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THE MULTISOLITON SOLUTION OF GENERALIZED BURGER'S EQUATION BY THE FORMAL LINEARIZATION METHOD

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Abstract. The formal linearization method is an efficient method for constructing multisoliton solution of some nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones. In this paper, we obtain multisoliton solution of generalization Burger's equation and the (3+1)-dimension Burger's equation and the Boussinesq equation by the formal linearization method.

1. Introduction

Many years ago there was interest in constructing solutions of nonlinear partial differential equations in the form of infinite series. The direct linearization of certain famous integrable nonlinear equations was carried out in [7]. Solutions of the KdV equation were connected with solutions of the Hopf equation by using formal series in [4]. Convergent exponential series were used in papers [1, 2, 3, 5, 8, 9] for constructing solutions of the Boltzmann equations. The possibility to use such series for some other equations was discussed in [2]. Fourier series were applied for constructing solutions of perturbed KdV equation in [6]. In this paper we consider the class of equations and systems containing arbitrary linear differential operators with constant coefficients and arbitrary nonlinear analytic functions of dependent variables and their derivatives up to some finite order in assumption that these equations possess a constant solution. The formal linearzation method is based on formal linearization of a nonlinear partial differential equation to the system of linear ordinary differential equations, describing some finite-dimensional subspace of the space of solutions of the linearized equation. It allows us to develop a very simple technique of finding the linearizing transformation and to apply the method to nonintegrable equations as well as to integrable ones, solutions have the form of exponential or Fourier series. Let us note that the similar approach with the

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different technique was independently developed in [10] for the wide class of evolution equations and in this case the convergence of constructed exponential series was investigated [10]. The aim of this paper is to find exact multisoliton solutions of generalized Burger's equation and the $(3+1)$-dimension Burger's equation and the Boussinesq equation by the formal linearization method.

2. The method of formal linearization

Let us consider equations of the following form

$$\hat{L}(D_{x_1}, D_{x_2}, D_{x_3})u(x_1, x_2, x_3) = N[u], \tag{1}$$

where

$$\hat{L}(D_{x_1}, D_{x_2}, D_{x_3}) = \sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}\sum_{k_3=0}^{K_3} l_{k_1k_2k_3} D_{x_1}^{k_1} D_{x_2}^{k_2} D_{x_3}^{k_3} \tag{2}$$

is a linear differential operator with constant coefficients and

$$N[u] = N(u, u_1, u_2, \ldots, u_p),$$
$$u_p = \frac{\partial^{p_1+p_2+p_3}u}{\partial {x_1}^{p_1}\partial {x_2}^{p_2}\partial {x_3}^{p_3}},\ p = (p_1, p_2, p_3),$$

is an arbitrary analytic function of u and of its derivatives up to some finite order p. We suppose that Eq.(1) possesses the constant solution. Without loss of generality we assume that

$$N[0] = 0,\ \frac{\partial N[0]}{\partial u} = 0,\ \frac{\partial N[0]}{\partial u_1} = 0, \ldots,\ \frac{\partial N[0]}{\partial u_p} = 0.$$

We consider Eq.(1) in connection with the equation linearized near a zero solution:

$$\hat{L}(D_{x_1}, D_{x_2}, D_{x_3})w(x_1, x_2, x_3) = 0. \tag{3}$$

Let L be the vector space of solutions of Eq.(3) and $P^N \subset L$ be the N-dimensional subspace with the basis

$$w_i = W_i \exp(\alpha_i \xi_i),$$
$$\xi_i = x_3 - a_i x_1 - b_i x_2,\ i = 1, \ldots, N.$$

Here a_i, b_i and W_i are some constants. The constants $\alpha_i = \alpha_i(a_i, b_i)$ are assumed to satisfy the dispersion relation

$$\hat{L}(-\alpha_i a_i, -\alpha_i b_i, \alpha_i) = 0.$$

The subspace $P^N = \{\sum_{i=1}^N C_i w_i | C_i = \text{constant}\}$ is specified by the system of N linear ordinary differential equations

$$\frac{dw_i}{d\xi_i} = \alpha_i w_i,\ i = 1, \ldots, N.$$

We use the following notation:

$$w_{(N)}^{\delta} = w_1^{\delta_1} w_2^{\delta_2} \cdots w_N^{\delta_N},$$

$$\delta = (\delta_1, \delta_2, \ldots, \delta_N),$$

$$|\delta| = \Sigma_{i=1}^{N} \delta_i.$$

It is obvious that the monomials $w_{(N)}^{\delta}$ are the eigenfunctions of the operator (2):

$$\hat{L}(D_{x_1}, D_{x_2}, D_{x_3}) w_{(N)}^{\delta} = \lambda_\delta w_{(N)}^{\delta}$$

with the eigenvalues

$$\lambda_\delta = \sum_{k_1=0}^{K_1} \sum_{k_2=0}^{K_2} \sum_{k_3=0}^{K_3} l_{k_1 k_2 k_3} (-\sum_{i=1}^{N} \alpha_i a_i \delta_i)^{k_1} (-\sum_{i=1}^{N} \alpha_i b_i \delta_i)^{k_2} (\sum_{i=1}^{N} \alpha_i \delta_i)^{k_3}.$$

Theorem 1. *If $\lambda_\delta \neq 0$ for every multiindex δ with positive integer components $\delta_i \in \mathbb{Z}_+, i = 1, \ldots, N$, satisfying the condition $|\delta| \neq 0, 1$, then Eq.*(1) *possesses solutions connected with solutions form P^N by the formal transformation*

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2, \ldots, w_N), \tag{4}$$

where

$$\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(N)}^{\delta} \tag{5}$$

are homogeneous polynomials of degree n in the variables w_i. This transformation is unique (*for the first term $\phi_1 \in P^N$ fixed*).

Remark 1. Here ε is the grading parameter, finally we can put $\varepsilon = 1$.

The proof of the theorem is constructive. Substituting (4) into (1), expanding $N[u]$ into the power series in ε, and then collecting equal powers of ε, we obtain the determining equations for the functions ϕ_n and show that if $\lambda_\delta \neq 0$, then these equations possess the solution (5) with the coefficients $(A_n)_\delta$ uniquely determined through the coefficients $(A_1)_\delta$ by the recursion relation. Thus, the theorem gives us the method for constructing particular solutions of Eq.(1).

3. Application

3.1. The generalized Burger's equation

The generalized Burger's equation is in the form

$$u_t + u_{xx} + u u_x + 2u u_{xxx} + 3u^2 u_x = 0, \tag{6}$$

where $u = u(t, x)$. Thus, we can write

$$\hat{L}(D_t, D_x) u(t, x) = -u u_x - 2u u_{xxx} - 3u^2 u_x, \tag{7}$$

$$\hat{L}(D_t, D_x) = D_t + D_x^2.$$

For simplicity we look for a solution of (7) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2), \tag{8}$$

where

$$w_i = W_i \exp[a_i(x - a_i t)],\ i = 1, 2$$

is the basis of the subspace $P^2 \subset L$ (let a_i and W_i be some real constants). Substituting (8) into (7) and collecting equal powers of ε we obtain the determining equations for the functions ϕ_n as follows

$$\hat{L}\phi_1 = 0,$$

$$\hat{L}\phi_n = -\sum_{k=1}^{n-1} \phi_k D_x \phi_{n-k} - 2\sum_{k=1}^{n-1} \phi_k D_x^3 \phi_{n-k} - 3\sum_{k=2}^{n-1} D_x \phi_{n-k} \sum_{l=1}^{k-1} \phi_l \phi_{k-l},\ n \geq 2.$$

These equations possess the solution $\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(2)}^\delta$, $\delta = (\delta_1, \delta_2)$, which can be rewritten in this case in the following form

$$\phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}\ (\phi_1 \in P^2), \tag{9}$$

the coefficients A_k^n can be found through A_0^1 and A_1^1 (we can assume that either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1$) by the recursion relation:
If $n \geq 2, 0 \leq k \leq n$, then

$$\begin{aligned} A_k^n = & -\frac{1}{\lambda_{(k,n-k)}}\{\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}(a_1 m + a_2(n-l-m))A_{k-m}^l A_m^{n-l} \\ & + 2\sum_{l=1}^{n-1}\sum_{m=0}^{n-l}(a_1 m + a_2(n-l-m))^3 A_{k-m}^l A_m^{n-l} \\ & + 3\sum_{l=2}^{n-1}\sum_{m=1}^{l-1}\sum_{p=0}^{n-l}\sum_{q=0}^{m}(a_1 p + a_2(n-l-p))A_p^{n-l} A_q^m A_{k-p-q}^{l-m}\}, \end{aligned}$$

if $k < 0$ or $k > n$, then $A_k^n = 0$.

$$\lambda_{(k,n-k)} = a_1^2 k(k-1) + a_2^2(n-k)((n-k)-1) + 2a_1 a_2 k(n-k).$$

If $a_1 \neq 0$ and $a_2 \neq 0$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in \mathbb{Z}_+$, $n \geq 2$, $0 \leq k \leq n$.

Remark 2. If $A_0^1 = 0$, then $\phi_1 \in P^1$ and we get from (8) the expansion for a 1-soliton solution. For obtaining the N-soliton solutions, we must take $\phi_1 \in P^N$.

3.2. The (3+1)-dimension Burger's equation

Let us consider the $(3+1)$-dimension Burger's equation

$$\begin{aligned} \hat{L}(D_t, D_y, D_z, D_x)u(t, y, z, x) &= -\alpha(uu_x + uu_y + uu_z), \\ \hat{L}(D_t, D_y, D_z, D_x) &= D_t - \beta(D_x^2 + D_y^2 + D_z^2). \end{aligned} \tag{10}$$

In this case, the subspace P^2 is generated by the functions

$$w_i = W_i \exp[-\frac{a_i}{\beta(1+b_i^2+c_i^2)}(x - a_i t - b_i y - c_i z)], \ i = 1, 2.$$

Our procedure give the solution

$$u = \sum_{n=1}^{\infty} \varepsilon^n \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}, \tag{11}$$

where if $n \geq 2, 0 \leq k \leq n$, then

$$\begin{aligned} A_k^n = & -\frac{\alpha}{\lambda_{(k,n-k)}} \sum_{l=1}^{n-1} \sum_{m=0}^{n-l} [-\frac{a_1(1-b_1-c_1)}{\beta(1+b_1^2+c_1^2)} m \\ & - \frac{a_2(1-b_2-c_2)}{\beta(1+b_2^2+c_2^2)}(n-l-m)] A_{k-m}^l A_m^{n-l}, \end{aligned}$$

if $k < 0$ or $k > n$, then $A_k^n = 0$.

$$\begin{aligned} \lambda_{(k,n-k)} = & \frac{a_1^2}{\beta(1+b_1^2+c_1^2)} k(1-k) + \frac{a_2^2}{\beta(1+b_2^2+c_2^2)}(n-k)(1-(n-k)) \\ & - \frac{2a_1 a_2}{\beta(1+b_1^2+c_1^2)(1+b_2^2+c_2^2)}(1 + b_1 b_2 + c_1 c_2) k(n-k). \end{aligned}$$

Here either $A_0^1 = A_1^1 = 1$or $A_0^1 = 0, A_1^1 = 1$. If $a_1 \neq 0$ and $a_2 \neq 0$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in \mathbb{Z}_+$, $n \geq 2$, $0 \leq k \leq n$.

In (11), if $A_0^1 = 0$, then we get

$$\begin{aligned} u &= \sum_{n=1}^{\infty} (-\frac{\alpha}{2a_1})^{n-1} (1-b_1-c_1)^{n-1} (\varepsilon w_1)^n \\ &= \frac{\varepsilon w_1}{1 + \frac{\alpha}{2a_1}(1-b_1-c_1)\varepsilon w_1} \\ &= \frac{a_1}{\alpha(1-b_1-c_1)} \frac{2w}{1+w}, \end{aligned} \tag{12}$$

where $w = \frac{\alpha}{2a_1}(1 - b_1 - c_1)\varepsilon w_1$.

In(t, y, z, x)-variables we have

$$u = \frac{a_1}{\alpha(1-b_1-c_1)}(1 - \tanh(\frac{a_1}{2\beta(1+b_1^2+c_1^2)}(x - a_1 t - b_1 y - c_1 z + x_0))), \tag{13}$$

where x_0 is arbitrary constant.

Then (11) is a 2-soliton solution of the $(3+1)$-dimension Burger's equation and (13) is a 1-soliton solution of the $(3+1)$-dimension Burger's equation.

3.3. The Boussinesq equation

Let us consider the Boussinesq equation

$$\begin{aligned} \hat{L}(D_t, D_x) u(t,x) &= -\alpha u_x^2 - \alpha u u_{xx}, \\ \hat{L}(D_t, D_x) &= D_t^2 - D_x^2 - \beta D_x^4. \end{aligned} \tag{14}$$

We look for a solution of (14) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2), \tag{15}$$

where

$$w_i = W_i \exp[\sqrt{\frac{a_i^2 - 1}{\beta}}(x - a_i t)],\ i = 1, 2$$

is the basis of the subspace $P^2 \subset L$ (let s_i and W_i be some real constants). Substituting (15) into (14) and collecting equal powers of ε we obtain the determining equations for the functions ϕ_n as follows:

$$\hat{L}\phi_1 = 0,$$

$$\hat{L}\phi_n = -\alpha(\sum_{k=1}^{n-1} D_x\phi_k D_x\phi_{n-k} + \sum_{k=1}^{n-1} \phi_k D_x^2 \phi_{n-k}),\ n \geq 2.$$

These equations possess the solution

$$\phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}\ (\phi_1 \in P^2),$$

the coefficients A_k^n can be found through A_0^1 and A_1^1 (we can assume that either $A_0^1 = A_1^1 = 1$ or $A_0^1 = 0, A_1^1 = 1$) by the recursion relation:
If $n \geq 2, 0 \leq k \leq n$, then

$$\begin{aligned} A_k^n = \frac{-\alpha}{\lambda_{(k,n-k)}} \{ & \sum_{l=1}^{n-1} \sum_{m=0}^{n-l} [\sqrt{\frac{a_1^2 - 1}{\beta}}(k - m) + \sqrt{\frac{a_2^2 - 1}{\beta}}(l - k + m)] \\ & [\sqrt{\frac{a_1^2 - 1}{\beta}} m + \sqrt{\frac{a_2^2 - 1}{\beta}}(n - l - m)] A_{k-m}^l A_m^{n-l} \\ & + \sum_{l=1}^{n-1} \sum_{m=0}^{n-l} [\sqrt{\frac{a_1^2 - 1}{\beta}} m + \sqrt{\frac{a_2^2 - 1}{\beta}}(n - l - m)]^2 A_{k-m}^l A_m^{n-l} \}, \end{aligned}$$

if $k < 0$ or $k > n$, then $A_k^n = 0$.

$$\begin{aligned} \lambda_{(k,n-k)} = & \frac{(a_1^2 - 1)^2}{\beta} k^2(1 - k^2) + \frac{(a_2^2 - 1)^2}{\beta}(n - k)^2(1 - (n - k)^2) \\ & - 6\frac{(a_1^2 - 1)(a_2^2 - 1)}{\beta} k^2 (n - k)^2 \\ & + 2\frac{\sqrt{(a_1^2 - 1)(a_2^2 - 1)}}{\beta}\{(a_1 a_2 - 1)k(n - k) \\ & - 2(a_1^2 - 1)k^3(n - k) - 2(a_2^2 - 1)k(n - k)^3\}. \end{aligned}$$

If $a_1 < -1, a_2 < -1,\ a_1 > 1, a_2 > 1$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n - k)$ with $k, n \in \mathbb{Z}_+$, $n \geq 2$, $0 \leq k \leq n$.

If $A_0^1 = 0$, then we get

$$
\begin{aligned}
u &= \varepsilon w_1 - \frac{\alpha}{6(1-a_1^2)}(\varepsilon w_1)^2 + \frac{\alpha^2}{48(1-a_1^2)^2}(\varepsilon w_1)^3 - \cdots \\
&= \frac{12(1-a_1^2)}{\alpha}\sum_{n=1}^{\infty}(-1)^{n+1} n w^n,
\end{aligned}
$$

where $w = \frac{\varepsilon\alpha}{12(1-a_1^2)} w_1$.

In(t, x)-variables we have

$$
u(t,x) = \pm\frac{3(1-a_1^2)}{\alpha}\operatorname{sech}^2\Big(\frac{1}{2}\sqrt{\frac{a_1^2-1}{\beta}}(x - a_1 t + x_0)\Big), \tag{16}
$$

where x_0 is arbitrary constant.

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