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STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In this paper, we determine some stability results concerning the 2-dimensional vector variable quadratic functional equation $f(x+y, z+w) + f(x-y, z-w) = 2f(x, z) + 2f(y, w)$ in intuitionistic fuzzy normed spaces (IFNS). We define the intuitionistic fuzzy continuity of the 2-dimensional vector variable quadratic mappings and prove that the existence of a solution for any approximately 2-dimensional vector variable quadratic mapping implies the completeness of IFNS.

1. Introduction and preliminaries

Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g. population dynamics [5], chaos control [9, 10], computer programming [11], nonlinear dynamical systems [12], nonlinear operators [21], statistical convergence [20], etc. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. In 1984, Katsaras [15] defined fuzzy norms on linear spaces and at the same year Wu and Fang also introduced a notion of fuzzy normed spaces and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. In [6], Biswas defined and studied fuzzy inner products on linear spaces. Since then some mathematicians have defined fuzzy metrics and norms on linear spaces from various points of view [4, 8, 16, 31, 33]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norms on linear spaces in such a manner that the corresponding induced fuzzy metrics are of Kramosil and Michalek type. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson by removing a regular condition. Following [3], we give the following notion of fuzzy norms.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x, c) = 0$ for $c \leq 0$;

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(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
(N3) $N(cx, t) = N\big(x, \frac{t}{|c|}\big)$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) For $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed linear space*. One may regard $N(x, t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t'.

There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm [23, 26, 27] seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness through the intuitionistic fuzzy norm. Stability problem of a functional equation was first posed by Ulam [32] which was answered by Hyers [13] and then generalized by Aoki [1] and Rassias [24] for additive mappings and linear mappings, respectively. Since then several stability problems for various functional equations have been investigated in [14, 25]; various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations were discussed in [17, 18, 19, 22, 29]; and some random stability results concerning Jensen and cubic functional equations were discussed in [7, 30]. The stability problem for the 2-dimensional vector variable quadratic functional equation was proved by the authors [2] for mappings $f : X \times X \to Y$, where X is a real normed space and Y is a Banach space. In this paper, we determine some stability results concerning the 2-dimensional vector variable quadratic functional equation

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w)$$

in intuitionistic fuzzy normed spaces. We define the intuitionistic fuzzy continuity of the 2-dimensional vector variable quadratic mappings and prove that the existence of a solution for any approximately 2-dimensional vector variable quadratic mapping implies the completeness of intuitionistic fuzzy normed spaces (IFNS). In this section we recall some notations and basic definitions used in this paper.

Definition 1.1 ([28])**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *continuous t-norm* if it satisfies the following conditions: (a) $*$ is associative and commutative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0, 1]$, (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 1.2 ([28])**.** A binary operation $\diamondsuit : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions: (a′) $\diamondsuit$ is associative and commutative, (b′) $\diamondsuit$ is continuous, (c′) $a \diamondsuit 0 = a$ for all $a \in [0, 1]$, (d′) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [27] have recently introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 1.3. The five-tuple $(X, \mu, \nu, *, \diamondsuit)$ is said to be an *intuitionistic fuzzy normed space* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times \mathbb{R}$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$,

(i) $\mu(x,t) + \nu(x,t) \leq 1$, (ii) $\mu(x,t) > 0$, (iii) $\mu(x,t) = 1$ if and only if $x = 0$, (iv) $\mu(\alpha x, t) = \mu\big(x, \frac{t}{|\alpha|}\big)$ for each $\alpha \neq 0$, (v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$, (vi) $\mu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous, (vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$, (viii) $\nu(x,t) < 1$, (ix) $\nu(x,t) = 0$ if and only if $x = 0$, (x) $\nu(\alpha x, t) = \nu\big(x, \frac{t}{|\alpha|}\big)$ for each $\alpha \neq 0$, (xi) $\nu(x,t) \diamondsuit \nu(y,s) \geq \nu(x+y, t+s)$, (xii) $\nu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous, (xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$. In this case (μ, ν) is called an *intuitionistic fuzzy norm.*

Example 1.4. Let $(X, \|\cdot\|)$ be a normed space, $a * b = ab$ and $a \diamondsuit b = \min\{a + b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) := \begin{cases} \frac{t}{t+\|x\|} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0 \end{cases} \qquad \text{and} \qquad \nu(x,t) := \begin{cases} \frac{\|x\|}{t+\|x\|} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamondsuit)$ is an IFNS.

Remark 1.5. In intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamondsuit)$, $\mu(x, \cdot)$ is non-decreasing and $\nu(x, \cdot)$ is non-increasing for all $x \in X$ (see [27]).

Let $(X, \mu, \nu, *, \diamondsuit)$ be an IFNS. A sequence $x = (x_k)$ is said to be *intuitionistic fuzzy convergent* to $L \in X$ if $\lim_{k\to\infty} \mu(x_k - L, t) = 1$ and $\lim_{k\to\infty} \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write $x_k \to L$ as $k \to \infty$. A sequence $x = (x_k)$ is said to be *intuitionistic fuzzy Cauchy sequence* if $\lim_{k\to\infty} \mu(x_{k+p} - x_k, t) = 1$ and $\lim_{k\to\infty} \nu(x_{k+p} - x_k, t) = 0$ for all $p \in \mathbb{N}$ and all $t > 0$. The IFNS $(X, \mu, \nu, *, \diamondsuit)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamondsuit)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamondsuit)$ and $(X, \mu, \nu, *, \diamondsuit)$ is also called an *intuitionistic fuzzy Banach space.*

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [27].

2. Intuitionistic fuzzy stability

Consider the functional equation

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w) \tag{2.1}$$

having quadratic forms $f(x,y) = ax^2 + bxy + cy^2$ as solutions. We begin with a generalized Hyers-Ulam type theorem in IFNS for the functional equation (2.1).

Theorem 2.1. *Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping such that, for some $0 < \alpha < 4$,*

$$\begin{cases} \mu'(\varphi(2x, 2y, 2z, 2w), t) \geq \mu'(\alpha\varphi(x,y,z,w),\, t), \\ \nu'(\varphi(2x, 2y, 2z, 2w), t) \leq \nu'(\alpha\varphi(x,y,z,w),\, t) \end{cases} \tag{2.2}$$

$$\lim_{n\to\infty} \mu'\big(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t\big) = 1$$

and

$$\lim_{n\to\infty} \nu'\big(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t\big) = 0$$

for all $x, y, z, w \in X$ *and all* $t > 0$. *Let* (Y, μ, ν) *be an intuitionistic fuzzy Banach space and let* $f : X \times X \to Y$ *be a mapping such that*

$$(2.3) \qquad \begin{cases} & \mu\big(f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w),\, t\big) \\ \geq & \mu'(\varphi(x,y,z,w),\, t), \\ & \nu\big(f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w),\, t\big) \\ \leq & \nu'(\varphi(x,y,z,w),\, t) \end{cases}$$

for all $x, y, z, w \in X$ *and all* $t > 0$. *Then there exists a unique mapping* $F : X \times X \to Y$ *satisfying* (2.1) *such that*

$$(2.4) \quad \begin{cases} \mu\left(F(x,y) - f(x,y) + \dfrac{1}{3} f(0,0),\, t\right) \geq *^{\infty} \mu'(\varphi(x,x,y,y),\, (4-\alpha)t), \\ \nu\left(F(x,y) - f(x,y) + \dfrac{1}{3} f(0,0),\, t\right) \leq \diamondsuit^{\infty} \nu'(\varphi(x,x,y,y),\, (4-\alpha)t) \end{cases}$$

for all $x, y \in X$ *and all* $t > 0$, *where* $*^{\infty} a := a * a * \cdots$ *and* $\diamondsuit^{\infty} a := a \diamondsuit a \diamondsuit \cdots$ *for all* $a \in [0,1]$.

Proof. Putting $y = x$ and $w = z$ in (2.3), we get

$$(2.5) \qquad \begin{cases} \mu\left(\dfrac{f(2x,2z) + f(0,0)}{4} - f(x,z),\, \dfrac{t}{4}\right) \geq \mu'(\varphi(x,x,z,z),\, t), \\ \nu\left(\dfrac{f(2x,2z) + f(0,0)}{4} - f(x,z),\, \dfrac{t}{4}\right) \leq \nu'(\varphi(x,x,z,z),\, t) \end{cases}$$

for all $x, z \in X$ and all $t > 0$. Replacing x by $2^n x$ and z by $2^n z$ in (2.5) and using (2.2), we get

$$\begin{aligned} &\mu\left(\frac{f(2^{n+1}x, 2^{n+1}z) + f(0,0)}{4^{n+1}} - \frac{f(2^n x, 2^n z)}{4^n},\, \frac{t}{4^{n+1}}\right) \\ &\geq \mu'(\varphi(2^n x, 2^n x, 2^n z, 2^n z),\, t) \geq \mu'\left(\varphi(x,x,z,z),\, \frac{t}{\alpha^n}\right) \end{aligned}$$

and

$$\begin{aligned} &\nu\left(\frac{f(2^{n+1}x, 2^{n+1}z) + f(0,0)}{4^{n+1}} - \frac{f(2^n x, 2^n z)}{4^n}, \frac{t}{4^{n+1}}\right) \\ &\leq \nu'(\varphi(2^n x, 2^n x, 2^n z, 2^n z),\, t) \leq \nu'\left(\varphi(x,x,z,z),\, \frac{t}{\alpha^n}\right) \end{aligned}$$

for all $x, z \in X$, all $n \geq 0$ and all $t > 0$. By replacing t by $\alpha^n t$, we have

(2.6)
$$\begin{cases} \mu\left(\dfrac{f(2^{n+1}x, 2^{n+1}z) + f(0,0)}{4^{n+1}} - \dfrac{f(2^n x, 2^n z)}{4^n}, \dfrac{\alpha^n t}{4^{n+1}}\right) \geq \mu'(\varphi(x,x,z,z), t), \\ \nu\left(\dfrac{f(2^{n+1}x, 2^{n+1}z) + f(0,0)}{4^{n+1}} - \dfrac{f(2^n x, 2^n z)}{4^n}, \dfrac{\alpha^n t}{4^{n+1}}\right) \leq \nu'(\varphi(x,x,z,z), t) \end{cases}$$

for all $x, z \in X$, all $n \geq 0$ and all $t > 0$. It follows from

$$\begin{aligned} &\sum_{k=0}^{n-1} \left[\frac{f(2^{k+1}x, 2^{k+1}z) + f(0,0)}{4^{k+1}} - \frac{f(2^k x, 2^k z)}{4^k}\right] \\ &= \frac{f(2^n x, 2^n z)}{4^n} - f(x,z) + \frac{1}{3}\left(1 - \frac{1}{4^n}\right) f(0,0) \end{aligned}$$

and (2.6) that

$$\begin{cases} & \mu\left(\dfrac{f\,(2^n x, 2^n z)}{4^n} - f(x,z) + \dfrac{1}{3}\left(1 - \dfrac{1}{4^n}\right) f(0,0), \displaystyle\sum_{k=0}^{n-1} \dfrac{\alpha^k t}{4^{k+1}}\right) \\ \geq & \displaystyle\prod_{k=0}^{n-1} \mu\left(\dfrac{f(2^{k+1}x, 2^{k+1}z) + f(0,0)}{4^{k+1}} - \dfrac{f(2^k x, 2^k z)}{4^k}, \dfrac{\alpha^k t}{4^{k+1}}\right) \\ \geq & *^n \mu'(\varphi(x,x,z,z), t), \\ & \nu\left(\dfrac{f(2^n x, 2^n z)}{4^n} - f(x,z) + \dfrac{1}{3}\left(1 - \dfrac{1}{4^n}\right) f(0,0), \displaystyle\sum_{k=0}^{n-1} \dfrac{\alpha^k t}{4^{k+1}}\right) \\ \leq & \displaystyle\coprod_{k=0}^{n-1} \nu\left(\dfrac{f(2^{k+1}x, 2^{k+1}z) + f(0,0)}{4^{k+1}} - \dfrac{f(2^k x, 2^k z)}{4^k}, \dfrac{\alpha^k t}{4^{k+1}}\right) \\ \leq & \diamondsuit^n \nu'(\varphi(x,x,z,z), t) \end{cases} \tag{2.7}$$

for all $x, z \in X$, all $n \in \mathbb{N}$ and all $t > 0$, where $\prod_{j=1}^n a_j := a_1 * a_2 * \cdots * a_n$, $\coprod_{j=1}^n a_j := a_1 \diamondsuit a_2 \diamondsuit \cdots \diamondsuit a_n$, $*^n a := \prod_{j=1}^n a = a * \cdots * a$ and $\diamondsuit^n a := \coprod_{j=1}^n a = a \diamondsuit \cdots \diamondsuit a$ for all $a, a_1, a_2, \ldots, a_n \in [0,1]$. By replacing x with $2^m x$ and z with $2^m z$ in (2.7), we have

$$\begin{aligned} &\mu\left(\frac{f(2^{n+m}x, 2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3 \cdot 4^m}\left(1 - \frac{1}{4^n}\right) f(0,0), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^{k+m+1}}\right) \\ &\geq *^n\, \mu'(\varphi(2^m x, 2^m x, 2^m z, 2^m z), t) \geq *^n \mu'\left(\varphi(x,x,z,z), \frac{t}{\alpha^m}\right) \end{aligned}$$

and

$$\begin{aligned} &\nu\left(\frac{f(2^{n+m}x, 2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3 \cdot 4^m}\left(1 - \frac{1}{4^n}\right) f(0,0), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^{k+m+1}}\right) \\ &\leq \diamondsuit^n \nu'(\varphi(2^m x, 2^m x, 2^m z, 2^m z), t) \leq \diamondsuit^n \nu'\left(\varphi(x,x,z,z), \frac{t}{\alpha^m}\right) \end{aligned}$$

for all $x, z \in X$, all $m \geq 0$, all $n \in \mathbb{N}$ and all $t > 0$. Thus we get

$$\mu\Bigg(\frac{f(2^{n+m}x, 2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3 \cdot 4^m}\left(1 - \frac{1}{4^n}\right) f(0,0), \sum_{k=m}^{n+m-1} \frac{\alpha^k t}{4^{k+1}}\Bigg)$$
$$\geq \ *^n\ \mu'(\varphi(x,x,z,z), t)$$

and

$$\nu\Bigg(\frac{f(2^{n+m}x, 2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3 \cdot 4^m}\left(1 - \frac{1}{4^n}\right) f(0,0), \sum_{k=m}^{n+m-1} \frac{\alpha^k t}{4^{k+1}}\Bigg)$$
$$\leq \diamondsuit^n \nu'(\varphi(x,x,z,z), t)$$

for all $x, z \in X$, all $m \geq 0$, all $n \in \mathbb{N}$ and all $t > 0$. Replacing t by $\frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^{k+1}}}$, we get

(2.8)
$$\begin{cases} & \mu\left(\frac{f(2^{n+m}x, 2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3 \cdot 4^m}\left(1 - \frac{1}{4^n}\right) f(0,0), t\right) \\ \geq & *^n \mu'\left(\varphi(x,x,z,z), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^{k+1}}}\right), \\ & \nu\left(\frac{f(2^{n+m}x, 2^{n+m}z)}{4^{n+m}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3 \cdot 4^m}\left(1 - \frac{1}{4^n}\right) f(0,0), t\right) \\ \leq & \diamondsuit^n \nu'\left(\varphi(x,x,z,z), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^{k+1}}}\right) \end{cases}$$

for all $x, z \in X$, all $m \geq 0$, all $n \in \mathbb{N}$ and all $t > 0$. Since $0 < \alpha < 4$ and $\sum_{k=0}^{\infty} \left(\frac{\alpha}{4}\right)^k < \infty$ and $\sum_{k=m}^{n+m-1} \left(\frac{\alpha^k}{4^{k+1}}\right)^k \to 0$ as $m \to \infty$ for all $n \in \mathbb{N}$. Thus $\frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^{k+1}}} \to \infty$ and $\mu'\left(\varphi(x,x,z,z), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^{k+1}}}\right) \to 1$ as $m \to \infty$ for all $x, z \in X$, all $n \in \mathbb{N}$ and all $t > 0$. Hence the Cauchy criterion for convergence in IFNS shows that $\left(\frac{f(2^n x, 2^n z)}{4^n}\right)$ is a Cauchy sequence in (Y, μ, ν) for all $x, z \in X$. Since (Y, μ, ν) is complete, this sequence converges to some point $F(x, z) \in Y$ for all $x, z \in X$. Put $m = 0$ in (2.8) to obtain

$$\mu\left(\frac{f(2^n x, 2^n z)}{4^n} - f(x,z) + \frac{1}{3}\left(1 - \frac{1}{4^n}\right) f(0,0),\ t\right)$$
$$\geq \ *^n\ \mu'\left(\varphi(x,x,z,z),\ \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+1}}}\right)$$

and

$$\nu\left(\frac{f(2^n x, 2^n z)}{4^n} - f(x,z) + \frac{1}{3}\left(1 - \frac{1}{4^n}\right) f(0,0),\ t\right)$$
$$\leq \diamondsuit^n \nu'\left(\varphi(x,x,z,z),\ \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{4^{k+1}}}\right)$$

for all $x, z \in X$, all $n \in \mathbb{N}$ and all $t > 0$. Taking the limit as $n \to \infty$ and using the definition of IFNS, we get

$$\mu\left(F(x,y) - f(x,y) + \frac{1}{3}f(0,0), t\right) \geq *^{\infty}\mu'(\varphi(x,x,y,y),(4-\alpha)t)$$

and

$$\nu\left(F(x,y) - f(x,y) + \frac{1}{3}f(0,0), t\right) \leq \Diamond^{\infty}\nu'(\varphi(x,x,y,y),(4-\alpha)t)$$

for all $x, y \in X$ and all $t > 0$. Replacing x, y, z, w and t in (2.3) by $2^n x$, $2^n y$, $2^n z$, $2^n w$ and $4^n t$, respectively, we have

$$\begin{aligned}&\mu\bigg(\frac{f(2^n x + 2^n y, 2^n z + 2^n w)}{4^n} + \frac{f(2^n x - 2^n y, 2^n z - 2^n w)}{4^n}\\ &\qquad - 2\frac{f(2^n x, 2^n z)}{4^n} - 2\frac{f(2^n y, 2^n w)}{4^n}, t\bigg)\\ &\geq \mu'(\varphi(2^n x, 2^n x, 2^n y, 2^n y), 4^n t)\end{aligned}$$

and

$$\begin{aligned}&\nu\bigg(\frac{f(2^n x + 2^n y, 2^n z + 2^n w)}{4^n} + \frac{f(2^n x - 2^n y, 2^n z - 2^n w)}{4^n}\\ &\qquad - 2\frac{f(2^n x, 2^n z)}{4^n} - 2\frac{f(2^n y, 2^n w)}{4^n}, t\bigg)\\ &\leq \nu'(\varphi(2^n x, 2^n x, 2^n y, 2^n y), 4^n t)\end{aligned}$$

for all $x, y, z, w \in X$, all $n \in \mathbb{N}$ and all $t > 0$. Since

$$\lim_{n\to\infty} \mu'(\varphi(2^n x, 2^n x, 2^n y, 2^n y), 4^n t) = 1$$

and

$$\lim_{n\to\infty} \nu'(\varphi(2^n x, 2^n x, 2^n y, 2^n y), 4^n t) = 0$$

for all $x, y \in X$ and all $t > 0$, we observe that F fulfills (2.1). To prove the uniqueness of the mapping F, assume that there exists a mapping $G : X \times X \to Y$ which also satisfies (2.1) and (2.4). For fix $x, y \in X$, clearly $F(2^n x, 2^n y) = 4^n F(x,y)$ and $G(2^n x, 2^n y) = 4^n G(x,y)$ for all $n \in \mathbb{N}$. It follows from (2.4) that

$$\begin{aligned}&\mu(F(x,y) - G(x,y), t)\\ &= \mu\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{G(2^n x, 2^n y)}{4^n}, t\right)\\ &\geq \mu\left(\frac{F(2^n x, 2^n y)}{4^n} - \frac{f(2^n x, 2^n y)}{4^n} + \frac{1}{3\cdot 4^n}f(0,0), \frac{t}{2}\right)\end{aligned}$$

$$
\begin{aligned}
& * \, \mu\left(-\frac{G(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 2^n y)}{4^n} - \frac{1}{3 \cdot 4^n} f(0,0),\ \frac{t}{2}\right) \\
\geq\ & *^2 *^\infty \mu'\left(\varphi(2^n x, 2^n x, 2^n y, 2^n y),\ \frac{4^n(4-\alpha)t}{2}\right) \\
\geq\ & *^2 *^\infty \mu'\left(\varphi(x, x, y, y),\ \frac{4^n(4-\alpha)t}{2\alpha^n}\right)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and all $t > 0$, and similarly

$$
\nu(F(x,y) - G(x,y), t) \leq \diamondsuit^2 \diamondsuit^\infty \nu'\left(\varphi(x, x, y, y),\ \frac{4^n(4-\alpha)t}{2\alpha^n}\right)
$$

for all $n \in \mathbb{N}$ and all $t > 0$. Since $\lim_{n\to\infty} \frac{4^n(4-\alpha)t}{2\alpha^n} = \infty$ for all $t > 0$, we get

$$
\lim_{n\to\infty} \mu'\left(\varphi(x, x, y, y),\ \frac{4^n(4-\alpha)t}{2\alpha^n}\right) = 1
$$

and

$$
\lim_{n\to\infty} \nu'\left(\varphi(x, x, y, y), \frac{4^n(4-\alpha)t}{2\alpha^n}\right) = 0
$$

for all $t > 0$. Therefore $\mu(F(x,y) - G(x,y), t) = 1$ and $\nu(F(x,y) - G(x,y), t) = 0$ for all $t > 0$. Hence $F(x,y) = G(x,y)$. $\square$

Example 2.2. Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and Z be a normed space. Denote by (μ, ν) and (μ', ν') the intuitionistic fuzzy norms given as in Example 1.1 on X and Z, respectively. Let $\|\cdot\|$ be the induced norm on X by the inner product $\langle \cdot, \cdot \rangle$ on X. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping defined by $\varphi(x, y, z, w) = 2\big(\|x\| + \|y\| + \|z\| + \|w\|\big) z_0$ for all $x, y, z, w \in X$, where z_0 is a fixed unit vector in Z. Define a mapping $f : X \times X \to X$ by $f(x, y) := \langle x, y + x_0 \rangle x_0$ for all $x, y \in X$, where x_0 is a fixed unit vector in X. Then

$$
\begin{aligned}
& \mu\big(f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w),\ t\big) \\
=\ & \mu(-2\langle y, x_0 \rangle x_0, t) = \frac{t}{t + 2\ |\langle y, x_0 \rangle|} \\
\geq\ & \frac{t}{t + 2\|y\|} \geq \frac{t}{t + 2\big(\|x\| + \|y\| + \|z\| + \|w\|\big)} = \mu'\big(\varphi(x, y, z, w), t\big)
\end{aligned}
$$

and

$$
\begin{aligned}
& \nu\big(f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w),\ t\big) \\
=\ & \frac{2\ |\langle y, x_0 \rangle|}{t + 2\ |\langle y, x_0 \rangle|} \leq \frac{2\|y\|}{t + 2\|y\|} \leq \frac{2\big(\|x\| + \|y\| + \|z\| + \|w\|\big)}{t + 2\big(\|x\| + \|y\| + \|z\| + \|w\|\big)} \\
=\ & \nu'\big(\varphi(x, y, z, w), t\big)
\end{aligned}
$$

for all $x, y, z, w \in X$ and all $t > 0$. Also

$$
\mu'\big(\varphi(2x, 2y, 2z, 2w), t\big) = \frac{t}{t + 4\big(\|x\| + \|y\| + \|z\| + \|w\|\big)} = \mu'\big(2\varphi(x, y, z, w), t\big)
$$

and

$$\nu'\big(\varphi(2x,2y,2z,2w),t\big) = \frac{4\big(\|x\|+\|y\|+\|z\|+\|w\|\big)}{t+4\big(\|x\|+\|y\|+\|z\|+\|w\|\big)} = \nu'\big(2\varphi(x,y,z,w),t\big)$$

for all $x,y,z,w \in X$ and all $t>0$. Thus

$$\begin{aligned}&\lim_{n\to\infty}\mu'\big(\varphi(2^nx,2^ny,2^nz,2^nw),4^nt\big)\\ &= \lim_{n\to\infty}\frac{4^nt}{4^nt+2^{n+1}\big(\|x\|+\|y\|+\|z\|+\|w\|\big)} = 1\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\nu'\big(\varphi(2^nx,2^ny,2^nz,2^nw),4^nt\big)\\ &= \lim_{n\to\infty}\frac{2^{n+1}\big(\|x\|+\|y\|+\|z\|+\|w\|\big)}{4^nt+2^{n+1}\big(\|x\|+\|y\|+\|z\|+\|w\|\big)} = 0\end{aligned}$$

for all $x,y,z,w \in X$ and all $t>0$. Hence the conditions of Theorem 2.1 for $\alpha = 2$ are fulfilled. Therefore, there is a unique quadratic mapping $F : X\times X \to X$ such that

$$\mu\big(F(x,y)-f(x,y),\,t\big) \ge \mu'\big(4(\|x\|+\|y\|)z_0,\,2t\big)$$

and

$$\nu\big(F(x,y)-f(x,y),\,t\big) \le \nu'\big(4(\|x\|+\|y\|)z_0,\,2t\big)$$

for all $x,y \in X$ and all $t>0$.

In the following theorem we consider the case $\alpha > 4$.

Theorem 2.3. *Let X be a linear space and let (Z,μ',ν') be an IFNS. Let $\varphi : X\times X\times X\times X \to Z$ be a mapping such that, for some $\alpha>4$,*

$$\mu'\left(\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}\right),\,t\right) \ge \mu'(\varphi(x,y,z,w),\,\alpha t),$$

$$\nu'\left(\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}\right),\,t\right) \le \nu'(\varphi(x,y,z,w),\,\alpha t),$$

$$\lim_{n\to\infty}\mu'\left(4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right),\,t\right) = 1$$

and

$$\lim_{n\to\infty}\nu'\left(4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\right),\,t\right) = 0$$

for all $x,y,z,w \in X$ and all $t>0$. Let (Y,μ,ν) be an intuitionistic fuzzy Banach space and let $f : X\times X \to Y$ be a φ-approximately quadratic mapping in the sense of (2.3) with $f(0,0)=0$. Then there exists a unique mapping $F : X\times X\to Y$ such that

$$\mu(F(x,y)-f(x,y),\,t) \ge *^{\infty}\mu'(\varphi(x,x,y,y),\,(\alpha-4)t)$$

and

$$\nu(F(x,y)-f(x,y),\,t) \le \Diamond^{\infty}\nu'(\varphi(x,x,y,y),\,(\alpha-4)t)$$

for all $x,y\in X$ and all $t>0$.

Proof. The techniques are similar to that of Theorem 2.1. Hence we present a sketch of proof. Put $y = x$ and $w = z$ in (2.3), we get

$$\mu(f(2x, 2z) - 4f(x, z),\, t) \geq \mu'(\varphi(x, x, z, z),\, t)$$

and

$$\nu(f(2x, 2z) - 4f(x, z),\, t) \leq \nu'(\varphi(x, x, z, z),\, t)$$

for all $x, z \in X$ and all $t > 0$. Thus we get

$$\mu\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right),\, t\right) \geq \mu'(\varphi(x, x, z, z),\, \alpha t)$$

and

$$\nu\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \leq \nu'(\varphi(x, x, z, z), \alpha t)$$

for all $x, z \in X$ and all $t > 0$. For all $x, z \in X$, all $m \geq 0$, all $n \in \mathbb{N}$ and all $t > 0$, we can deduce

$$(2.9) \qquad \begin{cases} & \mu\left(4^m f\left(\dfrac{x}{2^m}, \dfrac{z}{2^m}\right) - 4^{n+m} f\left(\dfrac{x}{2^{n+m}}, \dfrac{z}{2^{n+m}}\right), t\right) \\ \geq & *^n \mu'\left(\varphi(x, x, z, z), \dfrac{t}{\sum_{k=m}^{n+m-1} \frac{4^k}{\alpha^{k+1}}}\right), \\ & \nu\left(4^m f\left(\dfrac{x}{2^m}, \dfrac{z}{2^m}\right) - 4^{n+m} f\left(\dfrac{x}{2^{n+m}}, \dfrac{z}{2^{n+m}}\right), t\right) \\ \leq & \diamondsuit^n \nu'\left(\varphi(x, x, z, z), \dfrac{t}{\sum_{k=m}^{n+m-1} \frac{4^k}{\alpha^{k+1}}}\right). \end{cases}$$

Hence the sequence $\left(4^n f(\frac{x}{2^n}, \frac{z}{2^n})\right)$ is a Cauchy sequence in the intuitionistic fuzzy Banach space Y. Therefore, there is a mapping $F : X \times X \to Y$ defined by $F(x, y) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ for all $x, y \in X$. The system of inequalities (2.9) with $m = 0$ implies that

$$\mu(F(x, z) - f(x, z), t) \geq *^\infty \mu'(\varphi(x, x, z, z), (\alpha - 4)t)$$

and

$$\nu(F(x, z) - f(x, z), t) \leq \diamondsuit^\infty \nu'(\varphi(x, x, z, z), (\alpha - 4)t)$$

for all $x, z \in X$ and all $t > 0$. The rest of the proof is similar to the proof of Theorem 2.1. $\square$

3. Intuitionistic fuzzy continuity

Recently, the intuitionistic fuzzy continuity is discussed in [21]. In this section, we establish some interesting results of continuous mappings satisfying (2.1) approximately.

Definition 3.1. Let $g : \mathbb{R} \to X$ be a mapping, where $\mathbb{R}$ is endowed with the Euclidean topology and X is an intuitionistic fuzzy normed space equipped with intuitionistic fuzzy norm (μ, ν). Then $L \in X$ is said to be *intuitionistic fuzzy limit* of g at some $r_0 \in \mathbb{R}$ if and olny if for every $\varepsilon > 0$ and $\alpha, \beta \in (0, 1)$ there exists some $\delta = \delta(\varepsilon, \alpha, \beta) > 0$ such that $\mu(g(r) - L, \varepsilon) \geq \alpha$ and $\mu(g(r) - L, \varepsilon) \leq 1 - \beta$ whenever $0 < |r - r_0| < \delta$. In this case, we write $\lim_{r\to r_0} g(r) = L$, which also means that $\lim_{r\to r_0}\mu(g(r)-L,t)=0$ and $\lim_{r\to r_0}\nu(g(r)-L,t)=1$ or $\mu(g(r)-L,t)=1$ and $\nu(g(r)-L,t)=0$ as $r\to r_0$ for all $t>0$.

The mapping g is said to be *intuitionistic fuzzy continuous* at a point $r_0\in\mathbb{R}$ if and only if $\lim_{r\to r_0}\mu(g(r)-g(r_0),t)=1$ and $\lim_{r\to r_0}\nu(g(r)-g(r_0),t)=0$ for all $t>0$.

Theorem 3.2. *Let X be a normed space and (Y,μ,ν) be an intuitionistic fuzzy Banach space. Let (Z,μ',ν') be an IFNS and let $0<p<2$ and $z_0\in Z$. Let $f:X\times X\to Y$ be a mapping such that*

$$(3.1)\qquad \begin{cases} & \mu\big(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w),\,t\big)\\ \geq & \mu'\big((\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,\,t\big),\\ & \nu\big(f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w),\,t\big)\\ \leq & \nu'\big((\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,\,t\big)\end{cases}$$

for all $x,y,z,w\in X$ and all $t>0$. Assume that μ' and ν' satisfies

$$\lim_{n\to\infty}\mu'\big(2^{np}(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,\,4^nt\big)=1$$

and

$$\lim_{n\to\infty}\nu'\big(2^{np}(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0,\,4^nt\big)=0$$

for all $x,y,z,w\in X$ and all $t>0$. Then there exists a unique mapping $F:X\times X\to Y$ satisfying (2.1) *such that*

$$(3.2)\qquad \begin{cases}\mu(F(x,y)-f(x,y),t)\geq *^{\infty}\mu'\big(2(\|x\|^p+\|y\|^p)z_0,\,(4-2^p)t\big),\\ \nu(F(x,y)-f(x,y),t)\leq \diamondsuit^{\infty}\nu'\big(2(\|x\|^p+\|y\|^p)z_0,\,(4-2^p)t\big)\end{cases}$$

for all $x,y\in X$ and all $t>0$. Furthermore, if the mapping $g:\mathbb{R}\to Y$ defined by $g(r):=\frac{f(2^nrx,2^nry)}{4^n}$ is intuitionistic fuzzy continuous for some $x,y\in X$ and all $n\in\mathbb{N}$, then the mapping $r\to F(rx,ry)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous; *in this case, $F(rx,ry)=r^2F(x,y)$ for all $r\in\mathbb{R}$.*

Proof. Define $\varphi:X\times X\times X\times X\to Z$ by $\varphi(x,y,z,w)=(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)z_0$ for all $x,y,z,w\in X$. Existence and uniqueness of the mapping F satisfying (2.1) and (3.2) are deduced from Theorem 2.1. Note that, for all $x,y\in X$, all $n\in\mathbb{N}$ and all $t>0$, we have

$$(3.3)\qquad \begin{cases} & \mu\left(F(x,y)-\dfrac{f(2^nx,2^ny)}{4^n},\,t\right)\\ = & \mu\left(\frac{F(2^nx,2^ny)}{4^n}-\frac{f(2^nx,2^ny)}{4^n},\,t\right)\\ = & \mu(F(2^nx,2^ny)-f(2^nx,2^ny),\,4^nt)\\ \geq & *^{\infty}\mu'\big(2^{np+1}(\|x\|^p+\|y\|^p)z_0,4^n(4-2^p)t\big),\\ & \nu\left(F(x,y)-\frac{f(2^nx,2^ny)}{4^n},\,t\right)\\ \leq & \diamondsuit^{\infty}\nu'\big(2^{np+1}(\|x\|^p+\|y\|^p)z_0,\,4^n(4-2^p)t\big).\end{cases}$$

Putting $x = y = 0$ in (2.1), we get

$$\mu\left(F(0,0) - \frac{1}{4^n} f(0,0), t\right) \geq 1$$

and

$$\nu\left(F(0,0) - \frac{1}{4^n} f(0,0), t\right) \leq 0$$

for all $n \in \mathbb{N}$ and all $t > 0$.

Fix $x, y \in X$. From (3.3), we get

$$\mu\left(F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, t\right) \geq *^{\infty} \mu'\left((\|x\|^p + \|y\|^p) z_0, \frac{4^n(4 - 2^p)t}{2^{np+1}|r|^p}\right)$$

and

$$\mu\left(F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, t\right) \leq \Diamond^{\infty} \nu'\left((\|x\|^p + \|y\|^p) z_0, \frac{4^n(4 - 2^p)t}{2^{np+1}|r|^p}\right)$$

for all $r \in \mathbb{R} \setminus \{0\}$. Since $\lim_{n\to\infty} \frac{4^n(4-2^p)t}{2^{np+1}|r|^p} = \infty$ for all $t > 0$, we get

$$\lim_{n\to\infty} \mu\left(F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, \frac{t}{3}\right) = 1$$

and

$$\lim_{n\to\infty} \nu\left(F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, \frac{t}{3}\right) = 0$$

for all $r \in \mathbb{R} \setminus \{0\}$. Fix $r_0 \in \mathbb{R}$. By the intuitionistic fuzzy continuity of the mapping $t \to \frac{f(2^n tx, 2^n ty)}{4^n}$, we have

$$\lim_{n\to\infty} \mu\left(\frac{f(2^n rx, 2^n ry)}{4^n} - \frac{f(2^n r_0 x, 2^n r_0 y)}{4^{n_0}}, \frac{t}{3}\right) = 1$$

and

$$\lim_{n\to\infty} \nu\left(\frac{f(2^n rx, 2^n ry)}{4^n} - \frac{f(2^n r_0 x, 2^n r_0 y)}{4^{n_0}}, \frac{t}{3}\right) = 0.$$

It follows that

$$\begin{aligned}
&\mu\big(F(rx, ry) - F(r_0 x, r_0 y), t\big) \\
&\geq \mu\left(F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, \frac{t}{3}\right) \\
&\quad * \mu\left(\frac{f(2^n rx, 2^n ry)}{4^n} - \frac{f(2^n r_0 x, 2^n r_0 y)}{4^n}, \frac{t}{3}\right) \\
&\quad * \mu\left(\frac{f(2^n r_0 x, 2^n r_0 y)}{4^n} - F(r_0 x, r_0 y), \frac{t}{3}\right) \\
&\geq 1
\end{aligned}$$

and

$$\nu\big(F(rx, ry) - F(r_0 x, r_0 y), t\big) \leq 0$$

as $r \to r_0$ for all $t > 0$. Hence the mapping $r \to F(rx, ry)$ is intuitionistic fuzzy continuous.

Now, we use the intuitionistic fuzzy continuity of the mapping $r \to F(rx, ry)$ to establish that $F(sx, sy) = s^2F(x, y)$ for all $s \in \mathbb{R}$. Fix $s \in \mathbb{R}$ and $t > 0$. Then, for each $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, there exists $\delta > 0$ such that

$$\mu\left(F(rx, ry) - F(sx, sy),\, \frac{t}{3}\right) \geq \alpha$$

and

$$\nu\left(F(rx, ry) - F(sx, sy),\, \frac{t}{3}\right) \leq 1 - \alpha.$$

Choose a rational number r with $0 < |r - s| < \delta$ and $0 < |r^2 - s^2| < 1 - \alpha$. Then

$$\begin{aligned}
&\mu\big(F(sx, sy) - s^2F(x, y),\, t\big)\\
&\geq \mu\left(F(sx, sy) - F(rx, ry),\, \frac{t}{3}\right) * \mu\left(F(rx, ry) - r^2F(x, y),\, \frac{t}{3}\right)\\
&\quad * \mu\left(r^2F(x, y) - s^2F(x, y),\, \frac{t}{3}\right)\\
&\geq \alpha * 1 * \mu\left(F(x, y),\, \frac{t}{3(1-\alpha)}\right)
\end{aligned}$$

and

$$\nu\left(F(sx, sy) - s^2F(x, y),\, t\right) \leq (1-\alpha)\diamondsuit 0 \diamondsuit \nu\left(F(x, y),\, \frac{t}{3(1-\alpha)}\right).$$

Letting $\alpha \to 1$ and using the definition of IFNS, we get

$$\mu\left(F(sx, sy) - s^2F(x, y),\, t\right) = 1 \quad \text{and} \quad \nu\left(F(sx, sy) - s^2F(x, y),\, t\right) = 0.$$

Hence we conclude that

$$F(sx, sy) = s^2F(x, y). \qquad \square$$

In the following theorem we prove a result similar to Theorem 3.2 for the case $p > 2$.

Theorem 3.3. *Let X be a normed space and (Y, μ, ν) an intuitionistic fuzzy Banach space. Let (Z, μ', ν') be an IFNS and let $p > 2$ and $z_0 \in Z$. Let $f : X \times X \to Y$ be a mapping such that* (3.1) *for all $x, y, z, w \in X$ and all $t > 0$. Assume that μ' and ν' satisfies*

$$\lim_{n\to\infty} \mu'\big(2^{np}(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0,\, 4^n t\big) = 1$$

and

$$\lim_{n\to\infty} \nu'\big(2^{np}(\|x\|^p + \|y\|^p + \|z\|^r + \|w\|^p)z_0,\, 4^n t\big) = 0$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique mapping $F : X \times X \to Y$ satisfying (2.1) *such that*

$$\begin{cases} \mu\big(F(x,y) - f(x,y), t\big) \geq \mu'\big(2(\|x\|^p + \|y\|^p)z_0,\ (2^p - 4)t\big), \\ \nu\big(F(x,y) - f(x,y), t\big) \leq \nu'\big(2(\|x\|^p + \|y\|^p)z_0,\ (2^p - 4)t\big) \end{cases} \tag{3.4}$$

for all $x, y \in X$ and all $t > 0$. Furthermore, if for some $x, y \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(r) = f(2^n rx, 2^n ry)$ is intuitionistic fuzzy continuous, then the mapping $r \to F(rx, ry)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous; *in this case, $F(rx, ry) = r^2 F(x, y)$ for all $r \in \mathbb{R}$.*

Proof. Define a mapping $\varphi : X \times X \times X \times X \to Z$ by $\varphi(x, y, z, w) = (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0$ for all $x, y, z, w \in X$. Then

$$\mu'\left(\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2}\right), t\right) = \mu'\left(\frac{1}{2^{p-1}}(\|x\|^p + \|y\|^p)z_0,\ t\right)$$

for all $x, y \in X$ and all $t > 0$. Since $p > 2$, we have $2^p > 4$. By Theorem 2.3, there exists a unique mapping F satisfying (2.1) and (3.4). The rest of the proof can be done on the same lines as in Theorem 3.2. □

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