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ON THE HYERS-ULAM-RASSIAS STABILITY OF THE JENSEN EQUATION IN DISTRIBUTIONS

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Abstract. We consider the Hyers-Ulam-Rassias stability problem

$$\left\|2u \circ \frac{A}{2} - u \circ P_1 - u \circ P_2\right\| \le \varepsilon(|x|^p + |y|^p), \quad x, y \in \mathbb{R}^n$$

for the Schwartz distributions u, which is a distributional version of the Hyers-Ulam-Rassias stability problem of the Jensen functional equation

$$\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| \le \varepsilon(|x|^p + |y|^p), \quad x, y \in \mathbb{R}^n$$

for the function $f : \mathbb{R}^n \to \mathbb{C}$.

1. Introduction

The stability problems of functional equations go back to 1940 when S. M. Ulam proposed the following problem [24]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy),\, f(x)f(y)) \le \varepsilon.$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d(f(x),\, h(x)) \le \delta_\epsilon$$

for all $x \in G_1$

This problem was solved affirmatively by D. H. Hyers under the assumption that G_2 is a Banach space (see Hyers [12]). Since then, the stability problems of many other functional equations have been investigated [1, 2, 3, 4, 5, 6, 7, 9, 10, 16, 17, 18, 19, 20, 21, 22]. Among them, generalizing the well known stability theorem of D. H. Hyers, Th. M. Rassias [22] and Z. Gajda [9] showed the following stability theorem for the Cauchy equation:

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Theorem 1.1 ([9, 22]). *Let f be a mapping from a normed linear space X to a Banach space Y satisfying the inequality*

$$||f(x+y)-f(x)-f(y)|| \le \epsilon(\|x\|^p+\|y\|^p), \quad p\neq 1 \tag{1.1}$$

for all $x,\ y\in X$ ($x\neq 0$ and $y\neq 0$ if $p<0$). Then there exists a unique function $g: X\to Y$ satisfying

$$g(x+y)-g(x)-g(y)=0$$

such that

$$||f(x)-g(x)|| \le \frac{2\varepsilon}{|2^p-2|}\|x\|^p$$

for all $x\in X$ ($x\neq 0$ if $p<0$).

As a similar result, generalizing the Hyers-Ulam stability theorem for the Jensen functional equation of K. Kominek [14], S. M Jung [13] prove a Hyers-Ulam-Rassias stability theorem for the Jensen functional equation

$$\left|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right| \le \epsilon(\|x\|^p+\|y\|^p). \tag{1.2}$$

For more interesting results related to the Hyers-Ulam stability of Jensen functional equation we refer the reader to the results of J.-H. Bae, D.-O. Lee and W.-G. Park [1] and that of C.-G. Park [16, 17, 18] and C.-G. Park and W.-G. Park [19, 20].

In this paper, we consider the stability theorem for the Jensen functional equation (1.2) in the spaces of generalized functions such as the spaces $\mathcal{S}'$ and $\mathcal{D}'$ of tempered distributions and distributions of L. Schwartz, respectively. Making use of the pullbacks of generalized function we extend the inequality (1.2) to distributions u as follows:

$$\left\|2u\circ\frac{A}{2}-u\circ P_1-u\circ P_2\right\| \le \varepsilon(|x|^p+|y|^p) \tag{1.3}$$

for even integers $p\ge 2$, where $A(x,y)=x+y$, $P_1(x,y)=x$, $P_2(x,y)=y$, $x,y\in\mathbb{R}^n$, and $u\circ A$, $v\circ P_1$ and $w\circ P_2$ are the pullbacks of u,v,w by A,P_1 and P_2, respectively. Also $|\cdot|$ denotes the Euclidean norm and the inequality $\|\cdot\|\le\psi(x,y)$ in (1.3) means that $|\langle\cdot,\varphi\rangle|\le\|\psi\varphi\|_{L^1}$ for all test functions $\varphi\in C_c^\infty(\mathbb{R}^{2n})$ which will be introduced in Section 2.

As the main result, we prove the following: Let $u\in\mathcal{D}'$ satisfy

$$\left\|2u\circ\frac{A}{2}-u\circ P_1-u\circ P_2\right\| \le \epsilon(|x|^{2p}+|y|^{2p}) \tag{1.4}$$

for some integer $p>1$. Then there exist a unique $a\in\mathbb{C}^n$ and $c\in\mathbb{C}$ such that

$$\|u-a\cdot x-c\| \le \frac{2\epsilon}{4^p-2}|x|^{2p}.$$

2. Schwartz distributions

We briefly introduce the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions and the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. Here we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$. We also denote by $C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

Definition 2.1. A distribution u is a linear form on $C_c^\infty(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

Definition 2.2. We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{2.1}$$

for all α, $\beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions and the elements of the dual space $\mathcal{S}'$ are called *tempered distributions*.

It is well known that the following topological inclusions:

$$C_c^\infty \hookrightarrow \mathcal{S}, \quad \mathcal{S}' \hookrightarrow \mathcal{D}'.$$

Example 2.1 ([11, 23])**.** In the usual sense of differentiations, the derivatives of locally integrable functions make no sense, however, one can differentiate every locally integrable function in the space of Schwartz distributions. As a matter of fact, it is well known that every derivative $\partial^\alpha f$ of a locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$ defines a distribution via the relation

$$\langle \partial^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) \partial^\alpha \varphi(x) dx, \quad \varphi \in C_c^\infty(\mathbb{R}^n). \tag{2.2}$$

Also it is well known that every derivative $\partial^\alpha f$ of locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$ defines a distribution via the relation (2.2) provided that f satisfies the growth condition; there exist positive constants C and N such that

$$|f(x)| \leq C(1 + |x|)^N$$

for all $x \in \mathbb{R}^n$.

Example 2.2 ([11, 23])**.** Let H be the Heaviside function on $\mathbb{R}$ defined by $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$. Then it is easy to see that $H' = \delta$ where δ denotes the Dirac measure on $\mathbb{R}$ which is defined by

$$\langle \delta, \varphi \rangle = \varphi(0), \quad \varphi \in C_c^\infty(\mathbb{R}).$$

Also every finite sum $u = \sum_{|\alpha|\le m} \partial^\alpha \delta$ of derivatives of δ defines a tempered distribution.

We denote by Ω_j open subsets of $\mathbb{R}^{n_j}$ for $j = 1, 2$, with $n_1 \ge n_2$.

Definition 2.3. Let $u_j \in \mathcal{D}'(\Omega_j)$ and $\lambda : \Omega_1 \to \Omega_2$ be a smooth function such that for each $x \in \Omega_1$ the derivative $\lambda'(x)$ is surjective, that is, the Jacobian matrix $\nabla\lambda$ of λ has rank n_2. Then there exists a unique continuous linear map $\lambda^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ such that $\Lambda^* u = u \circ \lambda$ when u is a continuous function. We call $\lambda^* u$ the pullback of u by λ and often denoted by $u \circ \lambda$.

In particular if λ is a diffeomorphism (a bijection with λ, λ^{-1} smooth functions) the pullback $u \circ \lambda$ can be written as follows:

$$\langle u \circ \lambda, \varphi\rangle = \langle u, (\varphi \circ \lambda^{-1})(x)|(\nabla\lambda^{-1}(x)|\rangle. \tag{2.3}$$

As a matter of fact, the pullbacks $u \circ A, u \circ P_1$, $u \circ P_2$ can be written in a transparent way as

$$\langle u \circ A\,,\, \varphi(x, y)\rangle = \langle u\,,\, \int \varphi(x - y, y)\, dy\rangle, \tag{2.4}$$

$$\langle u \circ P_1\,,\, \varphi(x, y)\rangle = \langle u\,,\, \int \varphi(x, y)\, dy\rangle, \tag{2.5}$$

$$\langle u \circ P_2\,,\, \varphi(x, y)\rangle = \langle u\,,\, \int \varphi(x, y)\, dx\rangle \tag{2.6}$$

for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$.

We refer the reader to ([11], chapter VI) for pullbacks of distributions and to [11, 23] for more details of distributions and tempered distributions.

3. Main theorems

We denote by $\delta(x)$ the function on $\mathbb{R}^n$,

$$\delta(x) = \begin{cases} A \exp\left(-\frac{1}{\sqrt{1-|x|^2}}\right), & |x| < 1 \\ 0, & |x| \ge 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp\left(-\frac{1}{\sqrt{1 - |x|^2}}\right) dx\right)^{-1}.$$

It is easy to see that $\delta(x)$ is an infinitely differentiable function with support $\{x : |x| \le 1\}$. We employ the regularizing function $\delta_t(x) := t^{-n}\delta(x/t),\ t > 0$. Let $u \in \mathcal{D}'$. Then, for each $t > 0, (u * \delta_t)(x) = \langle u_y, \delta_t(x - y)\rangle$ is a smooth function of $x \in \mathbb{R}^n$ and $(u * \delta_t)(x) \to u$ as $t \to 0^+$ in the sense that

$$\lim_{t\to 0^+} \int (u * \delta_t)(x)\varphi(x)\, dx = \langle u, \varphi\rangle$$

for all $\varphi \in C_c^\infty$.

Lemma 3.1. *Let $u \in \mathcal{D}'$ satisfy the inequality*

$$\left\| 2u \circ \frac{A}{2} - u \circ P_1 - u \circ P_2 \right\| \le \epsilon(|x|^{2p} + |y|^{2p}) \tag{3.1}$$

for some integer $p > 1$. Then $u \in \mathcal{S}'$.

Proof. We denote by

$$\Psi(x, y, t, s) = \epsilon(|\xi|^{2p} * \delta_t(\xi))(x) + \epsilon(|\eta|^{2p} * \delta_s(\eta))(y).$$

Convolving $\delta_t(x)\delta_s(y)$ in each side of (3.1) the inequality (3.1) is converted to the following stability problem

$$|(u^* * \delta_t * \delta_s)(x + y) - (u * \delta_t)(x) - (u * \delta_s)(y)| \le \Psi(x, y, t, s) \tag{3.2}$$

for x, $y \in \mathbb{R}^n$, $t, s > 0$, where $\langle u^*, \varphi(x)\rangle = 2^{n+1}\langle u, \varphi(2x)\rangle$. From (3.2) it is easy to see that

$$f(x) := \limsup_{t \to 0^+}(u * \delta_t)(x)$$

exists. Letting $y = 0$ in (3.2) we have

$$|(u^* * \delta_t * \delta_s)(x) - (u * \delta_t)(x) - (u * \delta_s)(0)| \le \Psi(x, 0, t, s) \tag{3.3}$$

for $x \in \mathbb{R}^n$, $t, s > 0$. From (3.2) and (3.3) we have

$$\begin{aligned} &|(u * \delta_t)(x + y) - (u * \delta_t)(x) - (u * \delta_s)(y) + (u * \delta_s)(0)| \\ &\le \Psi(x, y, t, s) + \Psi(x + y, 0, t, s) \end{aligned} \tag{3.4}$$

for x, $y \in \mathbb{R}^n$, $t, s > 0$. Letting $s \to 0^+$ so that $(u * \delta_s)(y) \to f(y)$ in (3.4) we have

$$\begin{aligned} &|(u * \delta_t)(x + y) - (u * \delta_t)(x) - f(y) + f(0)| \\ &\le \Psi(x, y, t, 0^+) + \Psi(x + y, 0, t, 0^+) \end{aligned} \tag{3.5}$$

for x, $y \in \mathbb{R}^n$, $t, s > 0$. Putting $x = 0$ and letting $t \to 0^+$ so that $(u * \delta_t)(0) \to f(0)$ in (3.5) we have

$$\|u - f(y)\| \le 2\epsilon|y|^{2p}. \tag{3.6}$$

On the other hand, let

$$D(x, y, t) = (u * \delta_t)(x + y) - (u * \delta_t)(x) - f(y) + f(0).$$

Then we have

$$\begin{aligned} |f(x + y) - f(x) - f(y) + f(0)| \le& |D(x, y, t)| + | - D(0, x + y, t)| + |D(0, x, t)| \\ \le& \Psi(x, y, t, 0^+) + \Psi(x + y, 0, t, 0^+) \\ &+ \Psi(0, x + y, t, 0^+) + \Psi(x + y, 0, t, 0^+) \\ &+ \Psi(0, x, t, 0^+) + \Psi(x, 0, t, 0^+) \end{aligned}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Letting $t \to 0^+$ in the above inequality we have

$$|f(x + y) - f(x) - f(y) + f(0)| \le 3\epsilon|x + y|^{2p} + 3\epsilon|x|^{2p} + \epsilon|y|^{2p}. \tag{3.7}$$

By the results in [9, 10], there exists a unique function A satisfying

$$A(x+y) = A(x) + A(y) \tag{3.8}$$

such that

$$|f(x) - A(x) - f(0)| \le \frac{\epsilon(3\cdot 4^p + 4)}{4^p - 2}|x|^{2p}. \tag{3.9}$$

Indeed, let $F(x) = f(x) - f(0)$. Then A is given by a locally uniform limit of the sequence of the continuous functions $A_m(x) = 2^n F(2^{-n}x)$. Thus A is a continuous function. Thus the solution A of the Cauchy functional equation (3.8) has the form $A(x) = a\cdot x$ for some $a \in \mathbb{C}^n$. Now, from (3.6) and (3.9) we have

$$\|u - a\cdot x - f(0)\| \le K|x|^{2p}, \tag{3.10}$$

where $K = \frac{5\cdot 4^p \epsilon}{4^p - 2}$. It follows from (3.10) that u is a locally integrable function satisfying

$$|u(x)| \le |a\cdot x| + |f(0)| + K|x|^{2p}.$$

Thus $u \in \mathcal{S}'$. This completes the proof. □

Now we may employ the n-dimensional heat kernel $E_t(x)$ given by

$$E_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t),\ x \in \mathbb{R}^n,\ t > 0. \tag{3.11}$$

It is easy to see that the heat kernel $E_t(\cdot)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ for each $t > 0$. Let $u \in \mathcal{S}'$. Then its *Gauss transform*

$$\tilde{u}(x,t) = (u * E_t)(x) = \langle u_y, E_t(x-y)\rangle, \quad x \in \mathbb{R}^n,\ t > 0,$$

is well defined. As a matter of fact the following result holds [10]:

Lemma 3.2 ([15])**.** *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then its Gauss transform $\tilde{u}(x,t)$ is a C^∞–solution of the heat equation satisfying*:

(i) *There exist positive constants C, M, N and δ such that*

$$|\tilde{u}(x,t)| \le Ct^{-M}(1+|x|)^N \quad in \quad \mathbb{R}^n \times (0,\, \delta), \tag{3.12}$$

(ii) *$\tilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}$,*

$$\langle u, \varphi\rangle = \lim_{t\to 0^+}\int \tilde{u}(x,t)\varphi(x)\,dx.$$

Conversely, every C^∞–solution $U(x,t)$ of the heat equation satisfying the estimate (3.12) can be uniquely expressed as $U(x,t) = \tilde{u}(x,t)$ for some $u \in \mathcal{S}'$.

It is well known that the *weak semigroup property* of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x) \tag{3.13}$$

holds for convolution. This semigroup property will be very useful later.

Throughout the paper, we denote by

$$\mathcal{H}_{2p}(x,t) = [|\xi|^{2p} * E_t(\xi)](x,t).$$

Since $|x|^{2p} = \sum_{|\gamma|=p} \frac{p!}{\gamma!} x^{2\gamma}$ we have

$$\mathcal{H}_{2p}(x,t) = \sum_{|\gamma|=p} \frac{p!}{\gamma!}(2\gamma)! \sum_{0\le\alpha\le\gamma} \frac{t^{|\alpha|}x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!}.$$

Note that if $p = 1$ we have

$$\mathcal{H}_{2\gamma}(x,t) = |x|^2 + 2nt$$

and for $p = 1, 2, \dots$

$$\mathcal{H}_{2\gamma}(x,0) = |x|^{2p}.$$

We need the following:

Lemma 3.3. *Let $g : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be a continuous function satisfying the inequality*

$$|g(x+y,t+s) - g(x,t) - g(y,s)| \le \epsilon(\mathcal{H}_{2p}(x,t) + \mathcal{H}_{2p}(y,s)) \tag{3.14}$$

for some integer $p > 1$. Then there exist unique constants $a \in \mathbb{C}^n, b \in \mathbb{C}$ such that

$$|g(x,t) - a\cdot x - bt| \le \epsilon\psi_p(x,t)$$

for all $x \in \mathbb{R}^n,\ t > 0$, where

$$\psi_p(x,t) = \sum_{|\gamma|=p} \frac{p!}{\gamma!}(2\gamma)! \sum_{0\le\alpha\le\gamma} \frac{2^{|\alpha|+1}t^{|\alpha|}x^{2\gamma-2\alpha}}{(2^{|2\gamma|} - 2^{|\alpha|+1})\alpha!(2\gamma-2\alpha)!}\ .$$

Proof. We can follow the same approach as in [22, 9]. Indeed, replacing both x and y by $\frac{x}{2}$, both t and s by $\frac{t}{2}$ in (3.14) we have

$$|g(x,t) - 2g(2^{-1}x, 2^{-1}t)| \le 2\epsilon\mathcal{H}_{2p}(2^{-1}x, 2^{-1}t)$$

for all $x \in \mathbb{R}^n,\ t > 0$. Making use of the induction argument and triangle inequality we have

$$\begin{aligned} |g(x,t) - 2^m g(2^{-m}x, 2^{-m}t)| &\le \epsilon \sum_{j=1}^{m} 2^j \mathcal{H}_{2p}(2^{-j}x, 2^{-j}t) \\ &\le \epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}(2\gamma)! \sum_{0\le\alpha\le\gamma} a_{m,\alpha} \frac{t^{|\alpha|}x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!} \end{aligned} \tag{3.15}$$

for all $n \in \mathbb{N},\ x \in \mathbb{R}^n,\ t > 0$, where $a_{m,\alpha} = 2^{|\alpha|+1}(1 - 2^{(|\alpha|-|2\gamma|+1)m})/(2^{|2\gamma|} - 2^{|\alpha|+1})$.

Replacing $x,\ t$ by $2^{-m}x,\ 2^{-m}t$, respectively in (3.14) and multiplying 2^m in the result it follows from $p > 1$ that

$$A_m(x,t) := 2^m g(2^{-m}x, 2^{-m}t)$$

is a Cauchy sequence which converges locally uniformly, say to $A(x,t)$. Letting $m \to \infty$ in (3.15) we have

$$|g(x,t) - A(x,t)| \le \epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}(2\gamma)! \sum_{0\le\alpha\le\gamma} a_\alpha \frac{t^{|\alpha|}x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!} \tag{3.16}$$

for all $x \in \mathbb{R}^n,\ t > 0$, where $a_\alpha = 2^{|\alpha|+1}/(2^{|2\gamma|} - 2^{|\alpha|+1})$.

Replacing $x,\ y,\ t,\ s$ by $2^{-m}x,\ 2^{-m}y,\ 2^{-m}t,\ 2^{-m}s$ in (3.14), respectively, multiplying 2^m and letting $m \to \infty$ it follows from the fact $p > 1$ that

$$A(x+y, t+s) - A(x,t) - A(y,s) = 0 \tag{3.17}$$

for all $x, y \in \mathbb{R}^n,\ t, s > 0$. To prove the uniqueness of $A(x,t)$, let $B(x,t)$ be another function satisfying (3.12) and (3.13). Then it follows from (3.16), (3.17) and the triangle inequality that for all $n \in \mathbb{N}$,

$$\begin{aligned} |A(x,t) - B(x,t)| &\le m|A\left(\frac{x}{m}, \frac{t}{m}\right) - B\left(\frac{x}{m}, \frac{t}{m}\right)| \\ &\le 2\epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}(2\gamma)!\, m^{1-|\gamma|} \sum_{0\le\alpha\le\gamma} a_\alpha \frac{t^{|\alpha|}x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!} \end{aligned} \tag{3.18}$$

for all $x \in \mathbb{R}^n,\ t > 0$. Letting $m \to \infty$, we have $A(x,t) = B(x,t)$ for all $x \in \mathbb{R}^n,\ t > 0$. This proves the uniqueness.

Now it is well known that every continuous solution $A(x,t)$ of the Cauchy equation (1.4) has the form

$$A(x,t) = a \cdot x + bt$$

for some $a \in \mathbb{C}^n, b \in \mathbb{C}$. Thus we have

$$|g(x,t) - a\cdot x - bt| \le \epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}(2\gamma)! \sum_{0\le\alpha\le\gamma} a_\alpha \frac{t^{|\alpha|}x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!} \tag{3.19}$$

for all $x, y \in \mathbb{R}^n,\ t > 0$, where $a_\alpha = 2^{|\alpha|+1}/(2^{|2\gamma|} - 2^{|\alpha|+1})$. This completes the proof. □

Lemma 3.4. *Let $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be a continuous function satisfying the inequality*

$$\left|2f\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - f(x,t) - f(y,s)\right| \le \epsilon(\mathcal{H}_{2p}(x,t) + \mathcal{H}_{2p}(y,s)) \tag{3.20}$$

for some integer $p > 1$. Then there exist a unique $a \in \mathbb{C}^n$, a unique $b \in \mathbb{C}$ and complex constant c such that

$$|f(x,t) - a\cdot x - bt - c| \le 2\epsilon\, 4^p \psi_p(x,t)$$

for all $x \in \mathbb{R}^n,\ t > 0$.

Proof. Let $F(x,t) = 2f(x/2, t/4)$. Then we have

$$|F(x+y, t+s) - f(x,t) - f(y,s)| \le \epsilon(\mathcal{H}_{2p}(x,t) + \mathcal{H}_{2p}(y,s)). \tag{3.21}$$

Putting $y = 0$ in (3.21), it is easy to see that $c := \limsup_{s\to 0^+} f(0,s)$ exists. Putting $y = 0$ and letting $s \to 0^+$ in (3.21) so that $f(0,s) \to c$ we have

$$|F(x,t) - f(x,t) - c| \le \epsilon \mathcal{H}_{2p}(x,t). \tag{3.22}$$

Now it follows from (3.21) and (3.22) that

$$|G(x+y, t+s) - G(x,t) - G(y,s)| \le 2\epsilon(\mathcal{H}_{2p}(x,t) + \mathcal{H}_{2p}(y,s)), \tag{3.23}$$

where $G(x,t) = F(x,t) - 2c$. Thus it follows from Lemma 3.3 that there exist a unique $a \in \mathbb{C}^n$, a unique $b \in \mathbb{C}$ such that

$$|F(x,t) - a\cdot x - bt - 2c| \le 2\epsilon\, \psi_p(x,t). \tag{3.24}$$

Replacing x by $2x$, t by $4t$ in (3.24) and dividing by 2 in the result we have

$$\begin{aligned} |f(x,t) - a\cdot x - 2bt - c| \le& 2\epsilon\, |\psi_p(2x, 4t)| \\ \le& 2\epsilon 4^p \sum_{|\gamma|=p} \frac{p!}{\gamma!}(2\gamma)! \sum_{0\le\alpha\le\gamma} \frac{2^{|\alpha|+1}}{2^{|2\gamma|} - 2^{|\alpha|+1}} \frac{t^{|\alpha|} x^{2\gamma-2\alpha}}{\alpha!(2\gamma-2\alpha)!} \end{aligned} \tag{3.25}$$

for all $x \in \mathbb{R}^n$, $t > 0$. This completes the proof. □

Theorem 3.5. *Let $u \in \mathcal{D}'$ satisfy*

$$\left\| 2u \circ \frac{A}{2} - u\circ P_1 - u \circ P_2 \right\| \le \epsilon(|x|^{2p} + |y|^{2p}) \tag{3.26}$$

for some integer $p > 1$. Then there exist a unique $a \in \mathbb{C}^n$ and $c \in \mathbb{C}$ such that can be written uniquely in the form

$$\|u - a\cdot x - c\| \le 2\epsilon \frac{4^p}{4^p - 2}|x|^{2p}.$$

Proof. Convolving in each side of (3.26) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels we have in view of (2.3), (2.4), (2.5) and the semigroup property (3.13),

$$\begin{aligned} \left[\left(2u\circ\frac{A}{2}\right) * (E_t(\xi)E_s(\eta))\right](x,\, y) &= \langle 2^{n+1}u_\xi,\, \int E_t(x - 2\xi + \eta)E_s(y-\eta)\, d\eta\rangle \\ &= \langle 2^{n+1}u_\xi,\, (E_t * E_s)(x + y - 2\xi)\rangle \\ &= \langle 2^{n+1}u_\xi,\, E_{t+s}(x+y-2\xi)\rangle \\ &= \langle 2^{n+1}u_\xi,\, 2^{-n}E_{\frac{t+s}{4}}\left(\frac{x+y}{2} - \xi\right)\rangle \\ &= 2\tilde{u}\left(\frac{x+y}{2}, \frac{t+s}{4}\right). \end{aligned}$$

Similarly we have

$$[(u \circ P_1) * (E_t(\xi)E_s(\eta))](x,\, y) = \tilde{u}(x,t),$$
$$[(u \circ P_2) * (E_t(\xi)E_s(\eta))](x,\, y) = \tilde{u}(y,s),$$

where $\tilde{u}(x,t)$ are the Gauss transform of u. Thus the inequality (3.26) is converted to the stability problem

$$\left|2\tilde{u}\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - \tilde{u}(x,t) - \tilde{u}(y,s)\right| \le \epsilon(\mathcal{H}_{2p}(x,t) + \mathcal{H}_{2p}(y,s)).$$

Now applying Lemma 3.4 and letting $t \to 0^+$ we have

$$\|u - a\cdot x - c\| \le 2\epsilon\,\frac{4^p}{4^p-2}|x|^{2p}.$$ □

Since every locally integrable function $f(x)$ can be viewed as a distribution via the equation

$$\langle f, \varphi\rangle = \int f(x)\varphi(x)dx,$$

we have the following stability theorem for locally integrable functions in almost everywhere sense.

Theorem 3.6. *Let Ω_1, $\Omega_2 \subset \mathbb{R}^n$ such that $m(\mathbb{R}^n \setminus \Omega_1) = m(\mathbb{R}^n \setminus \Omega_2) = 0$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be locally integrable functions satisfying the inequality*

$$\left|2f\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - f(x,t) - f(y,s)\right| \le \epsilon(\mathcal{H}_{2p}(x,t) + \mathcal{H}_{2p}(y,s))$$

for all $x \in \Omega_1$, $y \in \Omega_2$. Then there exist a unique $a \in \mathbb{C}^n$, complex constants c and $\Omega \subset \mathbb{R}^n$ with $m(\mathbb{R}^n \setminus \Omega) = 0$ such that

$$\|f(x) - a\cdot x - c\| \le 2\epsilon\frac{4^p}{4^p-2}|x|^{2p}$$

for all $x \in \Omega$.

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