\right)\right)(\vec{\eta}) df(w)\\
&= \int_{C'_{a,b}[0,T]} \exp\left\{i(w,y)^{\sim} + i(w,[\vec{\eta}])_{C'_{a,b}}\right\}\Psi(\lambda; p(w)) df(w).
\end{aligned}
$$

From this and Definition 4.4, and by a careful examination, we can see that:

(i) $T_\lambda(F|X)(y,\vec{\eta}) = E(F(y+\lambda^{-1/2}\cdot)|X(\lambda^{-1/2}\cdot))(\vec{\eta})$ is an analytic function of λ throughout the domain $\mathrm{Int}(\Gamma_{q_0})$, where Γ_{q_0} is given by (3.2);

(ii) $T_q^{(1)}(F|X)(y,\vec{\eta}) = \lim_{\lambda\to -iq} T_\lambda(F|X)(y,\vec{\eta})$ exists for s-a.e. $y \in C_{a,b}[0,T]$ and $\vec{\eta} \in \mathbb{R}^n$, and is given by the equation (5.6) above; and

(iii) for $p \in (1,2]$, $T_q^{(p)}(F|X)(y,\vec{\eta}) = \text{l.i.m.}_{\lambda\to -iq} T_\lambda(F|X)(y,\vec{\eta})$ exists for s-a.e. $y \in C_{a,b}[0,T]$ and $\vec{\eta} \in \mathbb{R}^n$, and is given by the equation (5.6) above.

In evaluations of $\lim_{\lambda\to -iq} T_\lambda(F|X)(y,\vec{\eta})$ and $\text{l.i.m.}_{\lambda\to -iq} T_\lambda(F|X)(y,\vec{\eta})$, the dominating functions are given by $k_n(q_0; w)$ and $(2\int_{C'_{a,b}[0,T]} k_n(q_0;w)d|f|(w))^{p'}$, respectively. □

The following corollary follows from (5.6) with $p = 1$ and (4.6).

Corollary 5.4. *Let X, q_0 and F be as in Theorem 5.3. Then for all real q with $|q| > q_0$, the conditional generalized Feynman integral of F, $E^{\mathrm{anf}_q}(F|X)$ exists and is given by the formula*

$$
\begin{aligned}
E^{\mathrm{anf}_q}(F|X)(\vec{\eta}) &= T_q^{(1)}(F|X)(0,\vec{\eta})\\
&= \int_{C'_{a,b}[0,T]} \exp\left\{i(w,[\vec{\eta}])_{C'_{a,b}}\right\}\Psi(-iq; p(w)) df(w),
\end{aligned}
$$

where Ψ is given by the equation (5.5).

In our next theorem, by using the techniques of similar to those used in [14], we show that if we multiply $T_q^{(p)}(F|X)(y,\vec{\eta})$ by

$$
(5.7)\qquad \varpi(-iq;\vec{\eta}) \equiv \left(\frac{-iq}{2\pi}\right)^{\frac{n}{2}} \exp\left\{\frac{iq}{2}\sum_{j=1}^{n}\left[\eta_j - (-iq)^{-1/2}(g_j,a)_{C'_{a,b}}\right]^2\right\},
$$

the analytic extension of the Radon-Nykodym derivative evaluated at $\lambda = -iq$, and then integrate over $\mathbb{R}^n$ we obtain the GFFT $T_q^{(p)}(F)(y)$. However, to do so we also need the following summation procedure as in [15, p. 340]. Let

$$\overline{\int_{\mathbb{R}^n}} f(\vec{\eta})d\vec{\eta} = \lim_{M\to\infty}\int_{\mathbb{R}^n} f(\vec{\eta})\exp\left\{-\left(\sum_{j=1}^n \eta_j^2\right)\Big/ 2M\right\} d\vec{\eta} \tag{5.8}$$

whenever the expression on the right-hand side exists. But if $f \in L^1(\mathbb{R}^n)$, it is clear by the dominated convergence theorem that

$$\overline{\int_{\mathbb{R}^n}} f(\vec{\eta})d\vec{\eta} = \int_{\mathbb{R}^n} f(\vec{\eta})d\vec{\eta}.$$

To establish the equation (5.10) below, the following well-known integration formula is useful:

$$\int_{\mathbb{R}} \exp\{-\alpha u^2 + \beta u\}du = \sqrt{\frac{\pi}{\alpha}}\exp\left\{\frac{\beta^2}{4\alpha}\right\} \tag{5.9}$$

for complex numbers α and β with $\mathrm{Re}(\alpha) > 0$.

Theorem 5.5. *Let X, q_0 and F be as in Theorem 5.3. Then for all $p \in [1, 2]$ and all real q with $|q| > q_0$,*

$$T_q^{(p)}(F)(y) = \overline{\int_{\mathbb{R}^n}} T_q^{(p)}(F|X)(y,\vec{\eta})\varpi(-iq;\vec{\eta})d\vec{\eta} \tag{5.10}$$

for s-a.e. $y \in C_{a,b}[0, T]$, where ϖ is given by (5.7).

Proof. Since $\mathcal{F}_n^{q_0} \subset \mathcal{F}^{q_0}$, the GFFT $T_q^{(p)}(F)$ and the CGFFT $T_q^{(p)}(F|X)$ of F exist for all $p \in [1, 2]$ by Theorems 3.3 and 5.3, respectively. Thus we need only to verify the equality in equation (5.10).

Let q be a nonzero real number with $|q| > q_0$. By using (5.6), (5.8), the Fubini theorem, (5.9) and a simple calculation, we obtain that

$$\begin{aligned}
&\overline{\int_{\mathbb{R}^n}} T_q^{(p)}(F|X)(y,\vec{\eta})\varpi(-iq;\vec{\eta})d\vec{\eta}\\
&= \lim_{M\to\infty}\int_{\mathbb{R}^n} T_q^{(p)}(F|X)(y,\vec{\eta})\varpi(-iq;\vec{\eta})\exp\left\{-\sum_{j=1}^n \frac{\eta_j^2}{2M}\right\} d\vec{\eta}\\
&= \lim_{M\to\infty}\int_{C'_{a,b}[0,T]} \exp\{i(w,y)^\sim\}\Psi(-iq;p(w))\\
&\qquad \left[\int_{\mathbb{R}^n} \varpi(-iq;\vec{\eta})\times\exp\left\{-\sum_{j=1}^n\frac{\eta_j^2}{2M} + i\sum_{j=1}^n (g_j,w)_{C'_{a,b}}\eta_j\right\}d\eta_j\right] df(w) \qquad (5.11)\\
&= \lim_{M\to\infty}\int_{C'_{a,b}[0,T]} \left(\frac{-iq}{2\pi}\frac{2\pi M}{1-iqM}\right)^{\frac{n}{2}} \exp\{i(w,y)^\sim\}\Psi(-iq;p(w))
\end{aligned}$$

$$\times \exp\left\{ -\sum_{j=1}^{n} \frac{(g_j,a)^2_{C'_{a,b}}}{2} - \frac{M}{2(1-iqM)} \right.$$
$$\left. \times \sum_{j=1}^{n} \Big[(g_j,w)^2_{C'_{a,b}} - 2q(-iq)^{-1/2}(g_j,w)_{C'_{a,b}}(g_j,a)_{C'_{a,b}} + iq(g_j,a)^2_{C'_{a,b}} \Big] \right\} df(w).$$

But a long and tedious calculation shows that for a sufficiently large $M > 0$,

$$\left| \exp\{i(w,y)^{\sim}\}\Psi(-iq;p(w)) \exp\left\{ -\sum_{j=1}^{n} \frac{(g_j,a)^2_{C'_{a,b}}}{2} - \frac{M}{2(1-iqM)} \right. \right.$$
$$\left. \left. \times \sum_{j=1}^{n} \Big[(g_j,w)^2_{C'_{a,b}} - 2q(-iq)^{-1/2}(g_j,w)_{C'_{a,b}}(g_j,a)_{C'_{a,b}} + iq(g_j,a)^2_{C'_{a,b}} \Big] \right\} \right|$$

is dominated by $k_n(q_0;w)$.

Using (5.11), the dominated convergence theorem, (5.5), (5.1), (5.2), (3.3) and (3.4), we have the equation (5.10). □

The following corollary follows from (5.10) with $p = 1$ and (4.6).

Corollary 5.6. *Let X, q_0 and F be as in Theorem 5.3. Then for all real q with $|q| > q_0$,*

$$E^{\mathrm{anf}_q}[F] = T_q^{(1)}(F)(0) = \overline{\int_{\mathbb{R}^n}} E^{\mathrm{anf}_q}(F|X)(\vec{\eta})\varpi(-iq;\vec{\eta})d\vec{\eta},$$

where ϖ is given by the equation (5.7).

Remark 5.7. (i) Let $x_0 \in C'_{a,b}[0,T]$ and for each nonzero real number q, let $\delta_{-qx_0} \in \mathcal{M}(C'_{a,b}[0,T])$ be the Dirac measure concentrated at $-qx_0$. Then the functional $H_{-qx_0}(x) = \exp\{-iq(x_0,x)^{\sim}\}$ is an element of $\mathcal{F}(C_{a,b}[0,T])$, because

$$H_{-qx_0}(x) = \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\}d\delta_{-qx_0}(w).$$

Clearly, $H_{-qx_0} \in \mathcal{F}_n^r$ for all positive real number r.

For each $F \in \mathcal{F}(C_{a,b}[0,T])$, let

$$F^*(x) = F(x)\exp\{-iq(x_0,x)^{\sim}\}. \tag{5.12}$$

Since $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra, F^* is an element of $\mathcal{F}(C_{a,b}[0,T])$.

(ii) Using the equation (3.1), we can write $F^*(x)$ as follows:

$$\begin{aligned} F^*(x) &= \int_{C'_{a,b}[0,T]} \exp\{i(w-qx_0,x)^{\sim}\}df(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\{i(h,x)^{\sim}\}df^*_{qx_0}(h), \end{aligned} \tag{5.13}$$

where $f^*_{qx_0}$ is a measure in $\mathcal{M}(C'_{a,b}[0,T])$ such that $f^*_{qx_0}(E) \equiv f(E+qx_0)$ for $E \in \mathcal{B}(C'_{a,b}[0,T])$.

Let $F \in \mathcal{F}_n^{q_0}$ be given by the equation (3.1) and let F^* be given by the equation (5.12) with $|q| > q_0$. Then

$$\begin{aligned}
&\int_{C'_{a,b}[0,T]} k_n(q_0;h)d|f^*_{qx_0}|(h) \\
&= \int_{C'_{a,b}[0,T]} k_n(q_0;w-qx_0)d|f|(w) \\
&\le \int_{C'_{a,b}[0,T]} \exp\left\{\frac{(n+1)}{\sqrt{2q_0}}\big(\|w\|_{C'_{a,b}} + |q|\|x_0\|_{C'_{a,b}}\big)\|a\|_{C'_{a,b}}\right\}d|f|(w) \\
&= \big(k_n(q_0;x_0)\big)^{|q|}\int_{C'_{a,b}[0,T]} k_n(q_0;w)d|f|(w) \\
&< +\infty.
\end{aligned}$$

Thus we see that $F \in \mathcal{F}_n^{q_0}$ implies $F^* \in \mathcal{F}_n^{q_0}$.

For notational convenience, we will write

$$\vec{\eta} + (\vec{g}, x_0)_{C'_{a,b}} = (\eta_1 + (g_1,x_0)_{C'_{a,b}}, \dots, \eta_n + (g_n,x_0)_{C'_{a,b}})$$

for $\vec{\eta} \in \mathbb{R}^n$, $x_0 \in C'_{a,b}[0,T]$ and $\{g_1,\dots,g_n\} \subset C'_{a,b}[0,T]$.

In Theorem 5.8, we obtain a translation theorem for CGFFT of functionals in $\mathcal{F}(C_{a,b}[0,T])$.

Theorem 5.8. *Let X, q_0 and F be as in Theorem 5.3. Let $x_0 \in C'_{a,b}[0,T]$. Then for all real q with $|q| > q_0$,*

(5.14)

$$\begin{aligned}
&T_q^{(p)}(F|X)(y+x_0,\vec{\eta}) \\
&= \exp\left\{iq(x_0,y)^\sim + iq\big(x_0,[\vec{\eta}]+[x_0]\big)_{C'_{a,b}} + \frac{iq}{2}B(p(x_0)) + iq(-iq)^{-1/2}A(p(x_0))\right\} \\
&\quad \times T_q^{(p)}(F^*|X)\left(y, \vec{\eta} + (\vec{g},x_0)_{C'_{a,b}}\right)
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$, where F^* is given by the equation (5.12).

Proof. From (ii) of Remark 5.7 we know that $F^* \in \mathcal{F}_n^{q_0}$ and so $T_q^{(p)}(F^*|X)$ exists for all $p \in [1,2]$ by Theorem 5.3. Thus we need only to verify the equality in equation (5.14).

We first note that for all $w \in C'_{a,b}[0,T]$ and all $q \in \mathbb{R}$ with $|q| > q_0$,

$$(w,x_0)^\sim = \int_0^T D_t w dx_0(t) = \int_0^T D_t w D_t x_0 db(t) = (w,x_0)_{C'_{a,b}}, \tag{5.15}$$

$$\left[\vec{\eta} + (\vec{g},x_0)_{C'_{a,b}}\right] = [\vec{\eta}] + [x_0], \tag{5.16}$$

$$A(p(w-qx_0)) = A(p(w)) - qA(p(x_0)), \tag{5.17}$$

and

$$(5.18)\qquad \begin{aligned} &B(p(w-qx_0)) \\ &= B(p(w)) + q^2B(p(x_0)) - 2q(w,x_0)_{C'_{a,b}} + 2q\big(w,[x_0]\big)_{C'_{a,b}}. \end{aligned}$$

Next using (5.6) with F and $\vec{\eta}$ replaced with F^* and $\vec{\eta}+(\vec{g},x_0)_{C'_{a,b}}$ respectively, (5.16), (5.5) with λ and w replaced with $-iq$ and $h=w-qx_0$ respectively, (5.17), (5.18), (5.15) and (5.6), we have that for s-a.e. $y\in C_{a,b}[0,T]$ and $\vec{\eta}\in\mathbb{R}^n$,

$$(5.19)$$
$$\begin{aligned} &T_q^{(p)}(F^*|X)(y,\vec{\eta}+(\vec{g},x_0)_{C'_{a,b}}) \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{i(h,y)^\sim + i\big(h,[\vec{\eta}]+[x_0]\big)_{C'_{a,b}}\Big\}\Psi(-iq;p(h))df^*_{qx_0}(h) \\ &= \int_{C'_{a,b}[0,T]} \exp\Big\{i(w,y)^\sim - iq(x_0,y)^\sim + i\big(w,[\vec{\eta}]\big)_{C'_{a,b}} + i\big(w,[x_0]\big)_{C'_{a,b}} \\ &\qquad\qquad - iq\big(x_0,[\vec{\eta}]+[x_0]\big)_{C'_{a,b}}\Big\} \\ &\quad\times \exp\bigg\{i(-iq)^{-1/2}\Big(A(p(w)) - qA(p(x_0))\Big) \\ &\qquad\qquad -\frac{i}{2q}\bigg(B(p(w)) + q^2B(p(x_0)) - 2q(w,x_0)_{C'_{a,b}} + 2q\big(w,[x_0]\big)_{C_{a,b}}\bigg)\bigg\}df(w) \\ &= \exp\bigg\{-iq(x_0,y)^\sim - iq\big(x_0,[\vec{\eta}]+[x_0]\big)_{C'_{a,b}} - \frac{iq}{2}B(p(x_0)) - iq(-iq)^{-1/2}A(p(x_0))\bigg\} \\ &\quad\times\int_{C'_{a,b}[0,T]} \exp\big\{i(w,y+x_0)^\sim + i\big(w,[\eta]\big)_{C'_{a,b}}\big\}\Psi(-iq;p(w))df(w) \\ &= \exp\bigg\{-iq(x_0,y)^\sim - iq\big(x_0,[\vec{\eta}]+[x_0]\big)_{C'_{a,b}} - \frac{iq}{2}B(p(x_0)) - iq(-iq)^{-1/2}A(p(x_0))\bigg\} \\ &\quad\times T_q^{(p)}(F|X)(y+x_0,\vec{\eta}). \end{aligned}$$

Equation (5.14) follows from the equation (5.19). □

Corollary 5.9. *Let X, q_0 and F be as in Theorem 5.3. Let $x_0\in C'_{a,b}[0,T]$. Then for all real q with $|q|>q_0$,*

$$\begin{aligned} &E^{anf_q}(F|X)(y+x_0,\vec{\eta}) \\ &= \exp\bigg\{iq\,(x_0,[\vec{\eta}]+[x_0])_{C'_{a,b}} + \frac{iq}{2}B(p(x_0)) + iq(-iq)^{-1/2}A(p(x_0))\bigg\} \\ &\quad\times E^{anf_q}(F^*|X)\Big(\vec{\eta}+(\vec{g},x_0)_{C'_{a,b}}\Big) \end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$, where F^* is given by the equation (5.12).

6. Examples

In this section we present several important functionals to apply our results in previous sections.

Let $S : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ be the linear operator defined by

$$Sw(t) = \int_0^t w(s)db(s).$$

Then the adjoint operator S^* of S is given by

$$S^*w(t) = \int_0^t \big(w(T) - w(s)\big)db(s).$$

Using an integration by parts formula, we see that

$$(S^*b, x)^\sim = \int_0^T x(t)db(t).$$

Example 6.1. The functional

$$F_1(x) = \exp\left\{ i \int_0^T x(t)db(s) \right\} \tag{6.1}$$

is a functional under our consideration because

$$F_1(x) = \exp\{i(S^*b, x)^\sim\} = \int_{C'_{a,b}[0,T]} \exp\{i(w, x)^\sim\} d\delta_1(w),$$

where δ_1 is the Dirac measure concentrated at S^*b in $C'_{a,b}[0,T]$. Obviously, F_1 is an element of $\mathcal{F}_n^{q_0}$ for all $q_0 > 0$.

Example 6.2. Let $\mathcal{M}(\mathbb{R})$ be the class of complex-valued countably additive measures on $\mathcal{B}(\mathbb{R})$, the Borel class of $\mathbb{R}$. For $\nu \in \mathcal{M}(\mathbb{R})$, the Fourier transform $\widehat{\nu}$ of ν is a complex-valued function defined on $\mathbb{R}$ by the formula

$$\widehat{\nu}(u) = \int_{\mathbb{R}} \exp\{iuv\} d\nu(v).$$

Given m and σ^2 in $\mathbb{R}$ with $\sigma^2 > 0$, let ν_{m,σ^2} be the Gaussian measure given by

$$\nu_{m,\sigma^2}(B) = (2\pi\sigma^2)^{-1/2} \int_B \exp\left\{ -\frac{(v-m)^2}{2\sigma^2} \right\} dv, \quad B \in \mathcal{B}(\mathbb{R}).$$

Then $\nu_{m,\sigma^2} \in \mathcal{M}(\mathbb{R})$ and

$$\widehat{\nu_{m,\sigma^2}}(u) = \int_{\mathbb{R}} \exp\{iuv\} d\nu_{m,\sigma^2}(v) = \exp\left\{ -\frac{1}{2}\sigma^2 u^2 + imu \right\}.$$

Let $h \in C'_{a,b}[0,T]$ and let $\nu \in \mathcal{M}(\mathbb{R})$. Define $F_2 : C_{a,b}[0,T] \to \mathbb{C}$ by

$$\begin{aligned} F_2(x) &= \widehat{\nu_{m,\sigma^2}}((h,x)^\sim) \\ &= \exp\left\{ -\frac{1}{2}\sigma^2 \big[(h,x)^\sim\big]^2 + im(h,x)^\sim \right\}. \end{aligned} \tag{6.2}$$

Define a function $\phi : \mathbb{R} \to C'_{a,b}[0,T]$ by $\phi(v) = vh$ and let $f_2 = \nu_{m,\sigma^2} \circ \phi^{-1}$. It is quite clear that f_2 is in $\mathcal{M}(C'_{a,b}[0,T])$ and is supported by $[h]$, the subspace of $C'_{a,b}[0,T]$ spanned by $\{h\}$. Now for s-a.e. $x \in C_{a,b}[0,T]$,

$$\begin{aligned}
&\int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\} df_2(w) \\
&= \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\} d(\nu_{m,\sigma^2} \circ \phi^{-1})(w) \\
&= \int_{\mathbb{R}} \exp\{i(\phi(v),x)^{\sim}\} d\nu_{m,\sigma^2}(v) \\
&= \int_{\mathbb{R}} \exp\{i(h,x)^{\sim} v\} d\nu_{m,\sigma^2}(v) \\
&= F_2(x).
\end{aligned}$$

Thus F_2 is an element of $\mathcal{F}(C_{a,b}[0,T])$. Moreover $F \in \mathcal{F}_n^{q_0}$ for all $q_0 > 0$, because

$$\begin{aligned}
&\int_{C'_{a,b}[0,T]} k_n(q_0;w) d|f_2|(w) \\
&= \int_{C'_{a,b}[0,T]} k_n(q_0;w) df_2(w) \\
&= \int_{\mathbb{R}} \exp\big\{(n+1)(2q_0)^{-1/2}\|vh\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\big\} d\nu_{m,\sigma^2}(v) \\
&= \int_{\mathbb{R}} \exp\bigg\{ -\frac{(v-m)^2}{2\sigma^2} + (n+1)(2q_0)^{-1/2}\|h\|_{C'_{a,b}}\|a\|_{C'_{a,b}}|v| \bigg\} dv \\
&< +\infty.
\end{aligned}$$

Thus we can apply the results in previous sections to the functional F_2.

In particular, if we choose $h = S^*b$, $m = 0$ and $\sigma^2 = 2$ in the last expression of (6.2), then we have

$$F_3(x) = \exp\left\{ -\left(\int_0^T x(t)db(t) \right)^2 \right\}. \tag{6.3}$$

The functionals given by (6.1) and (6.3) are interpreted as the potential energy in quantum mechanics.

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Department of Mathematics
Dankook University
Cheonan 330-714, Korea
E-mail address: sejchang@dankook.ac.kr