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IMPLICATIVE SOFT IDEALS AND IMPLICATIVE IDEALISTIC SOFT BCK-ALGEBRAS

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Abstract. Molodtsov [5] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. In this paper we apply the notion of soft sets by Molodtsov to an implicative ideal of BCK-algebras. The notion of implicative soft ideals in BCK-algebras and implicative idealistic soft BCK-algebras is introduced, and related properties are investigated. Relations between implicative soft ideals and commutative (resp. positive implicative) soft ideals are discussed. Also, relations between implicative idealistic soft BCK-algebras and commutative (resp. positive implicative) idealistic soft BCK-algebras are provided.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [5]. Maji et al. [3] and Molodtsov [5] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [5] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set

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theory are progressing rapidly. Maji et al. [3] described the application of soft set theory to a decision making problem. Maji et al. [2] also studied several operations on the theory of soft sets. Chen et al. [1] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. As it is well-known, BCK-algebras are characterized as a class of algebras important in logics. So, it is worthy of note to study an application of soft sets in algebraic structures, herewith, a BCK-algebra, and then the ideas/results of our study have an important role as a new tool and/or motivation in considering uncertainties and any other applications. With this in mind, we will mainly concern on an application of a soft set to BCK-algebras. We use the notion of soft sets to obtain soft structure of implicative ideals in BCK-algebras. We introduce the notion of implicative soft ideals in BCK-algebras and implicative idealistic soft BCK-algebras, and investigate several properties. We give relations between implicative soft ideals and commutative (resp. positive implicative) soft ideals. We also provide relations between implicative idealistic soft BCK-algebras and commutative (resp. positive implicative) idealistic soft BCK-algebras.

2. Basic results on BCK-algebras and soft sets

A BCK-algebra is an important calss of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK-algebra X satisfies the following conditions:

(a1) $(\forall x \in X)\ (x * 0 = x)$,

(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,

(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,

(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *positive implicative* if it satisfies the following identity:

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * (y * z)). \tag{2.1}$$

A BCK-algebra X is said to be *commutative* if $x \wedge y = y \wedge x$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$. A BCK-algebra X is said to be *implicative* if $x = x * (y * x)$ for all $x, y \in X$. A nonempty subset S of a BCK-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK-algebra X is called an *ideal* of X related to a subalgebra S of X (briefly, S-*ideal* of X), denoted by $I \lhd S$, if it satisfies the following axioms:

(b1) $0 \in I$,

(b2) $(\forall x \in S)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

Any X-ideal I of a BCK-algebra X satisfies the following implication:

$$(\forall x \in X)\,(\forall y \in I)\,(x \leq y \Rightarrow x \in I). \tag{2.2}$$

Let S be a subalgebra of a BCK-algebra X and let I be a subset of X satisfying the condition (b1). Then I is called

(i) a *positive implicative ideal* of X related to S (briefly, *positive implicative* S-*ideal* of X), denoted by $I \lhd_{pi} S$, if it satisfies

$$(\forall x, y, z \in S)\,((x * y) * z \in I,\ y * z \in I \Rightarrow x * z \in I). \tag{2.3}$$

(ii) a *commutative ideal* of X related to S (briefly, *commutative* S-*ideal* of X), denoted by $I \lhd_c S$, if it satisfies

$$(\forall x, y \in S)\,(\forall z \in I)\,((x * y) * z \in I \Rightarrow x * (y * (y * x)) \in I). \tag{2.4}$$

(iii) an *implicative ideal* of X related to S (briefly, *implicative* S-*ideal* of X), denoted by $I \lhd_i S$, if it satisfies

$$(\forall x, y \in S)\,(\forall z \in I)\,((x * (y * x)) * z \in I \Rightarrow x \in I). \tag{2.5}$$

Note that a (positive) implicative X-ideal (resp. a commutative X-ideal) means a (positive) implicative ideal (resp. commutative ideal). Also, we have that every (positive) implicative (or, commutative) S-ideal of X is an S-ideal of X.

We refer the reader to the book [4] for further information regarding BCK-algebras.

Molodtsov [5] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and A a nonempty subset of E.

Definition 2.1 ([5])**.** A pair (α, A) is called a *soft set* over U, where α is a mapping given by

$$\alpha : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $\alpha(x)$ may be considered as the set of x-approximate elements of the soft set (α, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [5].

Definition 2.2 ([2])**.** Let (α, A) and (β, B) be two soft sets over a common universe U. The *intersection* of (α, A) and (β, B) is defined to be the soft set (ρ, C) satisfying the following conditions:

(i) $C = A \cap B$,

(ii) $(\forall x \in C)\ (\rho(x) = \alpha(x)$ or $\beta(x)$, (as both are same sets)).

In this case, we write $(\alpha, A)\widetilde{\cap}(\beta, B) = (\rho, C)$.

Definition 2.3 ([2])**.** Let (α, A) and (β, B) be two soft sets over a common universe U. The *union* of (α, A) and (β, B) is defined to be the soft set (ρ, C) satisfying the following conditions:

(i) $C = A \cup B$,
(ii) for all $x \in C$,

$$\rho(x) = \begin{cases} \alpha(x) & \text{if } x \in A \setminus B, \\ \beta(x) & \text{if } x \in B \setminus A, \\ \alpha(x) \cup \beta(x) & \text{if } x \in A \cap B. \end{cases}$$

In this case, we write $(\alpha, A)\widetilde{\cup}(\beta, B) = (\rho, C)$.

Definition 2.4 ([2])**.** If (α, A) and (β, B) are two soft sets over a common universe U, then "(α, A) *AND* (β, B)" denoted by $(\alpha, A)\widetilde{\wedge}(\beta, B)$ is defined by $(\alpha, A)\widetilde{\wedge}(\beta, B) = (\rho, A \times B)$, where $\rho(x, y) = \alpha(x) \cap \beta(y)$ for all $(x, y) \in A \times B$.

Definition 2.5 ([2])**.** If (α, A) and (β, B) are two soft sets over a common universe U, then "(α, A) *OR* (β, B)" denoted by $(\alpha, A)\widetilde{\vee}(\beta, B)$ is defined by $(\alpha, A)\widetilde{\vee}(\beta, B) = (\rho, A \times B)$, where $\rho(x, y) = \alpha(x) \cup \beta(y)$ for all $(x, y) \in A \times B$.

Definition 2.6 ([2])**.** For two soft sets (α, A) and (β, B) over a common universe U, we say that (α, A) is a *soft subset* of (β, B), denoted by $(\alpha, A)\widetilde{\subset}\, (\beta, B)$, if it satisfies:

(i) $A \subset B$,
(ii) For every $x \in A$, $\alpha(x)$ and $\beta(x)$ are identical approximations.

3. Implicative soft ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1. Let (α, A) be a soft set over X. Then (α, A) is called a *soft BCK-algebra* over X if $\alpha(x)$ is a BCK-subalgebra of X for all $x \in A$.

Definition 3.2. Let (α, A) be a soft BCK-algebra over X. A soft set (β, I) over X is called a *soft ideal* of (α, A), denoted by $(\beta, I)\widetilde{\lhd}(\alpha, A)$, if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)\ (\beta(x) \lhd \alpha(x))$.

Definition 3.3. Let (α, A) be a soft BCK-algebra over X. A soft set (β, I) over X is called a *positive implicative soft ideal* of (α, A), denoted by $(\beta, I)\widetilde{\lhd}_{pi}$ (α, A), if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)\ (\beta(x) \lhd_{pi} \alpha(x))$.

Definition 3.4. Let (α, A) be a soft BCK-algebra over X. A soft set (β, I) over X is called a *commutative soft ideal* of (α, A), denoted by $(\beta, I)\widetilde{\lhd}_c\, (\alpha, A)$, if it satisfies:

(i) $I \subset A$,

(ii) $(\forall x \in I)\ (\beta(x) \lhd_c \alpha(x))$.

Definition 3.5. Let (α, A) be a soft BCK-algebra over X. A soft set (β, I) over X is called an *implicative soft ideal* of (α, A), denoted by $(\beta, I)\widetilde{\lhd}_i\ (\alpha, A)$, if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)\ (\beta(x) \lhd_i \alpha(x))$.

For any $a \in X$ and a subset D of X, let

$$\frac{a}{D} := \{x \in X \mid x * a \in D\} \text{ and } \frac{a^2}{D} := \{x \in X \mid x * (x * a) \in D\}.$$

Let us illustrate this definition using the following examples.

Example 3.6. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (α, A) be a soft set over X, where $A = X$ and $\alpha : A \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(x) = \frac{x}{\{0,b\}}$ for all $x \in A$. Then (α, A) is a soft BCK-algebra over X since $\alpha(0) = \alpha(b) = \{0, b\}$, $\alpha(a) = \{0, a, b\}$, $\alpha(c) = \{0, b, c\}$ and $\alpha(d) = \{0, b, d\}$ are BCK-subalgebras of X. Let (β, I) be a soft set over X, where $I = \{a, c, d\}$ and $\beta : I \to \mathscr{P}(X)$ is a set-valued function defined by $\beta(x) = \frac{x^2}{\{0,a\}}$ for all $x \in I$. Then $\beta(a) = X \lhd_i \alpha(a)$, $\beta(c) = \{0, a, b, d\} \lhd_i \alpha(c)$ and $\beta(d) = \{0, a, b, c\} \lhd_i \alpha(d)$, and so (β, I) is an implicative soft ideal of (α, A).

Based on the fact that every ideal of X is an implicative ideal of X if and only if X is an implicative BCK-algebra (see [4]), we have the following proposition.

Proposition 3.7. *If (α, A) is a soft BCK-algebra over an implicative BCK-algebra X, then every soft ideal of (α, A) is implicative.*

Note that every implicative soft ideal is both a commutative soft ideal and a positive implicative soft ideal since every implicative S-ideal of X is both a commutative S-ideal of X and a positive implicative S-ideal of X. In the following examples, we know that there exists a commutative soft ideal (or, positive implicative soft ideal) which is not an implicative soft ideal.

Example 3.8. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

For $A = X$, let $\alpha : A \to \mathscr{P}(X)$ be a set-valued function defined by $\alpha(x) = \frac{x^2}{\{0,b\}}$ for all $x \in A$. Then (α, A) is a soft BCK-algebra over X. Let (β, I) be a soft set over X, where $I = \{0\} \subset A$ and $\beta : I \to \mathscr{P}(X)$ is a set-valued function defined by $\beta(x) = \frac{x}{\{0\}}$ for all $x \in I$. Then $\beta(0) = \{0\} \lhd_{pi} \alpha(0) = X$ and so $(\beta, I)\widetilde{\lhd}_{pi}(\alpha, A)$. But $\beta(0)$ is not an implicative ideal of X relative to $\alpha(0)$ since $(a * (b * a)) * 0 = 0 \in \beta(0)$ and $a \notin \beta(0)$, which means that (β, I) is not an implicative soft ideal of (α, A).

Example 3.9. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	b	d	0

Let (α, A) be a soft set over X, where $A = X$ and $\alpha : A \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(x) = \frac{x}{\{0,c,d\}}$ for all $x \in A$. Then (α, A) is a soft BCK-algebra over X. Now let (β, I) be a soft set over X, where $I = \{a, b, d\} \subset A$ and $\beta : I \to \mathscr{P}(X)$ is a set-valued function defined by $\beta(x) = \frac{x^2}{\{0,c\}}$ for all $x \in I$. Then $\beta(a) = \{0, b, d\} \lhd_c \alpha(a) = \{0, a, d\}$, $\beta(b) = \{0, a, c\} \lhd_c \alpha(b) = \{0, b, c\}$ and $\beta(d) = \{0, a, c\} \lhd_c \alpha(d) = \{0, b, c, d\}$, and hence (β, I) is a commutative soft ideal of (α, A). But $\beta(d)$ is not an implicative ideal of X related to $\alpha(d)$ since $(b * (d * b)) * a = 0 \in \beta(d)$ and $b \notin \beta(d)$, and so (β, I) is not an implicative soft ideal of (α, A).

Based on the fact that every ideal of X is implicative if and only if it is both positive implicative and commutative (see [4]), the following characterization of an implicative soft ideal is straightforward.

Theorem 3.10. *Let (β, I) be a soft ideal of a soft BCK-algebra (α, A) over X. Then (β, I) is implicative if and only if (β, I) is both positive implicative and commutative.*

Theorem 3.11. *Let (β, I) and (β, J) be soft sets over X such that $I \subset J$. If (β, J) is an implicative soft ideal of a soft BCK-algebra (α, A) over X, then so is (β, I).*

Proof. Straightforward. □

The converse of Theorem 3.11 is not valid in general as seen in the following example.

Example 3.12. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	c	b	0

Let (α, A) be a soft set over X, where $A = X$ and $\alpha : A \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(x) = \frac{x^2}{\{0,a,b\}}$ for all $x \in A$. Then (α, A) is a soft BCK-algebra over X. Now let (β, I) be a soft set over X, where $I = \{b, d\} \subset A$ and $\beta : I \to \mathscr{P}(X)$ is a set-valued function defined by $\beta(x) = \frac{x}{\{0,a\}}$ for all $x \in I$. Then (β, I) is an implicative soft ideal of (α, A). If we take $J = \{a, b, d\}$ and define a set-valued function $\beta : J \to \mathscr{P}(X)$ by $\beta(x) = \frac{x}{\{0,a\}}$ for all $x \in J$, then $\beta(a) = \{0, a\}$ is not an implicative ideal of X related to $\alpha(a) = X$ since $(b * (c * b)) * 0 = 0 \in \beta(a)$ and $b \notin \beta(a)$. Hence (β, J) is not an implicative soft ideal of (α, A).

Theorem 3.13. *Let (α, A) be a soft BCK-algebra over X. For any soft sets (β_1, I_1) and (β_2, I_2) over X where $I_1 \cap I_2 \neq \emptyset$, we have*

$$(\beta_1, I_1)\widetilde{\triangleleft}_i(\alpha, A),\ (\beta_2, I_2)\widetilde{\triangleleft}_i(\alpha, A) \ \Rightarrow\ (\beta_1, I_1)\widetilde{\cap}(\beta_2, I_2)\widetilde{\triangleleft}_i(\alpha, A).$$

Proof. Using Definition 2.2, we can write $(\beta_1, I_1)\widetilde{\cap}(\beta_2, I_2) = (\beta, I)$, where $I = I_1 \cap I_2$ and $\beta(x) = \beta_1(x)$ or $\beta_2(x)$ for all $x \in I$. Obviously, $I \subset A$ and $\beta : I \to \mathscr{P}(X)$ is a mapping. Hence (β, I) is a soft set over X. Since $(\beta_1, I_1)\widetilde{\triangleleft}_i(\alpha, A)$ and $(\beta_2, I_2)\widetilde{\triangleleft}_i(\alpha, A)$, we know that $\beta(x) = \beta_1(x)\triangleleft_i\alpha(x)$ or $\beta(x) = \beta_2(x)\triangleleft_i\alpha(x)$ for all $x \in I$. Hence

$$(\beta_1, I_1)\widetilde{\cap}(\beta_2, I_2) = (\beta, I)\widetilde{\triangleleft}_i(\alpha, A).$$

This completes the proof. □

Corollary 3.14. *Let (α, A) be a soft BCK-algebra over X. For any soft sets (β, I) and (δ, I) over X, we have*

$$(\beta, I)\widetilde{\triangleleft}_i(\alpha, A),\ (\delta, I)\widetilde{\triangleleft}_i(\alpha, A) \ \Rightarrow\ (\beta, I)\widetilde{\cap}(\delta, I)\widetilde{\triangleleft}_i(\alpha, A).$$

Proof. Straightforward. □

Theorem 3.15. *Let (α, A) be a soft BCK-algebra over X. For any soft sets (β, I) and (δ, J) over X in which I and J are disjoint, we have*

$$(\beta, I)\widetilde{\triangleleft}_i(\alpha, A),\ (\delta, J)\widetilde{\triangleleft}_i(\alpha, A) \ \Rightarrow\ (\beta, I)\widetilde{\cup}(\delta, J)\widetilde{\triangleleft}_i(\alpha, A).$$

Proof. Assume that $(\beta, I)\widetilde{\lhd}_i(\alpha, A)$ and $(\delta, J)\widetilde{\lhd}_i(\alpha, A)$. By means of Definition 2.3, we can write $(\beta, I)\widetilde{\cup}(\delta, J) = (\rho, U)$ where $U = I \cup J$ and for every $x \in U$,

$$\rho(x) = \begin{cases} \beta(x) & \text{if } x \in I \setminus J, \\ \delta(x) & \text{if } x \in J \setminus I, \\ \beta(x) \cup \delta(x) & \text{if } x \in I \cap J. \end{cases}$$

Since $I \cap J = \emptyset$, either $x \in I \setminus J$ or $x \in J \setminus I$ for all $x \in U$. If $x \in I \setminus J$, then $\rho(x) = \beta(x) \lhd_i \alpha(x)$ since $(\beta, I)\widetilde{\lhd}_i(\alpha, A)$. If $x \in J \setminus I$, then $\rho(x) = \delta(x) \lhd_i \alpha(x)$ since $(\delta, J)\widetilde{\lhd}_i(\alpha, A)$. Thus $\rho(x) \lhd_i \alpha(x)$ for all $x \in U$, and so $(\beta, I)\widetilde{\cup}(\delta, J) = (\rho, U)\widetilde{\lhd}_i(\alpha, A)$. □

If I and J are not disjoint in Theorem 3.15, then Theorem 3.15 is not true in general as seen in the following example.

Example 3.16. Let (α, A) be a soft BCK-algebra over X which is given in Example 3.8. Take $I := \{b, c, d\}$ and let (β, I) be a soft set over X which is given by $\beta(x) = \frac{x}{\{0\}}$ for all $x \in I$. Then $(\beta, I)\widetilde{\lhd}_{pi}(\alpha, A)$ and $(\beta, I)\widetilde{\lhd}_c(\alpha, A)$. It follows from Theorem 3.10 that $(\beta, I)\widetilde{\lhd}_i(\alpha, A)$. Now, let $J := \{b\}$ which is not disjoint with I, and let $\delta : J \to \mathscr{P}(X)$ be a set-valued function defined by $\delta(x) = \frac{x^2}{\{0\}}$ for all $x \in J$. Then $(\delta, J)\widetilde{\lhd}_{pi}(\alpha, A)$ and $(\delta, J)\widetilde{\lhd}_c(\alpha, A)$. We also have $(\delta, J)\widetilde{\lhd}_i(\alpha, A)$ by Theorem 3.10. But if $(\rho, U) := (\beta, I)\widetilde{\cup}(\delta, J)$, then $\rho(b) = \beta(b) \cup \delta(b) = \{0, a, b, c\}$ is not an implicative ideal of X related to $\alpha(b) = \{0, b, c, d\}$ since $(d * (0 * d)) * c = b \in \rho(b)$ and $d \notin \rho(b)$. Hence $(\rho, U) = (\beta, I)\widetilde{\cup}(\delta, J)$ is not an implicative soft ideal of (α, A).

4. Implicative idealistic soft BCK-algebras

Definition 4.1. Let (α, A) be a soft set over X. Then (α, A) is called an *idealistic soft BCK-algebra* over X if $\alpha(x)$ is an ideal of X for all $x \in A$.

Definition 4.2. Let (α, A) be a soft set over X. Then (α, A) is called a *positive implicative idealistic soft BCK-algebra* over X if it satisfies:

$$(\forall x \in A)\,(\alpha(x) \lhd_{pi} X). \tag{4.1}$$

Definition 4.3. Let (α, A) be a soft set over X. Then (α, A) is called a *commutative idealistic soft BCK-algebra* over X if it satisfies:

$$(\forall x \in A)\,(\alpha(x) \lhd_{c} X). \tag{4.2}$$

Definition 4.4. Let (α, A) be a soft set over X. Then (α, A) is called an *implicative idealistic soft BCK-algebra* over X if it satisfies:

$$(\forall x \in A)\,(\alpha(x) \lhd_{i} X). \tag{4.3}$$

Let us illustrate this definition using the following examples.

Example 4.5. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Let (α, A) be a soft set over X, where $A = \{0, a, c\}$ and $\alpha : A \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(x) = \frac{x}{\{0,a,d\}}$ for all $x \in A$. Then $\alpha(0) = \{0, a, d\} = \alpha(a)$ and $\alpha(c) = X$, which are implicative ideals of X. Hence (α, A) is an implicative idealistic soft BCK-algebra over X.

Note that every implicative idealistic soft BCK-algebra over X is both a commutative idealistic soft BCK-algebra over X and a positive implicative idealistic soft BCK-algebra over X. In the following examples, we know that there exists a commutative (or, positive implicative) idealistic soft BCK-algebra over X which is not an implicative idealistic soft BCK-algebra over X.

Example 4.6. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	a
b	b	a	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (α, A) be a soft set over X, where $A = \{b, c, d\}$ and $\alpha : A \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(x) = \frac{x}{\{0,d\}}$ for all $x \in A$. Then $\alpha(b) = \{0, a, b, d\}$, $\alpha(c) = \{0, c, d\}$ and $\alpha(d) = \{0, d\}$, which are commutative ideals of X. Hence (α, A) is a commutative idealistic soft BCK-algebra over X. But $\alpha(c)$ is not an implicative ideal of X since $(a * (b * a)) * 0 = 0 \in \alpha(c)$ and $a \notin \alpha(c)$, and so (α, A) is not an implicative idealistic soft BCK-algebra over X.

Example 4.7. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

Let (α, A) be a soft set over X, where $A = \{0, b, c, d\}$ and $\alpha : A \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(x) = \frac{x^2}{\{0,b\}}$ for all $x \in A$. Then (α, A) is a positive implicative idealistic soft BCK-algebra over X. But $\alpha(c) = \alpha(d) = \{0, b\}$ is not an implicative ideal of X since $(a * (c * a)) * 0 = 0 \in \{0, b\}$ and $a \notin \{0, b\}$, and hence (α, A) is not an implicative idealistic soft BCK-algebra over X.

Theorem 4.8. *A soft set (α, A) over X is an implicative idealistic soft BCK-algebra over X if and only if (α, A) is both a commutative idealistic soft BCK-algebra over X and a positive implicative idealistic soft BCK-algebra over X.*

Proof. Straightforward. □

Proposition 4.9. *Let (α, A) and (α, B) be soft sets over X where $B \subseteq A$. If (α, A) is an implicative idealistic soft BCK-algebra over X, then so is (α, B).*

Proof. Straightforward. □

The converse of Proposition 4.9 is not true in general as seen in the following example.

Example 4.10. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	0
c	c	c	c	0	0
d	d	d	d	d	0

Let (α, A) be a soft set over X, where $A = \{0, b, c, d\}$ and $\alpha : A \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(x) = \frac{x}{\{0,c\}}$ for all $x \in A$. Then $\alpha(0) = \alpha(c) = \{0, c\}$ is not an implicative ideal of X since $(a * (d * a)) * c = 0 \in \{0, c\}$ and $a \notin \{0, c\}$, and so (α, A) is not an implicative idealistic soft BCK-algebra over X. But if we take $B := \{b, d\} \subseteq A$, then $\alpha(b) = \{0, a, b, c\} \lhd_i X$ and $\alpha(d) = X \lhd_i X$. Hence (α, B) is an implicative idealistic soft BCK-algebra over X.

Theorem 4.11. *Let (α, A) and (β, B) be two implicative idealistic soft BCK-algebras over X. If $A \cap B \neq \emptyset$, then the intersection $(\alpha, A)\widetilde{\cap}(\beta, B)$ is an implicative idealistic soft BCK-algebra over X.*

Proof. Using Definition 2.2, we can write $(\alpha, A)\widetilde{\cap}(\beta, B) = (\delta, C)$, where $C = A \cap B$ and $\delta(x) = \alpha(x)$ or $\beta(x)$ for all $x \in C$. Note that $\delta : C \to \mathscr{P}(X)$ is a mapping, and therefore (δ, C) is a soft set over X. Since (α, A) and (β, B) are implicative idealistic soft BCK-algebras over X, it follows that $\delta(x) = \alpha(x)$ is an implicative ideal of X, or $\delta(x) = \beta(x)$ is an implicative ideal of X for all $x \in C$. Hence $(\delta, C) = (\alpha, A)\widetilde{\cap}(\beta, B)$ is an implicative idealistic soft BCK-algebra over X. □

Corollary 4.12. *Let* (α, A) *and* (β, A) *be two implicative idealistic soft BCK-algebras over* X. *Then their intersection* $(\alpha, A)\widetilde{\cap}(\beta, A)$ *is an implicative idealistic soft BCK-algebra over* X.

Proof. Straightforward. □

Theorem 4.13. *Let* (α, A) *and* (β, B) *be two implicative idealistic soft BCK-algebras over* X. *If* A *and* B *are disjoint, then the union* $(\alpha, A)\widetilde{\cup}(\beta, B)$ *is an implicative idealistic soft BCK-algebra over* X.

Proof. Using Definition 2.3, we can write $(\alpha, A)\widetilde{\cup}(\beta, B) = (\delta, C)$, where $C = A \cup B$ and for every $x \in C$,

$$\delta(x) = \begin{cases} \alpha(x) & \text{if } x \in A \setminus B, \\ \beta(x) & \text{if } x \in B \setminus A, \\ \alpha(x) \cup \beta(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \setminus B$ or $x \in B \setminus A$ for all $x \in C$. If $x \in A \setminus B$, then $\delta(x) = \alpha(x)$ is an implicative ideal of X since (α, A) is an implicative idealistic soft BCK-algebra over X. If $x \in B \setminus A$, then $\delta(x) = \beta(x)$ is an implicative ideal of X since (β, B) is an implicative idealistic soft BCK-algebra over X. Hence $(\delta, C) = (\alpha, A)\widetilde{\cup}(\beta, B)$ is an implicative idealistic soft BCK-algebra over X. □

Theorem 4.14. *If* (α, A) *and* (β, B) *are implicative idealistic soft BCK-algebras over* X, *then* $(\alpha, A)\widetilde{\wedge}(\beta, B)$ *is an implicative idealistic soft BCK-algebra over* X.

Proof. By means of Definition 2.4, we know that

$$(\alpha, A)\widetilde{\wedge}(\beta, B) = (\delta, A \times B),$$

where $\delta(x, y) = \alpha(x) \cap \beta(y)$ for all $(x, y) \in A \times B$. Since $\alpha(x)$ and $\beta(y)$ are implicative ideals of X, the intersection $\alpha(x) \cap \beta(y)$ is also an implicative ideal of X. Hence $\delta(x, y)$ is an implicative ideal of X for all $(x, y) \in A \times B$, and therefore $(\alpha, A)\widetilde{\wedge}(\beta, B) = (\delta, A \times B)$ is an implicative idealistic soft BCK-algebra over X. □

Definition 4.15. An implicative idealistic soft BCK-algebra (α, A) over X is said to be *trivial* (resp., *whole*) if $\alpha(x) = \{0\}$ (resp., $\alpha(x) = X$) for all $x \in A$.

Example 4.16. (1) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra which is described in Example 3.8. Consider $A = \{c, d\} \subset X$ and a set-valued function $\alpha : A \to \mathscr{P}(X)$ defined by $\alpha(x) = \frac{x}{\{0,b\}}$ for all $x \in A$. Then $\alpha(c) = \frac{c}{\{0,b\}} = X$ and $\alpha(d) = \frac{d}{\{0,b\}} = X$. Hence (α, A) is a whole implicative idealistic soft BCK-algebra over X.

(2) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra which is described in Example 4.10. For $A = \{b, d\} \subset X$, let a set-valued function $\alpha : A \to \mathscr{P}(X)$ be defined by $\alpha(x) = \frac{x}{\{0,c,d\}}$ for all $x \in A$. Then $\alpha(b) = \frac{b}{\{0,c,d\}} = X$ and $\alpha(d) = \frac{d}{\{0,c,d\}} = X$, and so (α, A) is a whole implicative idealistic soft BCK-algebra over X.

Example 4.17. (1) Let $X = \{0, a, b, c, d\}$ be the BCK-algebra which is described in Example 3.6. Let $\alpha : \{0\} \to \mathscr{P}(X)$ be a set-valued function given by $\alpha(0) = \frac{0}{\{0\}}$. Then $(\alpha, \{0\})$ is a trivial implicative idealistic soft BCK-algebra over X.

(2) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

If $\alpha : \{0\} \to \mathscr{P}(X)$ is a set-valued function defined by $\alpha(0) = \frac{0}{\{0\}}$, then $(\alpha, \{0\})$ is a trivial implicative idealistic soft BCK-algebra over X.

The following example shows that there exists a BCK-algebra X such that a soft set $(\alpha, \{0\})$ may not be a trivial implicative idealistic soft BCK-algebra over X, where $\alpha : \{0\} \to \mathscr{P}(X)$ is given by $\alpha(0) = \frac{0}{\{0\}}$.

Example 4.18. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	b	a	0	a
d	d	b	a	a	0

Let $\alpha : \{0\} \to \mathscr{P}(X)$ be a set-valued function given by $\alpha(0) = \frac{0}{\{0\}}$. Then $\alpha(0) = \{0\}$ is not an implicative ideal of X since $(a * (b * a)) * 0 = 0$ and $a \neq 0$. This means that $(\alpha, \{0\})$ is not a trivial implicative idealistic soft BCK-algebra over X.

Proposition 4.19. *Let (α, A) be a soft set over X defined by $\alpha(x) = \frac{x}{\{0\}}$ for all $x \in A$. Then*

(i) *(α, A) is a trivial idealistic soft BCK-algebra over X if and only if $A = \{0\}$.*

(ii) *Assume that $A = \{0\}$. Then (α, A) is implicative if and only if X is implicative.*

Proof. (i) If $A = \{0\}$, then $\alpha(0) = \{0\} \lhd X$. Hence (α, A) is a trivial idealistic soft BCK-algebra over X. Conversely, assume that $A \neq \{0\}$. Then there exists $a (\neq 0) \in A$, and so $\{0, a\} \subseteq \alpha(a)$ since $a * a = 0$. This is a contradiction.

(ii) Note that X is implicative if and only if $\{0\}$ is an implicative ideal of X. Hence it is straightforward. □

Lemma 4.20. *Let $f : X \to Y$ be an epimorphism of BCK-algebras. If I is an implicative ideal of X containing* $\ker(f)$*, then $f(I)$ is an implicative ideal of Y.*

Proof. Let I be an implicative ideal of X containing $\ker(f)$. It is clear that $0 \in f(I)$. Let $x', y', z' \in Y$ be such that $z' \in f(I)$ and $(x' * (y' * x')) * z' \in f(I)$. Then there exist $a, z \in I$ such that $f(z) = z'$ and $f(a) = (x' * (y' * x')) * z'$. Since f is onto, $f(x) = x'$ and $f(y) = y'$ for some $x, y \in X$. Then

$$\begin{aligned} & f(((x * (y * x)) * z) * a) = f((x * (y * x)) * z) * f(a) \\ = \;& ((x' * (y' * x')) * z') * ((x' * (y' * x')) * z') = 0, \end{aligned} \tag{4.4}$$

and so $((x * (y * x)) * z) * a \in \ker(f)$. Hence $((x * (y * x)) * z) * a = k$ for some $k \in \ker(f)$. Using (a3), we know that

$$((x * (y * x)) * z) * k \leq a. \tag{4.5}$$

Since $a \in I$ and I is an ideal, it follows from (a3) and (2.2) that

$$((x * (y * x)) * k) * z = ((x * (y * x)) * z) * k \in I \tag{4.6}$$

so from (b2) that $(x*(y*x))*k \in I$. Since $k \in \ker(f) \subseteq I$ and I is an implicative ideal, we have $x \in I$; hence $x' = f(x) \in f(I)$. Therefore $f(I)$ is an implicative ideal of Y. □

Let $f : X \to Y$ be a mapping of BCK-algebras. For a soft set (α, A) over X, $(f(\alpha), A)$ is a soft set over Y where $f(\alpha) : A \to \mathscr{P}(Y)$ is defined by $f(\alpha)(x) = f(\alpha(x))$ for all $x \in A$.

Lemma 4.21. *Let $f : X \to Y$ be an onto homomorphism of BCK-algebras. Assume that every implicative ideal of X contains* $\ker(f)$*. If (α, A) is an implicative idealistic soft BCK-algebra over X, then $(f(\alpha), A)$ is an implicative idealistic soft BCK-algebra over Y.*

Proof. For every $x \in A$, we have $f(\alpha)(x) = f(\alpha(x))$ is an implicative ideal of Y since $\alpha(x)$ is an implicative ideal of X and its onto homomorphic image is also an implicative ideal of Y (see Lemma 4.20). Hence $(f(\alpha), A)$ is an implicative idealistic soft BCK-algebra over Y. □

Theorem 4.22. *Let $f : X \to Y$ be an onto homomorphism of BCK-algebras and let (α, A) be an implicative idealistic soft BCK-algebra over X. Assume that every implicative ideal of X contains* $\ker(f)$.

- (i) *If $\alpha(x) \subset \ker(f)$ for all $x \in A$, then $(f(\alpha), A)$ is a trivial implicative idealistic soft BCK-algebra over Y.*
- (ii) *If (α, A) is whole, then $(f(\alpha), A)$ is a whole implicative idealistic soft BCK-algebra over Y.*

Proof. (i) By Lemma 4.21, $(f(\alpha), A)$ is an implicative idealistic soft BCK-algebra over Y. Assume that $\alpha(x) \subset \ker(f)$ for all $x \in A$. Then $f(\alpha)(x) = f(\alpha(x)) \subset f(\ker(f)) = \{0\} \subset f(\alpha)(x)$, and so $f(\alpha)(x) = \{0\}$ for all $x \in A$. It follows from Definition 4.15 that $(f(\alpha), A)$ is a trivial implicative idealistic soft BCK-algebra over Y.

(ii) Suppose that (α, A) is whole. Then $\alpha(x) = X$ for all $x \in A$, and so $f(\alpha)(x) = f(\alpha(x)) = f(X) = Y$ for all $x \in A$. It follows from Lemma 4.21 and Definition 4.15 that $(f(\alpha), A)$ is a whole implicative idealistic soft BCK-algebra over Y. □

5. Conclusions

We introduced the notion of implicative soft ideals in BCK-algebras and implicative idealistic soft BCK-algebras, and investigated related properties. We discussed relations between implicative soft ideals and commutative (resp. positive implicative) soft ideals. Also, we provided relations between implicative idealistic soft BCK-algebras and commutative (resp. positive implicative) idealistic soft BCK-algebras. Based on these results, we will apply soft sets to another type of ideals in BCI-algebras, and investigate relations between fuzzy type of ideals, rough type of ideals, and soft type of ideals.

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