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CONVERGENCE OF ISHIKAWA ITERATION WITH ERROR TERMS ON AN ARBITRARY INTERVAL

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Abstract. In this paper, a continuous real function on the real line is considered. The necessary and sufficient conditions for the convergence of the Ishikawa iteration with error terms for the functional are obtained.

1. Introduction and preliminaries

Iterative methods are popular tools to approximate fixed points of nonlinear mappings. Recall that the normal Mann's iteration was introduced by Mann [4] in 1953. Recently, construction of fixed points for nonlinear mappings via the normal Mann's iteration has been extensively investigated by many authors. Throughout this paper, we always assume that $\mathbb{R}$ denotes the real line. The normal Mann's iteration generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in \mathbb{R}, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n), \quad \forall n \geq 1, \tag{1.1}$$

where x_1 is an initial value, f is a real function and $\{\alpha_n\}$ is a sequence in $[0,1]$.

Next, we recall another popular iteration: Ishikawa iteration. Ishikawa iteration was introduced by Ishikawa [3] in 1974. Ishikawa iteration generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_1 \in \mathbb{R}, \\ y_n = (1-\beta_n)x_n + \beta_n f(x_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(y_n), \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where x_1 is an initial value, f is a real function and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.

In 1953, Mann proved that, if f is a continuous real function on a unit interval of a real line with a unique fixed point, then Mann iteration converges to the unique fixed point. In 1971, Franks and Marzed [2] removed the condition that f enjoys a unique fixed point. Subsequently, Rhoades [6] extended the result to Ishikawa Iteration, see also Borwein and Borwein [1]. To be more precise, he obtained the following result.

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Theorem R. *Let $f : [a, b] \to [a, b]$ be a continuous real function. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$x_1 \in [a, b], \quad x_{n+1} = (1 - t_n)x_n + t_n f(x_n) \quad \forall n \geq 1,$$

where t_n is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} t_n = \infty$ and $\lim_{n\to\infty} t_n = 0$. Then the sequence $\{x_n\}$ converges to a fixed point of f.

Recently, Qing and Qihou [5] further considered the problem and obtained a more general result on an arbitrary intervals. To be more precise, they obtained the following results.

Theorem QQ. *Let E be a closed interval on the real line* (*can be unbounded*) *and $f : E \to E$ a continuous function on E. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$\begin{cases} x_1 \in E, \\ y_n = (1 - \beta_n)x_n + \beta_n f(x_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f(y_n), \quad \forall n \geq 1, \end{cases}$$

If $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a fixed point of f.

In this paper, motivated by the above results, we continue to study the convergence problem of real functions. To be more precise, we shall consider the case that the real function f defined on the real line $\mathbb{R}$ by Ishikawa iteration with errors, which is efficient from the view of numerical computation.

In order to our main results, we still need the following lemma.

Lemma 1.1. *Let f be a continuous real function defined on $\mathbb{R}$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$\begin{cases} x_1 \in \mathbb{R}, \\ y_n = (1 - \beta_n)x_n + \beta_n f(x_n) + v_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f(y_n) + u_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$;

(b) $\sum_{n=1}^{\infty} |u_n| < \infty$ *and* $\lim_{n\to\infty} |v_n| = 0$.

If the sequence $\{x_n\}$ converges to some point p, then p is a fixed point of the real function f.

Proof. Suppose $f(p) \neq p$. Since $x_n \to p$ and f is continuous, we see that $f(x_n)$ is bounded. Note that

$$|y_n - p| \leq (1 - \beta_n)|x_n - p| + \beta_n|f(x_n) - p| + |v_n|.$$

From the conditions (a) and (b), we obtain that $y_n \to p$ as $n \to \infty$. Putting $r_n = f(y_n) - x_n + u_n$ for each $n \geq 1$, we see that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} (f(y_n) - x_n + u_n) = f(p) - p + 0 \neq 0.$$

Note that $x_{n+1} - x_n = \alpha_n (f(y_n) - x_n + u_n) + (1 - \alpha_n)u_n$. It follows that

$$x_n = \sum_{k=1}^{n-1} \{\alpha_k (f(y_k) - x_k + u_k) + (1 - \alpha_k)u_k\} + x_1 \leq \sum_{k=1}^{n-1} \alpha_k r_k + \sum_{k=1}^{n-1} u_k + x_1.$$

From the condition (a), we assert that the sequence $\{x_n\}$ is diverge. This derives a contradiction with $x_n \to p$. We, therefore, obtain that $f(p) = p$. This completes the proof. □

2. Main results

Now, we are in a position to prove our main results.

Theorem 2.1. *Let f be a continuous real function on $\mathbb{R}$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\text{(II)} \qquad \begin{cases} x_1 \in \mathbb{R}, \\ y_n = (1 - \beta_n)x_n + \beta_n f(x_n) + v_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f(y_n) + u_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$ and $\{v_n\}$ are real sequences satisfy the following restrictions.

(a) $0 \leq \alpha_n, \beta_n \leq 1$ *for each* $n \geq 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$;
(c) $\lim_{n\to\infty} \frac{|u_n|}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{|v_n|}{\beta_n} = 0$;
(d) $\sum_{n=1}^{\infty} |u_n| < \infty$ *and* $\sum_{n=1}^{\infty} |v_n| < \infty$.

If $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a fixed point of f.

Proof. Suppose that the sequence $\{x_n\}$ is not convergent. Let $a = \liminf_{n\to\infty} x_n$ and $b = \limsup_{n\to\infty} x_n$, respectively. It is obvious that $a < b$. Let m be a positive constant. If $a < m < b$, then we see that $f(m) = m$. Indeed, we have the following. Suppose that $f(m) \neq m$. we may, without loss of generality, suppose that $f(m) - m = \sigma > 0$. Since f is a continuous function, we see that there exists δ, where $0 < \delta < b - a$ such that

$$f(x) - x > \frac{\sigma}{2}, \quad \text{where } |x - m| < \delta. \tag{2.1}$$

Since $\{x_n\}$ is bounded and the real function f is continuous, we see that the sequence $\{f(x_n)\}$ is bounded. In view of the iteration (II), we see that the sequence $\{y_n\}$ is bounded, so is $\{f(y_n)\}$. Note that $x_{n+1} - x_n = \alpha_n (f(y_n) - x_n) + u_n$. It follows from the conditions (b) and (d) that

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0. \tag{2.2}$$

On the other hand, we have $y_n - x_n = \beta_n (f(x_n) - x_n) + v_n$. It follows from the conditions (b) and (d) that

$$\lim_{n\to\infty} (y_n - x_n) = 0. \tag{2.3}$$

In view of (2.2), (2.3) and the condition (c), we see that there exists N such that

$$|x_{n+1} - x_n| < \frac{\delta}{2},\ |y_n - x_n| < \frac{\delta}{2},\ |u_n| < \frac{\sigma}{2}\alpha_n \text{ and } |v_n| < \frac{\sigma}{2}\beta_n \tag{2.4}$$

for all $n > N$. From the assumption that $m < b = \limsup_{n\to\infty} x_n$, we see that there exists a constant k, where $k > N$ such that $x_k > m$. For the fixed k, we have the following two cases.

Case (1). $x_k > m + \frac{\delta}{2}$.

From (2.4), we see hat $x_{k+1} > x_k - \frac{\delta}{2} > m$. That is, $x_{k+1} > m$.

Case (2). $m < x_k < m + \frac{\delta}{2}$.

In view of (2.4), we see that $x_k - \frac{\delta}{2} < y_k < x_k + \frac{\delta}{2}$. This implies that $m - \frac{\delta}{2} < y_k < m + \delta$. It follows that $|y_k - m| < \delta$ and $|x_k - m| < \frac{\delta}{2} < \delta$. Thanks to (2.1), we see that

$$f(x_k) - x_k > \frac{\sigma}{2} \quad \text{and} \quad f(y_k) - y_k > \frac{\sigma}{2}. \tag{2.5}$$

Note that

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k f(y_k) + u_k = x_k + \alpha_k\left(f(y_k) - y_k\right) + \alpha_k(y_k - x_k) + u_k$$

and $y_k - x_k = \beta_k\left(f(x_k) - x_k\right) + v_k$. It follows that

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k\left(f(y_k) - y_k\right) + \alpha_k\left(\beta_k\left(f(x_k) - x_k\right) + v_k\right) + u_k \\ &> x_k + \frac{\sigma}{2}\alpha_k + \alpha_k\left(\frac{\sigma}{2}\beta_k + v_k\right) + u_k > x_k. \end{aligned}$$

Both Case (1) and Case (2) imply that $x_{k+1} > m$. In a similar way, we can obtain that $x_n > m$ for all $n > k$. Note that $\liminf_{n\to\infty} x_n = a < m$, which derives a contradiction. This proves that $f(m) = m$.

On the other hand, from the condition (d) and (2.3), we see that there exists N^* such that

$$|u_n| < \frac{b-a}{10},\ |v_n| < \frac{b-a}{10} \text{ and } |y_n - x_n| < \frac{b-a}{5} \tag{2.6}$$

for all $n > N^*$. For all $n > N^*$, we have the following two cases.

Case (a). There at least exists an $M > N^*$ such that

$$a < \frac{3a+2b}{5} < x_M < \frac{2a+3b}{5} < b. \tag{2.7}$$

Case (b). For all $n > N^*$,

$$x_n \le \frac{3a+2b}{5} \quad \text{or} \quad x_n \le \frac{3a+2b}{5}.$$

Next, we prove both Case (a) and Case (b) lead to contradictions.

Suppose that Case (a) holds. From (2.7), we see that $f(x_M) = x_M$. On the other hand, we have

$$\frac{4a+b}{5} < x_n < \frac{a+4b}{5}, \quad n \ge M+1. \tag{2.8}$$

Indeed, we have the following observation. From (2.6) and (2.7), we obtain that

$$a < \frac{4a+b}{5} < y_M < \frac{a+4b}{5} < b.$$

It follows that $f(y_M) = y_M$. Note that

$$\begin{aligned} x_{M+1} &= x_M + \alpha_M \left(f(y_M) - y_M\right) + \alpha_M \left(\beta_M \left(f(x_M) - x_M\right) + v_M\right) + u_M \\ &= x_M + \alpha_M \left(f(y_M) - y_M\right) + \alpha_M \beta_M \left(f(x_M) - x_M\right) + \alpha_M v_M + u_M \\ &= x_M + \alpha_M v_M + u_M. \end{aligned}$$

It follows from (2.6) that $|x_{M+1} - x_M| = |u_M + \alpha_M v_M| < \frac{b-a}{5}$. In view of (2.7), we arrive at $\frac{4a+b}{5} < x_{M+1} < \frac{a+4b}{5}$, which is the start of our mathematical induction.

Suppose that $\frac{4a+b}{5} < x_m < \frac{a+4b}{5}$ for some m. It follows from (2.6) that $a < y_m < b$. This implies that $f(x_m) = x_m$ and $f(y_m) = y_m$. It follows that

$$\begin{aligned} x_{m+1} &= x_m + \alpha_m(f(y_m) - y_m) + u_m + \alpha_m(\beta_m(f(x_m) - x_m) + v_m) \\ &= x_m + u_m + \alpha_m v_m, \end{aligned}$$

which yields that $|x_{m+1} - x_M| = |u_m| + |v_m| < \frac{b-a}{5}$. This is, $\frac{4a+b}{5} < x_{m+1} < \frac{a+4b}{5}$. This shows that (2.8) holds. It follows that $a < \liminf_{n\to\infty} x_n = a$, which is a contradiction. That is, Case (a) leads to a contradiction.

Suppose that Case (b) holds. But from (2.2), we see that there exits $\overline{N}$ such that

$$x_n \le \frac{3a+2b}{5} \qquad \forall n \ge \overline{N}, \quad \text{or} \quad x_n \ge \frac{2a+3b}{5}, \quad \forall n \ge \overline{N}.$$

If $x_n \le \frac{3a+2b}{5}$ for each $n \ge \overline{N}$, then we see that $b = \limsup_{n\to\infty} x_n \le \frac{3a+2b}{5} < b$, which is a contradiction.

If $x_n \ge \frac{2a+3b}{5}$ for each $n \ge \overline{N}$, then we see that $a = \liminf_{n\to\infty} x_n \ge \frac{2a+3b}{5} > a$, which is a contradiction. That is, Case (b) also leads to a contradiction.

Combining Case (a) with Case (b), we see that the assumption that the sequence $\{x_n\}$ is not convergent is not true. It follows that the sequence $\{x_n\}$ convergence to some point, say p. In view of Lemma 1.1, we can conclude the desired conclusion easily. This completes the proof. □

Theorem 2.2. *Let f be a continuous real function on $\mathbb{R}$. Let $\{x_n\}$ be a sequence generated by the iterative algorithm* (II), *where $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$ and $\{v_n\}$ are real sequences satisfy the following restrictions.*

(1) $0 \le \alpha_n, \beta_n \le 1$ *for each* $n \ge 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$;
(3) $\lim_{n\to\infty} \frac{|u_n|}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{|v_n|}{\beta_n} = 0$;
(4) $\sum_{n=1}^{\infty} |u_n| < \infty$ *and* $\sum_{n=1}^{\infty} |v_n| < \infty$.

Then $\{x_n\}$ converges to a fixed point of f if and only if the sequence $\{x_n\}$ is bounded.

Proof. If the sequence $\{x_n\}$ converges to a fixed point of f, then we can see that the sequence $\{x_n\}$ is bounded easily. If the sequence $\{x_n\}$ is bounded, then we can obtain that the sequence $\{x_n\}$ converges to a fixed point of f easily from Theorem 2.1. This completes the proof. □

Remark 2.3. Let $f : E \to E$, where E stands for $(-\infty, a]$, $[a, b]$ and $[b, +\infty)$, respectively, be a continuous real function. Define a function $\overline{f} : \mathbb{R} \to \mathbb{R}$ as follows:

If $E = (-\infty, a]$, then

$$\overline{f(x)} := \begin{cases} f(x), & x \le a, \\ f(a), & x > a. \end{cases}$$

If $E = [a, b]$, then

$$\overline{f(x)} := \begin{cases} f(a), & x < a, \\ f(x), & a \le x \le b, \\ f(b), & x > b. \end{cases}$$

If $E = [b, +\infty)$, then

$$\overline{f(x)} := \begin{cases} f(b), & x \le b, \\ f(x), & x \ge b. \end{cases}$$

It is clear that $\overline{f}$ is continuous, the range of $\overline{f}$ is E and $\overline{f(x)} = f(x)$ for all $x \in E$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_1 \in \mathbb{R}, \\ y_n = (1 - \beta_n)x_n + \beta_n \overline{f(x_n)} + v_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \overline{f(y_n)} + u_n, \quad \forall n \ge 1, \end{cases} \tag{2.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$ and $\{v_n\}$ are real sequences satisfy the following restrictions.

(1) $0 \le \alpha_n, \beta_n \le 1$ for each $n \ge 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$;
(3) $\lim_{n\to\infty} \frac{|u_n|}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{|v_n|}{\beta_n} = 0$;
(4) $\sum_{n=1}^{\infty} |u_n| < \infty$ and $\sum_{n=1}^{\infty} |v_n| < \infty$.

According to Theorem 2.1, we see that $\{x_n\}$ converges to a fixed point of $\overline{f(x)}$ under the assumption that the sequence $\{x_n\}$ is bounded. Suppose that $\overline{f(x_0)} = x_0$, from which it follows that $x_0 \in E$. Since the range of $\overline{f(x)}$ is E, we obtain that $\overline{f(x_0)} = f(x_0) = x_0$, that is, $\{x_n\}$ converges to a fixed point x_0 of $f(x)$.

Remark 2.4. Let $f : E \to E$, where E stands for $(-\infty, a]$, $[a, b]$ and $[b, +\infty)$, respectively, be a continuous real function. Define a function $\overline{f} : R \to R$ as in Remark 2.3. Let $\{x_n\}$ be a sequence generated by (2.9), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$ and $\{v_n\}$ are real sequences satisfy the following restrictions.

(1) $0 \leq \alpha_n, \beta_n \leq 1$ for each $n \geq 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$;
(3) $\lim_{n\to\infty} \frac{|u_n|}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{|v_n|}{\beta_n} = 0$;
(4) $\sum_{n=1}^{\infty} |u_n| < \infty$ and $\sum_{n=1}^{\infty} |v_n| < \infty$.

According to Theorem 2.2, we see that the sequence $\{x_n\}$ converges to a fixed point of f if and only if the sequence $\{x_n\}$ is bounded.

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