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LOCALLY-ZERO GROUPOIDS AND THE CENTER OF $BIN(X)$

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Abstract. In this paper we introduce the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X, and show that if $(X, \bullet) \in ZBin(X)$, then $x \neq y$ implies $\{x, y\} = \{x \bullet y, y \bullet x\}$. Moreover, we show that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid.

1. Preliminaries

The notion of the semigroup $(Bin(X), \square)$ was introduced by H. S. Kim and J. Neggers ([2]). Given binary operations "$*$" and "$\bullet$" on a set X, they defined a product binary operation "$\square$" as follows: $x \square y := (x * y) \bullet (y * x)$. This in turn yields a binary operation on $Bin(X)$, the set of all groupoids defined on X turning $(Bin(X), \square)$ into a semigroup with identity $(x * y = x)$, the left-zero-semigroup, and an analog of negative one in the right-zero-semigroup.

Theorem 1.1 ([2]). *The collection $(Bin(X), \square)$ of all binary systems (groupoids or algebras) defined on X is a semigroup, i.e., the operation $\square$ as defined in general is associative. Furthermore, the left-zero-semigroup is an identity for this operation.*

Example 1.2 ([2]). Let $(R, +, \cdot, 0, 1)$ be a commutative ring with identity and let $L(R)$ denote the collection of groupoids $(R, *)$ such that for all $x, y \in R$

$$x * y = ax + by + c,$$

where $a, b, c \in R$ are fixed constants. We shall consider such groupoids to be *linear groupoids*. Notice that $a = 1, b = c = 0$ yields $x * y = 1 \cdot x = x$, and thus the left-zero-semigroup on R is a linear groupoid. Now, suppose that $(R, *)$ and $(R, \bullet)$ are linear groupoids where $x*y = ax+by+c$ and $x \bullet y = dx+ey+f$. Then $x \square y = d(ax+by+c)+e(ay+bx = c)+f = (da+eb)x+(db+ea)y+(d+e)c+f$, whence $(R, \square) = (R, *)\square(R, \bullet)$ is also a linear groupoid, i.e., $(L(R), \square)$ is a semigroup with identity.

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Example 1.3 ([2])**.** Suppose that in $Bin(X)$ we consider all those groupoids $(X, *)$ with the *orientation property*: $x * y \in \{x, y\}$ for all x and y. Thus, $x * x = x$ as a consequence. If $(X, *)$ and $(X, \bullet)$ both have the orientation property, then for $x \,\square\, y = (x * y) \bullet (y * x)$ we have the possibilities: $x * x = x$, $y * y = y$, $x * y \in \{x, y\}$ and $y * x \in \{x, y\}$, so that $x \,\square\, y \in \{x, y\}$. It follows that if $OP(X)$ denotes this collection of groupoids, then $(OP(X), \square)$ is a subsemigroup of $(Bin(X), \square)$. In a sequence of papers Nebeský ([3, 4, 5]) has sought to associate with graphs (V, E) groupoids $(V, *)$ with various properties and conversely. He defined a *travel groupoid* $(X, *)$ as a groupoid satisfying the axioms: $(u * v) * u = u$ and $(u * v) * v = u$ implies $u = v$. If one adds these two laws to the orientation property, then $(X, *)$ is an OP-travel-groupoid. In this case $u * v = v$ implies $v * u = u$, i.e., $uv \in E$ implies $vu \in E$, i.e., the digraph (X, E) is a (simple) graph if $uu \notin E$, with $u * u = u$. Also, if $u \neq v$, then $u * v = u$ implies $(u * v) * v = u * v = u$ is impossible, whence $u * v = v$ and $uv \in E$, so that (X, E) is a complete (simple) graph.

2. The center of $Bin\,(X)$ protect

Let $ZBin(X)$ denote the collection of elements $(X, \bullet)$ of $Bin(X)$ such that $(X, *) \,\square\, (X, \bullet) = (X, \bullet) \,\square\, (X, *)$, $\forall (X, *) \in Bin(X)$, i.e., $ZBin\,(X) = \{(X, \bullet) \in Bin\,(X) \,|\, (X, *) \,\square\, (X, \bullet) = (X, \bullet) \,\square\, (X, *), \forall (X, *) \in Bin(X)\}$. We call $ZBin(X)$ the *center* of the semigroup $Bin(X)$.

Proposition 2.1. *The left-zero-semigroup and the right-zero-semigroup on X are both in $ZBin(X)$.*

Proof. Given a groupoid $(X, *)$, let $(X, \bullet)$ be a left-zero-semigroup. Then $(x \bullet y) * (y \bullet x) = x * y = (x * y) \bullet (y * x)$ for all $x, y \in X$, proving $(X, \bullet) \in ZBin(X)$. Similarly, it holds for the right-zero-semigroup. □

Proposition 2.2. *If $(X, \bullet) \in ZBin(X)$, then $x \bullet x = x$ for all $x \in X$.*

Proof. If $(X, \bullet) \in ZBin(X)$, then $(X, \bullet) \square (X, *) = (X, *) \square (X, \bullet)$ for all $(X, *) \in Bin(X)$. Let $(X, *) \in Bin(X)$ defined by $x * y = a$ for any $x, y \in X$ where $a \in X$. Then $(x \bullet y) * (y \bullet x) = a$ and $(x * y) \bullet (y * x) = a \bullet a$ for any $x, y \in X$. Hence we obtain $a \bullet a = a$. If we change $(X, *)$ in $Bin\,(X)$ so that $x * y = b$ for every $x, y \in X$ and b is any other element of X, then we find that $a \bullet a = a$ for any $a \in X$. □

Any set can be well-ordered by well-ordering principle, and a well-ordered set is linearly ordered. With this notion we prove the following.

Theorem 2.3. *If $(X, \bullet) \in ZBin(X)$, then $x \neq y$ implies $\{x, y\} = \{x \bullet y, y \bullet x\}$.*

Proof. Let $(X, <)$ be a linearly ordered set and let $(X, *) \in Bin(X)$ be defined by

$$x * y := \min\{x, y\}, \;\; \forall x, y \in X. \tag{1}$$

Then we have the following:

$$(2) \qquad (x * y) \bullet (y * x) = \begin{cases} x & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

Similarly, we have

$$(3) \qquad (x \bullet y) * (y \bullet x) = \min\{x \bullet y, y \bullet x\} \in \{x \bullet y, y \bullet x\}.$$

If $(X, \bullet) \in ZBin(X)$, then $x < y$ implies $x \in \{x \bullet y, y \bullet x\}$ for all $x, y \in X$. Similarly, if we define $(X, *) \in Bin(X)$ by $x * y := \max\{x, y\}$ for all $x, y \in X$, then $x < y$ implies $y \in \{x \bullet y, y \bullet x\}$ for all $x, y \in X$ when $(X, \bullet) \in ZBin(X)$. In any case, we obtain that if $(X, \bullet) \in ZBin(X)$, then

$$(4) \qquad x, y \in \{x \bullet y, y \bullet x\}.$$

We consider four cases: (i) $x < y, x \bullet y < y \bullet x$; (ii) $x < y, y \bullet x < x \bullet y$; (iii) $y < x, x \bullet y < y \bullet x$; (iv) $y < x, y \bullet x < x \bullet y$. Routine calculations give us the conclusion that $\{x, y\} = \{x \bullet y, y \bullet x\}$. □

Proposition 2.4. *Let $(X, \bullet) \in ZBin(X)$. If $x \neq y$ in X, then $(\{x, y\}, \bullet)$ is either a left-zero-semigroup or a right-zero-semigroup.*

Proof. Assume that $(X, \bullet)$ is not a left-zero-semigroup and $x \neq y$ in X. Then $(X, \bullet)$ has a subtable:

$\bullet$	x	y
x	x	y
y	a	y

where $a \in \{x, y\}$. Note that $x \bullet x = x, y \bullet y = y$ by Proposition 2.2. Let $(X, *) \in Bin(X)$ such that X has a subtable:

$*$	x	y
x	x	x
y	x	y

Since $(X, \bullet) \in ZBin(X)$, we have $(x * y) \bullet (y * x) = (x \bullet y) * (y \bullet x)$ and hence $x \bullet x = y * a$. If $a = x$, then $x \bullet x = y * x = x$. If $a = y$, then $x = x \bullet x = y * y = y$, a contradiction. Hence $(X, \bullet)$ should have a subtable:

$\bullet$	x	y
x	x	y
y	x	y

This means $(X, \bullet)$ should be a right-zero-semigroup. Similarly, if $(X, \bullet)$ is not a right-zero-semigroup, then it must have a 2×2 table of a left-zero-semigroup. □

Proposition 2.5. *If $(\{x, y\}, \bullet)$ is either a left-zero-semigroup or a right-zero-semigroup for any $x \neq y$ in X, then $(X, \bullet) \in ZBin(X)$.*

Proof. Given $(X,*) \in Bin(X)$, let $x \neq y$ in X. Consider $(x*y)\bullet(y*x)$ and $(x\bullet y)*(y\bullet x)$. If we assume that $(\{x,y\},\bullet)$ is a left-zero-semigroup, then $(x*y)\bullet(y*x) = x*y = (x\bullet y)*(y\bullet x)$. Similarly, if we assume that $(\{x,y\},\bullet)$ is a right-zero-semigroup, then $(x*y)\bullet(y*x) = y*x = (x\bullet y)*(y\bullet x)$. Hence $(X,\bullet) \in ZBin(X)$. □

Example 2.6. Let $X := \{a,b,c\}$ with the following table:

$\bullet$	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

Then $(X,\bullet)$ is neither a left-zero-semigroup nor a right-zero-semigroup, while it has the following subtables:

$\bullet$	a	b
a	a	a
b	b	b

$\bullet$	a	c
a	a	c
c	a	c

$\bullet$	b	c
b	b	b
c	c	c

By applying Proposition 2.5, we can see that $(X,\bullet) \in ZBin(X)$.

Proposition 2.7. *Let $Ab(X)$ be the collection of all commutative binary systems on X. Then $(Ab(X) \cap ZBin(X), \square)$ is a right ideal of $(ZBin(X), \square)$.*

Proof. Let $(X,\bullet) \in ZBin(X)$ and $(X,*) \in (Ab(X) \cap ZBin(X))$. Then by Proposition 2.2, we have $x\square y = (x*y)\bullet(y*x) = (x*y)\bullet(x*y) = x*y$. Also, by Proposition 2.2, we get $y\square x = (y*x)\bullet(x*y) = (y*x)\bullet(y*x) = y*x$. Therefore, $(X,*)\,\square\,(X,\bullet) \in (Ab(X) \cap ZBin(X))$ and so $(Ab(X) \cap ZBin(X))\,\square ZBin(X) \subseteq (Ab(X) \cap ZBin(X))$. □

3. Locally-zero groupoids

A groupoid $(X,\bullet)$ is said to be *locally-zero* if (i) $x\bullet x = x$ for all $x \in X$; (ii) for any $x \neq y$ in X, $(\{x,y\},\bullet)$ is either a left-zero-semigroup or a right-zero-semigroup.

Using Propositions 2.2, 2.4 and 2.5 we obtain the following.

Theorem 3.1. *A groupoid $(X,\bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid.*

Given any two elements $x,y \in X$, there exists exactly one left-zero-semigroup and one right-zero-semigroup, and so if we apply Theorem 3.1 we have the following corollary.

Corollary 3.2. *If $|X| = n$, there are $2^{\binom{n}{2}}$ different (but may not be isomorphic) locally-zero groupoids.*

For example, if $n = 3$, there are $2^3 = 8$ such groupoids, i.e.,

$\bullet$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

$\bullet$	a	b	c
a	a	a	a
b	b	b	c
c	c	b	c

$\bullet$	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

$\bullet$	a	b	c
a	a	b	a
b	a	b	b
c	c	c	c

$\bullet$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

$\bullet$	a	b	c
a	a	a	c
b	b	b	c
c	a	b	c

$\bullet$	a	b	c
a	a	b	a
b	a	b	c
c	c	b	c

$\bullet$	a	b	c
a	a	b	c
b	a	b	b
c	a	c	c

Corollary 3.3. *The collection of all locally-zero groupoids on X forms a subsemigroup of $(Bin(X), \square)$.*

Proof. Let $x \neq y$ in X. If $(\{x, y\}, \bullet)$ is a left-zero-semigroup and $(\{x, y\}, *)$ is a right-zero semigroup, then $x \square y = (x \bullet y) * (y \bullet x) = x * y = y$, $y \square x = (y \bullet x) * (x \bullet y) = y * x = x$, i.e., $(\{x, y\}, \square)$ is a right-zero-semigroup. Similarly, we can prove the other three cases, i.e.,

$\square$	L	R
L	L	R
R	R	L

where L and R denote the "left-zero-semigroup" and the "right-zero-semigroup", respectively. This proves that the collection of all locally-zero groupoids on X forms a subsemigroup of $(Bin(X), \square)$. $\square$

Using Corollary 3.3, we can see that $(X, \bullet)\square(X, *)$ belongs to the center $ZBin(X)$ of $Bin(X)$ for any $(X, \bullet), (X, *) \in ZBin(X)$.

Proposition 3.4. *Not all locally-zero groupoids are semigroups.*

Proof. Consider $(X, \bullet)$ where $X := \{a, b, c\}$ and "$\bullet$" is given by the following table:

$\bullet$	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

Then it is easy to see that $(X, \bullet)$ is locally-zero. Consider the subtables:

$\bullet$	a	b
a	a	a
b	b	b

$\bullet$	a	c
a	a	c
c	a	c

$\bullet$	b	c
b	b	b
c	c	c

and notice that $(\{a, b\}, \bullet)$, $(\{a, c\}, \bullet)$ and $(\{b, c\}, \bullet)$ are a left-, a right- and a left-zero-semigroup, respectively. But $(a \bullet b) \bullet c = a \bullet c = c$, while $a \bullet (b \bullet c) = a \bullet b = a$. Hence $(X, \bullet)$ fails to be a semigroup and the result follows. $\square$

Proposition 3.5. *Let $(X, \bullet)$ be a locally-zero groupoid. If $(X, \bullet)$ is a semigroup, then it is either a left-zero-semigroup or a right-zero-semigroup.*

Proof. Suppose that $(X, \bullet)$ is a semigroup. Then $(x \bullet y) \bullet z = x \bullet (y \bullet z)$ for all $x, y, z \in X$. By Theorem 3.1, $(X, \bullet) \in ZBin(X)$, and then by Proposition 2.4, $(\{x, y\}, \bullet)$ is either a left- or a right-zero-semigroup for any $x \neq y$. In fact, $(\{x, z\}, \bullet)$ and $(\{y, z\}, \bullet)$ are also either a left- or a right-zero-semigroup for any $x \neq z$ and any $y \neq z$, respectively. Assume that $(\{x, y\}, \bullet)$, $(\{x, z\}, \bullet)$ and $(\{y, z\}, \bullet)$ are a left-, a right- and a left-zero-semigroup, respectively. Then, $(x \bullet y) \bullet z = x \bullet z = z$ while $x \bullet (y \bullet z) = x \bullet y = x$, a contradiction. Similarly, this leads to a contradiction if we assume that $(\{x, y\}, \bullet)$, $(\{x, z\}, \bullet)$ and $(\{y, z\}, \bullet)$ are a right-, a left- and a right-zero-semigroup, respectively. Now suppose that $(\{x, y\}, \bullet)$, $(\{x, z\}, \bullet)$ and $(\{y, z\}, \bullet)$ are a left-, a left- and a right-zero-semigroup, respectively. Then, $(y \bullet x) \bullet z = y \bullet z = z$ while $y \bullet (x \bullet z) = y \bullet x = y$, a contradiction. Moreover, this leads to a contradiction if we assume that $(\{x, y\}, \bullet)$, $(\{x, z\}, \bullet)$ and $(\{y, z\}, \bullet)$ are a right-, a right- and a left-zero-semigroup, respectively. Hence, the only two other cases are when all three subgroupoids are either all left- or all right-zero-semigroups. Therefore, $(X, \bullet)$ is either a left- or a right-zero-semigroup. □

Proposition 3.6. *Let $(X, \bullet)$ be a locally-zero groupoid. Then*

$$(X, \bullet) \,\square\, (X, \bullet) = (X, \square)$$

is the left-zero-semigroup on X.

Proof. Suppose that $(\{x, y\}, \bullet)$ is the right-zero-semigroup. Then $x \square y = (x \bullet y) \bullet (y \bullet x) = y \bullet x = x$. On the other hand, if $(\{x, y\}, \bullet)$ is the left-zero-semigroup, then $x \square y = (x \bullet y) \bullet (y \bullet x) = x \bullet y = x$. Thus in both cases, $x \square y = x$ for all $x \in X$ making $(X, \square)$ the left-zero-semigroup. □

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