Commun. Korean Math. Soc. **26** (2011), No. 2, pp. 215–227
DOI 10.4134/CKMS.2011.26.2.215

WEAKLY SUFFICIENT SETS FOR WEIGHTED SPACES $h_\Phi^{-\infty}(\mathbb{B})$

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Abstract. In this paper we introduce a class $h_\Phi^{-\infty}(\mathbb{B})$ of weighted spaces of harmonic functions in the unit ball $\mathbb{B}$ of $\mathbb{R}^n$. We define weakly sufficient sets in this space and give an explicit construction of countable sets of such a type. Various examples of weight functions are also discussed.

1. Introduction

Let $\mathcal{H}(\mathbb{B})$ be the space of harmonic functions in the unit ball $\mathbb{B} = \{x \in \mathbb{R}^n; |x| < 1\}$. To a function $\varphi\colon [0,1) \to (1,+\infty)$, called weight function (or *weight*), we associate the following normed space

$$h_\varphi = \left\{ f \in \mathcal{H}(\mathbb{B}); \|f\|_\varphi := \sup_{x\in\mathbb{B}} \frac{|f(x)|}{\varphi(|x|)} < +\infty \right\}.$$

Let $\Phi = \big(\varphi_p\big)_{p=1}^\infty$ denote the increasing sequence of weights. For simplicity, we use $\|f\|_p$ instead of $\|f\|_{\varphi_p}$. We also replace h_{φ_p} by h_φ^{-p}. Then $h_\varphi^{-p} \subset h_\varphi^{-(p+1)}$, and let

$$h_\Phi^{-\infty} := \bigcup_{p\geq 1} h_\varphi^{-p}.$$

This weighted space can be endowed with some topological structure, namely the *inner inductive limit* of h_φ^{-p}:

$$(h_\Phi^{-\infty}, \tau) = \liminf h_\varphi^{-p}.$$

In the case $\varphi_p(r) = (1-r)^{-p}$, $h_\Phi^{-\infty}$ is the well-known function space $h^{-\infty}(\mathbb{B})$ whose members have the polynomial growth condition at the boundary. The space $h^{-\infty}(\mathbb{B})$ (and for more general domains in $\mathbb{R}^n$) has been considered rather extensively (see, e.g., [3], [12], [13],..., and references therein).

Received July 7, 2009; Revised January 7, 2011.
2010 *Mathematics Subject Classification.* 31B05, 46E10.
Key words and phrases. weighted space, harmonic function, inductive limit, weakly sufficient set.

Now let S be a subset of $\mathbb{B}$. Define

$$h_\varphi^{-p,S} := \left\{ f \in h_\Phi^{-\infty}; \|f\|_{p,S} := \sup_{x \in S} \frac{|f(x)|}{\varphi_p(|x|)} < +\infty \right\}.$$

Notice that the inclusion relations $h_\varphi^{-p} \subset h_\varphi^{-p,S} \subset h_\Phi^{-\infty}$ always hold. Hence, it follows immediately that

$$h_\Phi^{-\infty} = \bigcup_{p \geq 1} h_\varphi^{-p} \subset \bigcup_{p \geq 1} h_\varphi^{-p,S} \subset h_\Phi^{-\infty}$$

and consequently that

$$h_\Phi^{-\infty} = \bigcup_{p \geq 1} h_\varphi^{-p} = \bigcup_{p \geq 1} h_\varphi^{-p,S}.$$

Therefore, we can endow $h_\Phi^{-\infty}$ with another weaker inner inductive limit topology of semi-normed spaces $h_\varphi^{-p,S}$:

$$(h_\Phi^{-\infty}, \tau_S) = \lim \operatorname{ind} h_\varphi^{-p,S}.$$

Definition 1.1. A set $S \subseteq \mathbb{B}$ is said to be *weakly sufficient* for the space $h_\Phi^{-\infty}$ if two topologies τ and τ_S are equivalent.

Notice that always $\|f\|_{p,S} \leq \|f\|_p$. In case S is much smaller than $\mathbb{B}$ in some sense, $\|f\|_{p,S}$ will be considerably smaller than $\|f\|_p$ in general. Therefore it is surprising that there is a discrete, even countable, subset S that is weakly sufficient, for instance.

It should be noted that, for holomorphic functions, the existence of discrete weakly sufficient sets related closely to the representation of functions by series (see, e.g., [5], [9], [11], [14],..., and references therein).

The main target of this article is a presentation of an explicit method for the construction of countable, weakly sufficient sets for $h_\Phi^{-\infty}$ (for every $n \geq 1$). The construction exploits a method introduced in [7] (for $n = 1$ see also [8]).

2. Preliminaries

In this section we present the regularity property of the space $(h_\Phi^{-\infty}, \tau)$ that allows to get some equivalent conditions for weak sufficiency of the set in this space. The last result is essentially needed in the sequel. It should be noted that results in this section in fact can be obtained by applying [1], where a more general situation has been considered. Nevertheless, we present it here for completeness of exposition as well as for reader's convenience.

Lemma 2.1. *The space $(h_\Phi^{-\infty}, \tau)$ is regular, i.e., each bounded subset of $(h_\Phi^{-\infty}, \tau)$ is contained and bounded in some space h_φ^{-p}.*

Proof. By [10], it suffices to show that the unit ball $U_p = \{f \in h_\varphi^{-p}; \|f\|_p \leq 1\}$ of h_φ^{-p} is closed in $(h_\Phi^{-\infty}, \tau)$. Indeed, let $(f_k) \subset U_p$ converge in $(h_\Phi^{-\infty}, \tau)$ to some element f_0. Then $|f_k(x)| \to |f_0(x)|, \ \forall x \in \mathbb{B}$, and therefore,

$$\frac{|f_k(x)|}{\varphi_p(|x|)} \to \frac{|f_0(x)|}{\varphi_p(|x|)}, \ \forall x \in \mathbb{B}.$$

Since $\frac{|f_k(x)|}{\varphi_p(|x|)} \leq \|f_k\|_p \leq 1, \ \forall x \in \mathbb{B}$, it follows that $\frac{|f_0(x)|}{\varphi_p(|x|)} \leq 1, \ \forall x \in \mathbb{B}$. So $\|f_0\|_p \leq 1$ and $f_0 \in U_p$. □

Thanks to the regularity of $(h_\Phi^{-\infty}, \tau)$, we can have some simplifications of weak sufficiency.

Proposition 2.2. *For the space $h_\Phi^{-\infty}(\mathbb{B})$ the following statements are equivalent*:

(a) *S is weakly sufficient set, that is, $(h_\Phi^{-\infty}, \tau_S) = (h_\Phi^{-\infty}, \tau)$.*

(b)

$$\forall p \ \exists m = m(p) : \ h_\varphi^{-p,S} \hookrightarrow h_\varphi^{-m},$$

that is,

$$\forall p \ \exists m = m(p) \ \exists C = C(p) > 0 : \|f\|_m \leq C\|f\|_{p,S}, \ \forall f \in h_\varphi^{-p,S} \tag{2.1}$$

(*Here the sign $\hookrightarrow$ denotes the continuous embedding*).

(c)

$$\forall p \ \exists m = m(p) \ \exists C = C(p) > 0 : \ \|f\|_m \leq C\|f\|_{p,S}, \ \forall f \in h_\Phi^{-\infty}. \tag{2.2}$$

Proof. (a) $\Rightarrow$ (b): The regularity of $h_\Phi^{-\infty}$ is used in this part.

As the unit ball $E_{p,S}$ of $h_\varphi^{-p,S}$ is bounded in $h_\varphi^{-p,S}$, and $h_\varphi^{-p,S} \hookrightarrow (h_\Phi^{-\infty}, \tau_S)$, it is bounded in $(h_\Phi^{-\infty}, \tau_S)$. Since S is a weakly sufficient set for $h_\Phi^{-\infty}$, i.e., $\tau = \tau_S$, and $(h_\Phi^{-\infty}, \tau)$, by Lemma 2.1, is regular, $E_{p,S}$ is contained and bounded in some h_φ^{-m}, which means that the embedding from $h_\varphi^{-p,S}$ into h_φ^{-m} is continuous. Hence, $h_\varphi^{-p,S} \hookrightarrow h_\varphi^{-m}$.

(b) $\Rightarrow$ (a): It is straightforward. Since $h_\varphi^{-p,S} \hookrightarrow h_\varphi^{-m}$, and $h_\varphi^{-m} \hookrightarrow (h_\Phi^{-\infty}, \tau)$, $h_\varphi^{-p,S} \hookrightarrow (h_\Phi^{-\infty}, \tau), \ \forall p > 0$. Hence, $(h_\Phi^{-\infty}, \tau_S) = \lim\operatorname{ind} h_\varphi^{-p,S} \hookrightarrow (h_\Phi^{-\infty}, \tau)$, which shows that $\tau_S \geq \tau$, while always $\tau \geq \tau_S$.

(b) $\Leftrightarrow$ (c): The implication (c) $\Rightarrow$ (b) is trivial because $h_\varphi^{-p,S} \subset h_\Phi^{-\infty}$. Now, suppose that (b) holds and consider an arbitrary element $f \in h_\Phi^{-\infty}$. If $f \in h_\varphi^{-p,S}$, then (2.2) holds, due to (2.1). If $f \notin h_\varphi^{-p,S}$, then (2.2) is also valid, as in this case the right-hand side of (2.2) is ∞. □

3. Non-discrete weakly sufficient sets

In this section we show that under some conditions imposed on weight functions the union of concentric spheres inside the ball forms a (non-discrete) weakly sufficient set for the space $h_\Phi^{-\infty}(\mathbb{B})$. The method of "extracting" from this union a discrete weakly sufficient set will be treated in the next section.

We have the following, so-called "toy-problem", result.

Proposition 3.1. *Suppose that a sequence* $0 < (r_k) \uparrow 1$ *and weight functions* (φ_p) *satisfy the following conditions*:

$$\forall p : \varphi_p(r) \uparrow \text{ with respect to } r, \tag{3.1}$$

$$\forall p \ \exists m \ \exists C > 0 \ \exists k_0 : \varphi_p(r_{k+1}) \leq C\varphi_m(r_k), \ \forall k \geq k_0. \tag{3.2}$$

The concentric set

$$\mathcal{S} = \bigcup_{k=1}^{\infty} \{x \in \mathbb{B} : |x| = r_k\},$$

is weakly sufficient for the space $h_{\Phi}^{-\infty}(\mathbb{B})$.

Proof. First we show that S is a set of uniqueness for the space $h_{\Phi}^{-\infty}(\mathbb{B})$, that is, $f \in h_{\Phi}^{-\infty}(\mathbb{B})$ and $f(x) = 0$ for all $x \in S$ imply that $f = 0$.[1]

Indeed, choose any point x in the ball and take k large enough, so that x belongs to the ball $\mathbb{B}_k = \{|x| < r_k\}$ whose boundary is $\mathcal{S}_k$. Since f vanishes on $\mathcal{S}_k$, it vanishes in all of $\mathbb{B}_k$, for the Dirichlet problem in $\mathbb{B}_k$ has a unique solution. Hence f vanishes at x.

Next we prove the following set inclusion: any $h_{\varphi}^{-p,S}$ is contained in a certain h_{φ}^{-m}. Indeed, given p, let m and k_0 be as in the condition (3.2). For any $f \in h_{\varphi}^{-p,S}$, if $|x| < r_{k_0}$, then

$$\frac{|f(x)|}{\varphi_m(|x|)} \leq |f(x)| \leq \sup_{|x| \leq r_{k_0}} |f(x)|.$$

Furthermore, if $|x| \geq r_{k_0}$, let k be the unique integer such that $r_k < |x| \leq r_{k+1}$. Then, by conditions (3.1) and (3.2), we have

$$\begin{aligned} \frac{|f(x)|}{\varphi_m(|x|)} &\leq \frac{\sup_{|x|=r_{k+1}} |f(x)|}{\varphi_m(|x|)} \leq \frac{\sup_{|x|=r_{k+1}} |f(x)|}{\varphi_m(r_k)} \\ &\leq C\frac{\sup_{|x|=r_{k+1}} |f(x)|}{\varphi_p(r_{k+1})} \leq C||f||_{p,S}. \end{aligned}$$

Combining the two estimates yields that $f \in h_{\varphi}^{-m}$.

Thus the set S said above is a set of uniqueness for the space $h_{\Phi}^{-\infty}(\mathbb{B})$, and the so-called set-inclusion property holds: $\forall p \geq 1 \ \exists m \geq 1 : h_{\varphi}^{-p,S} \subset h_{\varphi}^{-m}$. By [1], S is a weakly sufficient set for the space $h_{\Phi}^{-\infty}(\mathbb{B})$. □

We give some examples of weight functions that satisfy all above-mentioned conditions. It is easily verified that the standard weights (said in Introduction)

$$\varphi_p(r) = (1-r)^{-p}, \ 0 < r < 1, \ p = 1, 2, \ldots,$$

with $r_k = \frac{1}{k+1}, \ k \in \mathbb{N}$, work well.

[1] I owe Nikolai Tarkhanov for this.

4. An explicit construction of weakly sufficient sequences

First we note, by Proposition 2.2(c), that a set $S \subset \mathbb{B}$ will be weakly sufficient for the space $h_\Phi^{-\infty}$ if and only if

$$\forall p\ \exists m = m(p)\ \exists C = C(p) > 0,$$
$$\sup_{x\in\mathbb{B}} \frac{|f(x)|}{\varphi_m(|x|)} \le C \sup_{x\in S} \frac{|f(x)|}{\varphi_p(|x|)},\ \forall f \in h_\Phi^{-\infty}.$$

This fact for a set to be weakly sufficient in the space $h_\Phi^{-\infty}$ will be used in our construction of a desired sequence (λ_k) in the next section.

Bellow we provide some properties of functions from the space $h_\Phi^{-\infty}$.

For $0 < r < 1$ denote by $\mathbb{B}_r$ the set $\{x \in \mathbb{R}^n : |x| < r\}$ and by $\mathcal{S}_r$ its boundary. Also let $\mathcal{M}_f(r) = \sup_{x\in\mathcal{S}_r} |f(x)|$. Note that by the maximum and minimum principles for harmonic functions, we have if $r < s$ that $\mathcal{M}_f(r) \le \mathcal{M}_f(s)$.

The following result, inspired by [7], is needed in the sequel.[2]

Proposition 4.1. *For any numbers $0 < a < b < 1$ and any points $x, y \in \overline{\mathbb{B}}_a$, if $f \in \mathcal{H}(\mathbb{B})$, then*

$$|f(x) - f(y)| \le \frac{2n|x-y|}{b-a}\mathcal{M}_f(b).$$

Proof. We will make use of a corollary to Harnack's inequalities [2, Cor. 1.4.2], stating that if a function h is nonnegative and harmonic on a ball $B(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$, then the gradient ∇h satisfies

$$|\nabla h(x_0)| \le \frac{nh(x_0)}{r}. \tag{4.1}$$

Now, suppose $f \in \mathcal{H}(\mathbb{B})$ and $0 < a < b < 1$. Then, by the minimum principle, f is bounded below by $-\mathcal{M}_f(b)$ on $\mathbb{B}_b$, so $\hat{f} = f + \mathcal{M}_f(b)$ is nonnegative on $\mathbb{B}_b$. Taking $x, y \in \overline{\mathbb{B}}_a$ and letting ℓ be the line segment from x to y, we have for all $t \in \ell$ that $\hat{f}$ is harmonic and nonnegative on $B(t, b-a)$. Hence, using the inequality (4.1), we get

$$|\nabla f(t)| = |\nabla \hat{f}(t)| \le \frac{n}{b-a}\hat{f}(t) \le \frac{n}{b-a} \cdot 2\mathcal{M}_f(b).$$

The result then follows since by calculus,

$$|f(y) - f(x)| = \left| \int_\ell \nabla f \cdot dr \right| \le \sup_{t\in\ell} |\nabla f(t)| \cdot \text{length}(\ell).$$ □

The following result plays an important role in our discussion.

Proposition 4.2. *Let weight functions φ_p satisfy the following condition*

(C1) $\exists\ \psi(r) > 0$ *on* $(0, 1)$, $\psi(r) \to \infty$ *as* $r \to 1$, *and* $\liminf_{r\to 1} \frac{\log \varphi_p(r)}{\psi(r)} = p.$

[2] I owe Anders Gustavsson for this.

Consider a sequence $0 < (r_k) \uparrow 1$. *For each natural number* $\ell \geq 1$, *if* $f \in h_\varphi^{-p}$, *then* $\forall q > p$

$$\liminf_{k\to\infty} \left(\frac{\mathcal{M}_f(r_{k+\ell})}{\varphi_q(r_{k+\ell})} \Big/ \frac{\mathcal{M}_f(r_k)}{\varphi_q(r_k)} \right) \leq 1. \tag{4.2}$$

We note that in fact, as one can see in the proof of Theorem 4.4 below, only the case $\ell = 1$ is needed for our purpose.

Also notice that condition (C1) implies condition (3.2).

Proof. Take and fix an arbitrary natural number $\ell \geq 1$.

As $f \in h_\varphi^{-p}$, there is a constant $C > 0$ such that

$$\log \mathcal{M}_f(r_{k\ell}) \leq \log C + \log \varphi_p(r_{k\ell}), \ \forall k \geq 1,$$

or, equivalently that

$$\frac{\log \mathcal{M}_f(r_{k\ell})}{\psi(r_{k\ell})} \leq \frac{\log C}{\psi(r_{k\ell})} + \frac{\log \varphi_p(r_{k\ell})}{\psi(r_{k\ell})}.$$

From this it follows, by condition (C1), that

$$\liminf_{k\to\infty} \frac{\log \mathcal{M}_f(r_{k\ell})}{\psi(r_{k\ell})} \leq p. \tag{4.3}$$

Suppose that the inequality (4.2) is false. Then there exists $q_0 > p$ such that for all k large enough, say $k \geq k_0\ell$, we have

$$\frac{\mathcal{M}_f(r_{k+\ell})}{\varphi_{q_0}(r_{k+\ell})} > \frac{\mathcal{M}_f(r_k)}{\varphi_{q_0}(r_k)}.$$

This means that for all $k \geq 1$

$$\frac{\mathcal{M}_f(r_{(k_0+k)\ell})}{\varphi_{q_0}(r_{(k_0+k)\ell})} > \frac{\mathcal{M}_f(r_{[k_0+(k-1)]\ell})}{\varphi_{q_0}(r_{[k_0+(k-1)]\ell})} > \cdots > \frac{\mathcal{M}_f(r_{k_0\ell})}{\varphi_{q_0}(r_{k_0\ell})}.$$

This in turn implies that

$$\log \mathcal{M}_f(r_{(k_0+k)\ell}) - \log \varphi_{q_0}(r_{(k_0+k)\ell}) > \log \mathcal{M}_f(r_{k_0\ell}) - \log \varphi_{q_0}(r_{k_0\ell}).$$

Hence,

$$\frac{\log \mathcal{M}_f(r_{(k_0+k)\ell})}{\psi(r_{(k_0+k)\ell})} > \frac{\log \varphi_{q_0}(r_{(k_0+k)\ell})}{\psi(r_{(k_0+k)\ell})} + \frac{\log \mathcal{M}_f(r_{k_0\ell})}{\psi(r_{(k_0+k)\ell})} - \frac{\log \varphi_{q_0}(r_{k_0\ell})}{\psi(r_{(k_0+k)\ell})}, \ \forall k \geq 1.$$

Consequently, again by condition (C1), we obtain

$$\liminf_{k\to\infty} \frac{\log \mathcal{M}_f(r_{k\ell})}{\psi(r_{k\ell})} \geq q_0 > p.$$

This contradicts (4.3), and completes our proof of the proposition. □

We now proceed to the main result of this paper.

Suppose that the weight functions satisfy, besides condition (C1), the following additional properties:

(C2) $\forall p : \varphi_p$ is an increasing function on $(0, 1)$.

(C3) $\forall p : \sup_{k\geq 1} \dfrac{\varphi_p(r_{k+1})}{\varphi_p(r_k)} = R_p < +\infty.$

We present the following explicit construction in the form of an algorithm. This construction exploits a method in [8] for one variable and developed in [7] for several variables. It should be noted that a similar kind of construction was also presented in [4], [6].

Algorithm 4.3. *Let us construct a sequence* $S = \big(\lambda_k\big)_{k=1}^{\infty} \subset \mathbb{B}$ *in the following manner*:

Step 1. *Take a sequence of natural numbers* $(s_k) \uparrow \infty$ *so that*

$$\lim_{k\to\infty} \frac{1}{s_k(r_{k+1} - r_k)} = 0. \tag{4.4}$$

This can always be done, for example, if $0 < (\rho_k) \uparrow \infty$, then we can choose

$$s_k = \text{the intergal part of } \Big(\frac{\rho_k}{r_{k+1} - r_k}\Big),\ k = 1, 2, \dots .$$

Step 2. *Take and fix some natural number* $N_0 \geq 1$ *and on each sphere* $\mathcal{S}_k = \{x \in \mathbb{R}^n; |x| = r_k\}$ *with* $k = N_0, N_0 + 1, \dots$, *mark* ℓ_k *points* $x_{k,j}$ $(j = 1, 2, \dots, \ell_k)$, *forming an* $1/s_k$*-net of* $\mathcal{S}_k$.

Step 3. *We re-numerate the obtained system of points* $\{x_{k,j}; 1 \leq j \leq \ell_k, k \geq N_0\}$ *under one sequence, denoted by* $S = \big(\lambda_k\big)_{k=1}^{\infty}$, *writing first all the points with* $k = N_0$, *and then with* $k = N_0 + 1$, *etc.*

The meaning of a choice of the number N_0 just is that the elements (λ_k) can be chosen arbitrarily far away from the origin of coordinates, being in the unit ball $\mathbb{B}$.

Theorem 4.4. *The sequence* $S = \big(\lambda_k\big)_{k=1}^{\infty}$ *constructed in Algorithm 4.3 above is weakly sufficient for the space* $h_\Phi^{-\infty}(\mathbb{B})$.

Proof. We notice that a set $S \subset \mathbb{B}$ will be weakly sufficient for the space $h_\Phi^{-\infty}$ if and only if

$$\forall p\ \exists m = m(p)\ \exists C = C(p) > 0,$$

$$\sup_{x\in\mathbb{B}} \frac{|f(x)|}{\varphi_m(|x|)} \leq C \sup_{x\in S} \frac{|f(x)|}{\varphi_p(|x|)},\ \text{i.e.,}\ \big(h_\varphi^{-p,S}, \|\ \|_{p,S}\big) \hookrightarrow \big(h_\varphi^{-m}, \|\ \|_m\big),\ \forall f \in h_\varphi^{-p,S}.$$

This fact for a set to be weakly sufficient in the space $h_\Phi^{-\infty}$ will be used in our construction of a desired sequence (λ_k) in this theorem.

Our goal is to show that for any p given, we can choose the corresponding m as $p + 1$. In other words, we will show that $\|f\|_p$ is bounded by a constant times $\|f\|_{p-1,S}$.

Using (4.4), given p, we can find a natural number N_p, such that

$$\frac{2nR_{p-1}}{s_k(r_{k+1} - r_k)} < \frac{1}{4}, \quad \text{whenever } k \geq N_p. \tag{4.5}$$

Let f be an arbitrary element in $h_{\varphi}^{-(p-1),S}$. We notice that

$$\|f\|_p = \sup_{0<r<1} \frac{\mathcal{M}_f(r)}{\varphi_p(r)}.$$

We estimate this quantity.

Take and fix a natural number $N_0 > N_p$. On the one hand, for $r \in (0, r_{N_0}]$ we have:

$$\frac{\mathcal{M}_f(r)}{\varphi_p(r)} \leq \frac{\mathcal{M}_f(r_{N_0})}{\varphi_p(r)} \leq \mathcal{M}_f(r_{N_0}) \leq \frac{\mathcal{M}_f(r_{N_0})}{\varphi_{p-1}(r_{N_0})} \cdot \varphi_p(r_{N_0}).$$

On the other hand, for $r \in (r_k, r_{k+1}]$ $(k \geq N_0)$ we also have:

$$\begin{aligned} \frac{\mathcal{M}_f(r)}{\varphi_p(r)} &\leq \frac{\mathcal{M}_f(r_{k+1})}{\varphi_p(r)} \leq \frac{\mathcal{M}_f(r_{k+1})}{\varphi_p(r_k)} \text{ by (C2)} \\ &\leq \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \cdot \frac{\varphi_p(r_{k+1})}{\varphi_p(r_k)} \leq R_p \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \text{ by (C3).} \end{aligned}$$

From these estimates, putting $C_p = \max\{R_p; \varphi_p(r_{N_0})\}$, we obtain for all $N \geq N_0$:

$$\begin{aligned} \sup_{0<r<r_N} \frac{\mathcal{M}_f(r)}{\varphi_p(r)} &= \max\left\{ \sup_{0<r\leq r_{N_0}} \frac{\mathcal{M}_f(r)}{\varphi_p(r)}; \sup_{r_{N_0}<r\leq r_N} \frac{\mathcal{M}_f(r)}{\varphi_p(r)} \right\} \\ &\leq C_p \max\left\{ \frac{\mathcal{M}_f(r_{N_0})}{\varphi_{p-1}(r_{N_0})}; \sup_{N_0\leq k\leq N-1} \frac{M_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \right\} \\ &= C_p \sup_{N_0\leq k\leq N} \frac{\mathcal{M}_f(r_k)}{\varphi_{p-1}(r_k)}. \end{aligned}$$

Let $y_k \in \mathcal{S}_k$ with $|f(y_k)| = \mathcal{M}_f(r_k)$. Then by the construction of points $x_{k,j}$ $(j = 1, \ldots, \ell_k)$, there exists $x_{k,j_0} \in \mathcal{S}_k$ such that $|y_k - x_{k,j_0}| < \frac{1}{s_k}$.

By Proposition 4.1 we have

$$\begin{aligned} M_f(r_k) - |f(x_{k,j_0})| &= |f(y_k)| - |f(x_{k,j_0})| \leq |f(y_k) - f(x_{k,j_0})| \\ &\leq \frac{2n|y_k - x_{k,j_0}|}{r_{k+1} - r_k} \mathcal{M}_f(r_{k+1}) \leq \frac{2n\mathcal{M}_f(r_{k+1})}{s_k(r_{k+1} - r_k)}. \end{aligned}$$

Hence,

$$\frac{\mathcal{M}_f(r_k)}{\varphi_{p-1}(r_k)} \leq \frac{|f(x_{k,j_0})|}{\varphi_{p-1}(r_k)} + \frac{2n}{s_k(r_{k+1} - r_k)} \cdot \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_k)}.$$

Consequently,

$$\begin{aligned} \mathcal{Y}_N &:= \sup_{N_0\leq k\leq N} \frac{\mathcal{M}_f(r_k)}{\varphi_{p-1}(r_k)} \\ &\leq \sup_{N_0\leq k\leq N} \frac{|f(x_{k,j_0})|}{\varphi_{p-1}(r_k)} + \sup_{N_0\leq k\leq N} \left\{ \frac{2n}{s_k(r_{k+1} - r_k)} \cdot \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_k)} \right\} \\ &:= \mathcal{A} + \mathcal{B}. \end{aligned}$$

Since $x_{k,j_0} \in \mathcal{S}_k$ we have $|f(x_{k,j_0})| \le \sup_{\lambda_j \in \mathcal{S}_k} |f(\lambda_j)|$. Therefore,

$$\mathcal{A} \le \sup_{N_0 \le k \le N} \sup_{\lambda_j \in \mathcal{S}_k} \frac{|f(\lambda_j)|}{\varphi_{p-1}(r_k)} := \mathcal{P}_N.$$

Furthermore,

$$\frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_k)} = \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \cdot \frac{\varphi_{p-1}(r_{k+1})}{\varphi_{p-1}(r_k)} \le R_{p-1} \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \text{ by (C3).}$$

Hence,

$$\begin{aligned}
\mathcal{B} &\le \sup_{N_0 \le k \le N} \left\{ \frac{2nR_{p-1}}{s_k(r_{k+1}-r_k)} \cdot \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \right\} \\
&\le \sup_{N_0 \le k \le N-1} \left\{ \frac{2nR_{p-1}}{s_k(r_{k+1}-r_k)} \cdot \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \right\} \\
&\quad + \frac{2nR_{p-1}}{s_N(r_{N+1}-r_N)} \cdot \frac{\mathcal{M}_f(r_{N+1})}{\varphi_{p-1}(r_{N+1})} \\
&\le \sup_{N_0-1 \le k \le N-1} \left\{ \frac{2nR_{p-1}}{s_k(r_{k+1}-r_k)} \cdot \frac{\mathcal{M}_f(r_{k+1})}{\varphi_{p-1}(r_{k+1})} \right\} \\
&\quad + \frac{2nR_{p-1}}{s_N(r_{N+1}-r_N)} \cdot \frac{\mathcal{M}_f(r_{N+1})}{\varphi_{p-1}(r_{N+1})}.
\end{aligned}$$

But from our choice of N_p in (4.5) and by the definition of $\mathcal{Y}_\mathcal{N}$, we can estimate the first term above and obtain

$$\mathcal{B} \le \frac{1}{4}\mathcal{Y}_N + \frac{2nR_{p-1}}{s_N(r_{N+1}-r_N)} \cdot \frac{\mathcal{M}_f(r_{N+1})}{\varphi_{p-1}(r_{N+1})}.$$

Combining the last two inequalities for $\mathcal{A}$ and $\mathcal{B}$ yields

$$\mathcal{Y}_N \le \mathcal{A} + \mathcal{B} \le \mathcal{P}_N + \frac{1}{4}\mathcal{Y}_N + \frac{2nR_{p-1}}{s_N(r_{N+1}-r_N)} \cdot \frac{\mathcal{M}_f(r_{N+1})}{\varphi_{p-1}(r_{N+1})},$$

or equivalently,

$$\frac{3}{4}\mathcal{Y}_N \le \mathcal{P}_N + \frac{2nR_{p-1}}{s_N(r_{N+1}-r_N)} \cdot \frac{\mathcal{M}_f(r_{N+1})}{\varphi_{p-1}(r_{N+1})}. \tag{4.6}$$

Now, since $f \in h_\varphi^{-(p-1),S} \subset h_\Phi^{-\infty} = \bigcup_s h_\varphi^{-s}$, there is s_0 such that $f \in h_\varphi^{-s_0}$. Take and fix a number $q > s_0$. By Proposition 4.2 with $\ell = 1$, there is a sequence $(k_j) \uparrow +\infty$ such that

$$\frac{\mathcal{M}_f(r_{k_j+1})}{\varphi_q(r_{k_j+1})} \le 2\frac{\mathcal{M}_f(r_{k_j})}{\varphi_q(r_{k_j})}, \ \forall j \ge 1.$$

Then we have for (4.6) with $N = k_j$ $(j = 1, 2, \ldots)$:

$$\begin{aligned}
\frac{3}{4}\mathcal{Y}_{k_j} &\le \mathcal{P}_{k_j} + \frac{2nR_{p-1}}{s_{k_j}(r_{k_j+1} - r_{k_j})} \cdot \frac{\mathcal{M}_f(r_{k_j+1})}{\varphi_{p-1}(r_{k_j+1})} \\
&\le \mathcal{P}_{k_j} + \frac{4nR_{p-1}}{s_{k_j}(r_{k_j+1} - r_{k_j})\varphi_{p-1}(r_{k_j+1})} \cdot \frac{\mathcal{M}_f(r_{k_j})\varphi_q(r_{k_j+1})}{\varphi_q(r_{k_j})} \\
&= \mathcal{P}_{k_j} + \frac{4nR_{p-1}\varphi_q(r_{k_j+1})}{s_{k_j}(r_{k_j+1} - r_{k_j})\varphi_q(r_{k_j})} \cdot \frac{\mathcal{M}_f(r_{k_j})}{\varphi_{p-1}(r_{k_j+1})} \\
&\le \mathcal{P}_{k_j} + \frac{4nR_{p-1}\varphi_q(r_{k_j+1})}{s_{k_j}(r_{k_j+1} - r_{k_j})\varphi_q(r_{k_j})} \cdot \frac{\mathcal{M}_f(r_{k_j})}{\varphi_{p-1}(r_{k_j})}.
\end{aligned}$$

Note, by (C2) and (4.4), that

$$\lim_{k\to\infty} \frac{4nR_{p-1}\varphi_q(r_{k+1})}{s_k(r_{k+1} - r_k)\varphi_q(r_k)} = 0.$$

Then for all j large enough we have

$$\frac{4nR_{p-1}\varphi_q(r_{k_j+1})}{s_{k_j}(r_{k_j+1} - r_{k_j})\varphi_q(r_{k_j})} < \frac{1}{4}.$$

Therefore, we have

$$\frac{3}{4}\mathcal{Y}_{k_j} \le \mathcal{P}_{k_j} + \frac{1}{4} \cdot \frac{\mathcal{M}_f(r_{k_j})}{\varphi_{p-1}(r_{k_j})} \le \mathcal{P}_{k_j} + \frac{1}{4}\mathcal{Y}_{k_j},$$

which can be written as

$$\mathcal{Y}_{k_j} \le 2\mathcal{P}_{k_j} \ \text{ for all } j \text{ large enough.}$$

Taking into account that $\mathcal{X}_N := \sup_{0<r\le r_N} \frac{\mathcal{M}_f(r)}{\varphi_p(r)} \le C_p\mathcal{Y}_N$, we arrive at the following inequalities

$$\mathcal{X}_{k_j} \le C_p\mathcal{Y}_{k_j} \le 2C_p\mathcal{P}_{k_j} \ \text{ for all } j \text{ large enough.} \tag{4.7}$$

We note that $\|f\|_{p,S}$ can be written as follows

$$\|f\|_{p,S} = \sup_{k\ge1} \frac{|f(\lambda_k)|}{\varphi_p(|\lambda_k|)} = \sup_{\ell\ge1} \sup_{\lambda_k\in\mathcal{S}_\ell} \frac{|f(\lambda_k)|}{\varphi_p(|\lambda_k|)}.$$

Now letting in (4.7) $k_j \to \infty$ we obtain

$$\|f\|_p \le 2C_p \sup_{\ell\ge N} \sup_{\lambda_k\in\mathcal{S}_\ell} \frac{|f(\lambda_k)|}{\varphi_{p-1}(|\lambda_k|)} \le 2C\|f\|_{p-1,S}.$$

Thus we always have

$$\|f\|_p \le 2C_p\|f\|_{p-1,S}, \ \forall f \in h_\varphi^{-(p-1),S}.$$

The theorem is proved completely. □

5. Examples of weight functions

As we have seen from previous sections weight functions $\Phi = \big(\varphi_p(r)\big)$ are supposed to satisfy conditions (C1)-(C3). Condition (C2), each weight is a increasing function, appeared quite naturally for a general setting of our problem. Concerning the rest two conditions (C1) and (C3), they seem to be imposed due to the techniques used in the present paper. The weight functions are required to satisfy some conditions in connection with the existence of a function $\psi(r)$, and moreover, with the sequence $0 < (r_k) \uparrow 1$ on which the countable sets (λ_k) are chosen. Therefore, it is of interest to consider some issues related to that.

We provide various examples of weight functions (φ_p) together with the corresponding ψ in condition (C1), and discuss conditions on (r_k) under which the requirements in (C3) are satisfied.

For a sequence $0 < (r_k) \uparrow 1$ consider the following three conditions:

$$\sup_{k\geq 1}\left(\frac{1}{1-r_{k+1}} - \frac{1}{1-r_k}\right) < +\infty.$$

$$\lim_{k\to\infty}\frac{1-r_{k+1}}{1-r_k} = 1.$$

$$\lim_{k\to\infty}\frac{|\log(1-r_{k+1})|}{|\log(1-r_k)|} = 1.$$

We can check that each of these conditions implies the next, and both converse implications fail.

Indeed, suppose that the first condition holds. Then there is $L > 0$ such that

$$0 < \frac{1}{1-r_{k+1}} - \frac{1}{1-r_k} \leq L, \ \forall k \geq 1,$$

which is equivalent to

$$0 < 1 - \frac{1-r_{k+1}}{1-r_k} \leq L(1-r_{k+1}),$$

and the second condition follows, by the squeeze principle.

A counterexample for the failure of the converse implication is say $r_k = 1 - \frac{1}{k^\alpha}$ with $\alpha > 1$.

Now if we have the second condition, then

$$\begin{aligned}
\lim_{k\to\infty}\left[\frac{|\log(1-r_{k+1})|}{|\log(1-r_k)|} - 1\right] &= \lim_{k\to\infty}\left[\frac{\log(1-r_{k+1})}{\log(1-r_k)} - 1\right] \\
&= \lim_{k\to\infty}\frac{\log(1-r_{k+1}) - \log(1-r_k)}{\log(1-r_k)} \\
&= \lim_{k\to\infty}\frac{\log\frac{1-r_{k+1}}{1-r_k}}{\log(1-r_k)} = 0,
\end{aligned}$$

that is the third condition is satisfied.

Furthermore, for $r_k = 1 - \frac{1}{2^k}$ the third holds, while the second one fails.

We note that $r_k = 1 - \frac{1}{(k+1)^\alpha}, \alpha > 0$, satisfies the third condition, also $r_k = 1 - e^{-e^{\sqrt{k}}}$ works, etc.

We can also consider weaker conditions than those said above:

$$\limsup_{k\to\infty} \frac{1-r_k}{1-r_{k+1}} < +\infty, \tag{5.1}$$

$$\limsup_{k\to\infty} \frac{|\log(1-r_k)|}{|\log(1-r_{k+1})|} < +\infty. \tag{5.2}$$

Now return back to weight functions.

Example 5.1. Consider $\varphi_p(r) = (1-r)^{-p},\ p > 0$.

In this case we get the space $h^{-\infty}(\mathbb{B})$ of harmonic functions on the ball with polynomial growth. Choosing $\psi(r) = |\log(1-r)|$, we have

$$\liminf_{r\to 1} \frac{\log \varphi_p(r)}{\psi(r)} = \liminf_{r\to 1} \frac{-p\log(1-r)}{|\log(1-r)|} = p,$$

that is, (C1) is fulfilled.

Now for $(r_k) \uparrow 1$ we impose condition (5.1). Then we have

$$\sup_{k\geq 1} \frac{\varphi_p(r_{k+1})}{\varphi_p(r_k)} = \sup_{k\geq 1} \left(\frac{1-r_k}{1-r_{k+1}} \right)^p < +\infty,$$

that is, (C3) is satisfied.

Applying Theorem 4.4 we get the explicit construction of (countable) weakly sufficient sets for the well-known space $h^{-\infty}(\mathbb{B})$.

Example 5.2. Consider $\varphi_p(r) = e^{\frac{p}{1-r}},\ p > 0$.

The corresponding space $h_\Phi^{-\infty}(\mathbb{B})$ consists of harmonic functions on the ball with "exponential growth".

In this case, we can take $\psi(r) = \frac{1}{1-r}$ to have condition (C1) fulfilled.

How to choose (r_k) for condition (C3)? We note that

$$\begin{aligned}
\sup_{k\geq 1} \frac{\varphi_p(r_{k+1})}{\varphi_p(r_k)} &= \sup_{k\geq 1} \exp\left\{ p\left(\frac{1}{1-r_{k+1}} - \frac{1}{1-r_k} \right) \right\} \\
&= \exp\left\{ p \sup_{k\geq 1} \left(\frac{1}{1-r_{k+1}} - \frac{1}{1-r_k} \right) \right\} \\
&= \exp\left\{ p \sup_{k\geq 1} \frac{r_{k+1}-r_k}{(1-r_k)(1-r_{k+1})} \right\}.
\end{aligned}$$

Therefore, (r_k) must satisfy the first condition said above.

For simplicity denote $a_k = \frac{1}{1-r_k}$. Then we have an increasing sequence of positive numbers $(a_k) \uparrow +\infty$, and the first condition looks as

$$\sup_{k\geq 1}(a_{k+1} - a_k) < +\infty.$$

Notice that $a_k = k^\alpha,\ \alpha \leq 1$, works well.

Acknowledgment. The author thanks the referees for their useful remarks and comments that led to the improvement of this paper.

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