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SUPERSTABILITY OF MULTIPLICATIVE LINEAR MAPPINGS

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Abstract. Let A and B be Banach algebras with unit. Here we prove that an approximate algebra homomorphism $f : A \to B$, in the sense of Rassias, is an algebra homomorphism.

1. Introduction

The stability problem of functional equations appeared at first by Ulam [13] in 1940. In the next year, Hyers [8] studied a version of this problem. In 1978, Th. M. Rassias [12] extended the result of Hyers:

Let X and Y be Banach spaces. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$. Then there exists a unique $\mathbb{R}$-linear mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in X$.

In 1994, Gavruta [7] provided a generalization of Rassias theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ in (1.1) by a general control function $\phi : X \times X \to [0, \infty)$ satisfying

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty$$

for all $x, y \in X$. During the last decades, several stability problems of functional equations have been investigated in spirit of Hyers-Ulam-Rassias (see [4, 9]).

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The aim of this paper is to examine the superstability problem of algebra homomorphisms. The following result, which is called the superstability of ring homomorphisms, was obtained by Bourgin [2]:

Suppose A and B are Banach algebras with unit. If $f : A \to B$ is a surjective mapping such that

$$\|f(a+b) - f(a) - f(b)\| \leq \epsilon \ (\forall a, b \in A),$$
$$\|f(ab) - f(a)f(b)\| \leq \delta \ (\forall a, b \in A)$$

for some $\epsilon \geq 0$ and $\delta \geq 0$, then f is a ring homomorphism, that is, $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in A$.

In 2002, Badora [1] proved the Hyers-Ulam-Rassias stability of ring homomorphisms, which generalized the result of Bourgin. During the last years, the various problems of the stability of homomorphisms have been investigated (See, e.g. [5, 10, 11]). C. G. Park has studied the stability and superstability of algebra homomorphisms on C^*-algebras in [11]. In this paper, we generalize the results of Park's paper. In fact, we prove that an approximate algebra homomorphism between Banach algebras, in the sense of Rassias, is an algebra homomorphism.

Throughout the paper, A and B denote Banach algebras with unit. Here, a linear mapping $f : A \to B$ is called an algebra homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in A$.

2. Main results

In this section, we state and prove our main results which generalize the results of C. G. Park [11, Corollary 3.2 and Theorem 3.3]. We apply the following theorem [6, Theorem 1] or [3, Proposition 1], to obtain an additive function from an approximate one.

Theorem 2.1. *Let X and Y be Banach spaces and $\alpha \in \mathbb{C} \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping for which there exists a function $\phi : X \to [0, \infty)$ such that*

$$\widetilde{\phi}(x) := \sum_{n=0}^{\infty} |\alpha|^{-n} \phi(\alpha^n x) < \infty,$$
$$\|\alpha^{-1} f(\alpha x) - f(x)\| \leq \phi(x)$$

for all $x \in X$. Then $F(x) := \lim_{n\to\infty} \frac{f(\alpha^n x)}{\alpha^n}$ exists and $F(\alpha x) = \alpha F(x)$ for all $x \in X$.

Theorem 2.2. *Let $\gamma, \beta \in \mathbb{C}\setminus\{0\}$ with $\alpha := \gamma+\beta \neq 0$. Suppose that $f : X \to Y$ is a mapping for which there exists a function $\phi : X \times X \to [0, \infty)$ such that*

$$\sum_{n=0}^{\infty} |\alpha|^{-n} \phi(\alpha^n x, \alpha^n y) < \infty, \tag{2.1}$$

$$\|f(\gamma x + \beta y) - \gamma f(x) - \beta f(y)\| \leq \phi(x, y) \tag{2.2}$$

for all $x, y \in X$. *Then* $F(x) := \lim_{n\to\infty} \frac{f(\alpha^n x)}{\alpha^n}$ *exists,* $F(\alpha x) = \alpha F(x)$ *and* $F(\gamma x + \beta y) = \gamma F(x) + \beta F(y)$ *for all* $x, y \in X$.

Proof. Define $\psi(x) = |\alpha|^{-1}\phi(x, x)$ for all $x \in X$. We have

$$\sum_{n=0}^{\infty} |\alpha|^{-n}\psi(\alpha^n x) = \sum_{n=0}^{\infty} |\alpha|^{-n}|\alpha|^{-1}\phi(\alpha^n x, \alpha^n x) < \infty,$$

$$\|\alpha^{-1}f(\alpha x) - f(x)\| = |\alpha|^{-1}\|f(\alpha x) - \alpha f(x)\| \le |\alpha|^{-1}\phi(x, x) = \psi(x).$$

By Theorem 2.1, $F(x) := \lim_{n\to\infty} \frac{f(\alpha^n x)}{\alpha^n}$ exists and $F(\alpha x) = \alpha F(x)$ for all $x \in X$.

Replacing $\alpha^n x, \alpha^n y$ by x, y in (2.2), we get

$$\|f(\alpha^n(\gamma x + \beta y)) - \gamma f(\alpha^n x) - \beta f(\alpha^n y)\| \le \phi(\alpha^n x, \alpha^n y),$$

so

$$\left\|\frac{f(\alpha^n(\gamma x + \beta y))}{\alpha^n} - \gamma\frac{f(\alpha^n x)}{\alpha^n} - \beta\frac{f(\alpha^n y)}{\alpha^n}\right\| \le \frac{1}{|\alpha|^n}\phi(\alpha^n x, \alpha^n y)$$

for all $x, y \in X$. By taking the limit as $n \to \infty$, we have $F(\gamma x + \beta y) = \gamma F(x) + \beta F(y)$ for all $x, y \in X$. □

Theorem 2.3. *Let* $\gamma, \beta \in \mathbb{C}\setminus\{0\}$ *with* $\alpha := \gamma + \beta \ne 0$. *Suppose that* $f : A \to B$ *is a mapping with* $f(0) = 0$ *for which there exist the functions* $\phi, \psi : A \times A \to [0, \infty)$ *such that*

$$\sum_{n=0}^{\infty} |\alpha|^{-n}\phi(\alpha^n x, \alpha^n y) < \infty, \tag{2.3}$$

$$\lim_{n\to\infty} \alpha^{-n}\psi(\alpha^n x, y) = 0,$$

$$\|f(\gamma x + \beta y) - \gamma f(x) - \beta f(y)\| \le \phi(x, y), \tag{2.4}$$

$$\|f(xy) - f(x)f(y)\| \le \psi(x, y) \tag{2.5}$$

for all $x, y \in A$. *Assume that* $F(1) := \lim_{n\to\infty} \frac{f(\alpha^n 1)}{\alpha^n}$ *is invertible. Then* $f : A \to B$ *is an additive and multiplicative mapping.*

Proof. By Theorem 2.2, $F(x) := \lim_{n\to\infty} \frac{f(\alpha^n x)}{\alpha^n}$ exists for all $x \in A$.

Replacing $\alpha^n x$ by x in (2.5), we get

$$\|f(\alpha^n xy) - f(\alpha^n x)f(y)\| \le \psi(\alpha^n x, y),$$

hence

$$\left\|\frac{f(\alpha^n xy)}{\alpha^n} - \frac{f(\alpha^n x)}{\alpha^n}f(y)\right\| \le \frac{1}{|\alpha|^n}\psi(\alpha^n x, y)$$

for all $x, y \in A$ and $n \in \mathbb{N}$. By taking the limit as $n \to \infty$, we have

$$F(xy) = F(x)f(y). \tag{2.6}$$

Let $x, y \in A$ and $n \in \mathbb{N}$. Since $F(\alpha x) = \alpha F(x)$, by (2.6) we have

$$F(x)f(\alpha^n y) = F(\alpha^n xy) = F(\alpha^n x)f(y) = \alpha^n F(x)f(y).$$

Therefore,

$$F(x)\frac{f(\alpha^n y)}{\alpha^n} = F(x)f(y).$$

Sending n to ∞, we get

$$F(x)F(y) = F(x)f(y) \tag{2.7}$$

for all $x, y \in A$.

Let $x = 1$ in (2.7). We have $F(1)F(y) = F(1)f(y)$. Since $F(1)$ is invertible, we obtain $F = f$. Therefore, by (2.6) f is multiplicative. By Theorem 2.2, we have

$$f(\gamma x + \beta y) = \gamma f(x) + \beta f(y) \tag{2.8}$$

for all $x, y \in A$.

Put $y = 0$ and replace x by $\gamma^{-1}x$ in (2.8). We have $f(x) = \gamma f(\gamma^{-1}x)$. Similarly, $f(y) = \beta f(\beta^{-1}y)$.

Replacing x by $\gamma^{-1}x$ and y by $\beta^{-1}y$ in (2.8), we have $f(x+y) = f(x)+f(y)$ for all $x, y \in A$. □

Remark 2.4. It easily can be replaced the condition that $\lim_{n\to\infty} \frac{f(\alpha^n.1)}{\alpha^n}$ is invertible in Theorem 2.3 by the condition that $F(A) \bigcap invB \neq \varnothing$, however this condition is essential. For example the function $f(x) = \sin x$ satisfies (2.4), (2.5) in which $\gamma = \beta = 1$ and $\phi(x, y) = \psi(x, y) = 3$, but f is neither additive nor multiplicative!

Theorem 2.5. *Suppose that $f : A \to B$ is a mapping with $f(0) = 0$ for which there exist the functions $\phi, \psi : A \times A \to [0, \infty)$ such that*

$$\widetilde{\phi}(x, y) := \sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty, \tag{2.9}$$

$$\lim_{n\to\infty} 2^{-n}\psi(2^n x, y) = 0, \tag{2.10}$$

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| \leq \phi(x, y), \tag{2.11}$$

$$\|f(xy) - f(x)f(y)\| \leq \psi(x, y)$$

for all $x, y \in A$ and all $\lambda \in \mathbb{T} := \{\mu \in \mathbb{C} : |\mu| = 1\}$. Assume that $F(1) := \lim_{n\to\infty} \frac{f(2^n.1)}{2^n}$ is invertible in B. Then the mapping f is an algebra homomorphism.

Proof. Assuming $\lambda = 1$ and applying Theorem 2.3, we have that $f : A \to B$ is an additive and multiplicative mapping. Replacing x by $2^n x$ and putting $y = 0$ in (2.11), we get

$$\|f(2^n\lambda x) - \lambda f(2^n x)\| \leq \phi(2^n x, 0)$$

for all $x \in A$ and all $\lambda \in \mathbb{T}$ and all $n \in \mathbb{N}$. Since f is additive, we have $f(2^n x) = 2^n f(x)$, and so we obtain

$$\|f(\lambda x) - \lambda f(x)\| \leq 2^{-n}\phi(2^n x, 0).$$

By taking the limit as $n \to \infty$, we get $f(\lambda x) = \lambda f(x)$ for all $x \in A$ and $\lambda \in \mathbb{T}$. Therefore, by the same reasoning as in the proof of [11, Theorem 2.1], the additive mapping f is linear. □

Corollary 2.6. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exist $p, q \in (-\infty, 1)$ and $\epsilon > 0$ such that*

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| \leq \epsilon(\|x\|^p + \|y\|^q),$$

$$\|f(xy) - f(x)f(y)\| \leq \epsilon(\|x\|^p + \|y\|^q) \tag{2.12}$$

for all $x, y \in A$ and all $\lambda \in \mathbb{T}$. Assume that $\lim_{n\to\infty} \frac{f(2^n.1)}{2^n}$ is invertible. Then f is an algebra homomorphism.

Proof. Define $\phi(x, y) = \psi(x, y) = \epsilon(\|x\|^p + \|y\|^q)$ and apply Theorem 2.5. □

We have replaced the condition that f satisfies $f(xy) = f(x)f(y)$ for all $x, y \in A$ in [11, Corollary 3.2] by the condition that f satisfies inequality (2.12). Therefore, Corollary 2.6 is a generalization of [11, Corollary 3.2].

Theorem 2.7. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exist the functions $\phi, \psi : A \times A \to [0, \infty)$ satisfying* (2.9), (2.10) *and* (2.5) *such that*

$$\|f(ix + iy) - if(x) - if(y)\| \leq \phi(x, y)$$

for all $x, y \in A$. Assume that $\lim_{n\to\infty} \frac{f(2^n 1)}{2^n}$ is invertible. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then f is an algebra homomorphism.

Proof. It is easy to see that the mappings ϕ and ψ, satisfying (2.9) and (2.10), respectively, satisfy

$$\sum_{n=0}^{\infty} |2i|^{-n} \phi((2i)^n x, (2i)^n y) < \infty,$$

$$\lim_{n\to\infty} |2i|^{-n} \psi((2i)^n x, y) = 0$$

for all $x, y \in A$, respectively. By Theorem 2.2, $\lim_{n\to\infty} \frac{f((2i)^n.1)}{(2i)^n}$ exists. Since the sequence $\{\frac{f(2^{4n}.1)}{2^{4n}}\}$ is the subsequence of $\{\frac{f((2i)^n.1)}{(2i)^n}\}$, we have $\lim_{n\to\infty} \frac{f((2i)^n.1)}{(2i)^n}$ is invertible in B. Now by Theorem 2.3, $f : A \to B$ is an additive and multiplicative mapping.

Fix $x \in A$ and $\lambda \in \mathbb{C}$. We define the mapping $h : \mathbb{R} \to B$ by $h(t) = f(tx)$. It is clear that h is additive and so it is $\mathbb{R}$-linear, since the mapping h is continuous in $t \in \mathbb{R}$. Therefore, f is $\mathbb{R}$-linear.

There exist $a, b \in \mathbb{R}$ such that $\lambda = a + ib$. So

$$f(\lambda x) = f(ax + ibx) = af(x) + bf(ix) = af(x) + bif(x) = \lambda f(x).$$

Therefore, f is linear. □

Corollary 2.8. *Let $p, q \in (-\infty, 1)$ and $\epsilon > 0$. Suppose $f : A \to B$ is a mapping with $f(0) = 0$ satisfying (2.12) such that*

$$\|f(ix + iy) - if(x) - if(y)\| \leq \epsilon(\|x\|^p + \|y\|^q)$$

for all $x, y \in A$. Assume that $\lim_{n\to\infty} \frac{f(2^n.1)}{2^n}$ is invertible. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then f is an algebra homomorphism.

Proof. Define $\phi(x, y) = \psi(x, y) = \epsilon(\|x\|^p + \|y\|^q)$ and apply Theorem 2.7. $\square$

Remark 2.9. We have proved Theorem 2.7 and Corollary 2.8, under the assumptions that

i) the mapping f satisfies

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| \leq \phi(x, y)$$

only for $\lambda = i$ and

ii) f is an approximately multiplicative mapping. (Compare [11, Theorem 3.3]).

Remark 2.10. In all statements, the condition that control function ψ satisfies $\lim_{n\to\infty} 2^{-n}\psi(2^n x, y) = 0$ can be replaced by the condition that ψ satisfies $\lim_{n\to\infty} 2^{-n}\psi(x, 2^n y) = 0$.

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