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HEREDITARY HEMIMORPHY OF $\{-k\}$-HEMIMORPHIC TOURNAMENTS FOR $k \geq 5$

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Abstract. Let $T = (V, A)$ be a tournament. With every subset X of V is associated the subtournament $T[X] = (X, A \cap (X \times X))$ of T, induced by X. The dual of T, denoted by T^*, is the tournament obtained from T by reversing all its arcs. Given a tournament $T^{'} = (V, A^{'})$ and a non-negative integer k, T and $T^{'}$ are $\{-k\}$-hemimorphic provided that for all $X \subset V$, with $|X| = k$, $T[V-X]$ and $T^{'}[V-X]$ or $T^*[V-X]$ and $T^{'}[V-X]$ are isomorphic. The tournaments T and $T^{'}$ are said to be hereditarily hemimorphic if for all subset X of V, the subtournaments $T[X]$ and $T^{'}[X]$ are hemimorphic. The purpose of this paper is to establish the hereditary hemimorphy of the $\{-k\}$-hemimorphic tournaments on at least $k + 7$ vertices, for every $k \geq 5$.

1. Introduction

A *tournament* $T = (V, A)$ (or $(V(T), A(T))$) consists of a finite *vertex* set V with an *arc* set A of ordered pairs of distinct vertices, satisfying: for $x, y \in V$, with $x \neq y$, $(x, y) \in A$ if and only if $(y, x) \notin A$, in this case we write $x \to y$. Given two sets of vertices A and B, write $A \to B$ to mean that there is an arc from any element of A to any element of B. For singletons, just write $a \to B$ for $\{a\} \to B$ and with $A \to b$ as well. The *cardinality* of T is that of V. This cardinality $|V|$ is also denoted by $|T|$. For each $x \in V$, we denote by $N^+(x)$ (resp. $N^-(x)$) the set $\{y \in V : (x, y) \in A\}$ (resp. $\{y \in V : (y, x) \in A\}$). The *score* of a vertex x (in T), denoted by $s_T(x)$, is the cardinality of $N^+(x)$. The *dual* of T is the tournament $T^* = (V, A^*)$ defined by: for all $x, y \in V$, $(y, x) \in A^*$ if and only if $(x, y) \in A$. With every subset X of V is associated the *subtournament* $T[X] = (X, A \cap (X \times X))$ of T induced by X. The subtournament $T[V - X]$ is also denoted by $T - X$. For each $x \in V$, the subtournament $T - \{x\}$ is denoted by $T - x$. Say that a set W of vertices satisfies a property if the subtournament $T[W]$ enjoys it. A *transitive tournament* or a *total order* is a tournament T such that for all $x, y, z \in V(T)$, if $(x, y) \in A(T)$ and $(y, z) \in A(T)$, then

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$(x,z) \in A(T)$. If $T = (\{x_1, \dots, x_n\}, A)$ is a total order such that: $(x_i, x_j) \in A$ if and only if $i < j$, T is said to be the total order: $x_1 < \cdots < x_n$ and x_1 (resp. x_n) is denoted by $\min(T)$ (resp. $\max(T)$). In another respect, given two tournaments $T = (V, A)$ and $T^{'} = (V^{'}, A^{'})$, a bijection f from V onto $V^{'}$ is an *isomorphism* from T onto T' provided that for any $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A^{'}$. The tournaments T and $T^{'}$ are then said to be isomorphic, which is denoted by $T \simeq T^{'}$, if there exists an isomorphism from T onto $T^{'}$. A tournament T is said to be *selfdual* if T and T^* are isomorphic. A *hemimorphism* [13] from T onto $T^{'}$ is either an isomorphism from T onto $T^{'}$ or an isomorphism from T^* onto $T^{'}$. Consider two tournaments $T = (V, A)$ and $T^{'} = (V, A^{'})$, with $|V| = n \geq 2$, a non-negative integer k and a set F of non-negative integers. The tournaments T and $T^{'}$ are $\{k\}$*-hypomorphic* (resp. $\{k\}$*-hemimorphic*), whenever for every subset X of V such that $|X| = k$, the subtournaments $T[X]$ and $T^{'}[X]$ are isomorphic (resp. hemimorphic). Say that T and $T^{'}$ are $\{-k\}$-hypomorphic (resp. $\{-k\}$-hemimorphic), whenever for every subset X of V such that $|X| = n - k$, the subtournaments $T[X]$ and $T^{'}[X]$ are isomorphic (resp. hemimorphic). We call the tournament T $\{-k\}$-selfdual if it is $\{-k\}$-hypomorphic to T^*. The tournament T is $\{k\}$*-monomorphic* (resp. $\{-k\}$*-monomorphic*) whenever for every subsets X, Y of V such that $|X| = |Y| = k$ (resp. $|X| = |Y| = n - k$), the subtournaments $T[X]$ and $T[Y]$ are isomorphic. The tournaments T and $T^{'}$ are F-hypomorphic (resp. F-hemimorphic), if for every $k \in F$, the tournaments T and $T^{'}$ are $\{k\}$-hypomorphic (resp. $\{k\}$-hemimorphic). The tournament T is F*-reconstructible* [15, 16, 17] (resp. F*-half-reconstructible* [13]) provided that every tournament F-hypomorphic (resp. F-hemimorphic) to T is isomorphic (resp. hemimorphic) to T. The $\{1, \dots, k\}$-hypomorphy (resp. $\{1, \dots, k\}$-hemimorphy) is also denoted by the $(\leq k)$-hypomorphy (resp. $(\leq k)$-hemimorphy). The tournament T is called $(\leq k)$*-selfdual*, if it is $(\leq k)$-hypomorphic to T^*. The tournaments T and $T^{'}$ are *hereditarily isomorphic* if for all $X \subseteq V$, $T[X]$ and $T^{'}[X]$ are isomorphic. Note that for the tournaments, the notion of the "hereditary isomorphy" was firstly given by K. B. Reid and C. Thomassen in [23]. From that, we introduce in this paper, the notion of "hereditary hemimorphy" in this way: the tournaments T and $T^{'}$ are *hereditarily hemimorphic* if for all $X \subseteq V$, $T[X]$ and $T^{'}[X]$ are hemimorphic.

Y. Boudabbous and G. Lopez [7] showed that if two tournaments T and $T^{'}$ are (≤ 7)-hemimorphic, then T and $T^{'}$ are hereditarily hemimorphic. Thus, by the combinatorial lemma introduced by M. Pouzet [21], it follows immediately that: Given an integer $k \geq 7$, if two tournaments T and $T^{'}$ defined on the same vertex set V, with $|V| \geq k + 7$, are $\{-k\}$-hemimorphic, then T and $T^{'}$ are hereditarily hemimorphic. The aim of this paper is to prove the hereditary hemimorphy of the $\{-k\}$-hemimorphic tournaments on at least $k + 7$ vertices where $k \in \{5, 6\}$.

The text is organized as follows: Section 2 recalls some results concerning hereditary isomorphy and hereditary hemimorphy. Then in Section 3: first, we give some definitions (strong connectivity, indecomposability,...), second, we recall Gallai's decomposition and finally, we give an overview about tournaments without diamonds. We consider Lopez's difference classes of $(\leq k)$-hypomorphy in Section 4. In Section 5, we recall and establish some results used in the proof of the main theorem. In the last section we prove our result.

2. Hereditary isomorphy, hereditary hemimorphy

The $(\leq k)$-reconstruction was introduced by R. Fraïssé in 1970 [10]. In 1972, G. Lopez [15, 16] showed that the tournaments (indeed the finite binary relations) are (≤ 6)-reconstructible. One may deduce the next corollary.

Corollary 2.1 ([15, 16, 17]). *Two (≤ 6)-hypomorphic tournaments are hereditarily isomorphic.*

In 1993, J. G. Hagendorf raised the $(\leq k)$-half-reconstruction and solved it with G. Lopez [13]. In fact, they proved that if two finite binary relations R and $R^{'}$ are (≤ 12)-hemimorphic, then either R and $R^{'}$ are (≤ 6)-hypomorphic or R^{*} and $R^{'}$ are (≤ 6)-hypomorphic. From that, they obtained in particular that the finite binary relations are (≤ 12)-half-reconstructible. Concerning the tournaments, in 1995, Y. Boudabbous and G. Lopez [7] showed that they are (≤ 7)-half-reconstructible.

The four corollaries below follow directly from the preceding results.

Corollary 2.2. *Two (≤ 7)-hemimorphic tournaments are hereditarily hemimorphic.*

Corollary 2.3. *If T and $T^{'}$ are (≤ 7)-hemimorphic tournaments, then either $T^{'}$ and T are (≤ 6)-hypomorphic or $T^{'}$ and T^{*} are (≤ 6)-hypomorphic.*

Corollary 2.4. *Two tournaments T and $T^{'}$ are hereditarily hemimorphic if and only if either $T^{'}$ and T are (≤ 6)-hypomorphic or $T^{'}$ and T^{*} are (≤ 6)-hypomorphic.*

Corollary 2.5. *Two tournaments T and $T^{'}$ are hereditarily hemimorphic if and only if either $T^{'}$ and T are hereditarily isomorphic or $T^{'}$ and T^{*} are hereditarily isomorphic.*

Later, M. Pouzet [1, 2] introduced the $\{-k\}$-reconstruction. P. Ille [14] (resp. G. Lopez and C. Rauzy [19]) proved that the tournaments on at least 11 (resp. 10) vertices are $\{-5\}$-reconstructible (resp. $\{-4\}$-reconstructible).

Y. Boudabbous improved these results by the following one that we frequently use in this paper.

Theorem 2.6 ([5]). *Let $k \in \{4, 5\}$. Two $\{-k\}$-hypomorphic tournaments, which have at least $6 + k$ vertices, are hereditarily isomorphic.*

We use also the following lemma obtained by H. Bouchaala and Y. Boudabbous.

Lemma 2.7 ([4])**.** *If a tournament T without diamonds on at least 9 vertices is $\{-3\}$-hypomorphic to T^*, then T is hereditarily isomorphic to T^*.*

In 1998, Y. Boudabbous and J. Dammak [6] introduced the $\{-k\}$-half-reconstruction and proved that: For $k \in \{5, 6\}$, the tournaments which have at least $7+k$ vertices are $\{-k\}$-half-reconstructible.

Notation 2.8. Given a tournament $T = (V, A)$, $X \subset V$ and a tournament H, we denote by $n(T, H; X) := |\{F \subseteq V, X \subset F$ and $T[F]$ is hemimorphic to $H\}|$.

The "Combinatorial lemma" due to M. Pouzet [21], makes a link between the problems of ($\leq k$)-reconstruction (resp. ($\leq k$)-half-reconstruction) and those of the $\{-k\}$-reconstruction (resp. $\{-k\}$-half-reconstruction) and allows us to deduce the next corollaries.

Corollary 2.9 ([21])**.** *Given two tournaments $T = (V, A)$ and $T' = (V, A')$ for each integer $0 < p < |V|$, if T and T' are $\{p\}$-hypomorphic (resp. $\{p\}$-hemimorphic), then for $q = 1, \ldots, \min(p, |V| - p)$, T and T' are $\{q\}$-hypomorphic (resp. $\{q\}$-hemimorphic).*

Corollary 2.10 ([22])**.** *Given positive integers n, p, h such that $0 \leq p < n$ and $1 \leq h \leq n - p$, a tournament H with h vertices and two $\{-p\}$-hemimorphic tournaments $T = (V, A)$ and $T' = (V, A')$ with $|V| = n$. Then for all subset X of V which has at most p vertices, $n(T, H; X) = n(T', H; X)$.*

Using Corollary 2.2 and Corollary 2.9, we find the following result.

Corollary 2.11. *Given an integer $k \geq 7$ and two tournaments T and T' defined on the same vertex set V, with $|V| \geq k + 7$, if T and T' are $\{-k\}$-hemimorphic, then T and T' are hereditarily hemimorphic.*

The goal of the present work is to study the above result where $k = 5$. We obtain the following theorem.

Theorem 2.12. *Let T and T' be two tournaments defined on the same vertex set V, with $|V| \geq 12$. If T and T' are $\{-5\}$-hemimorphic, then T and T' are hereditarily hemimorphic.*

Consider two $\{-6\}$-hemimorphic tournaments T and T' defined on the same vertex set V, with $|V| \geq 13$, a subset X of V such that $|X| \leq 7$ and a vertex $y \in V - X$. By applying the above result on the two subtournaments $T' - y$ and $T - y$, we deduce that $T'[X]$ and $T[X]$ are hemimorphic. Thus, T' and T are (≤ 7)-hemimorphic and then by Corollary 2.2, we obtain the following.

Corollary 2.13. *Let T and T' be two tournaments defined on the same vertex set V, with $|V| \geq 13$. If T and T' are $\{-6\}$-hemimorphic, then T and T' are hereditarily hemimorphic.*

3. Indecomposability, Gallai's decomposition and tournament without diamonds

3.1. Strong connectivity. Indecomposability. Interval partition. Dilation

Given a tournament $T = (V, A)$, define a relation $\mathcal{R}$ on V as follows: for all $x \in V$, $x\mathcal{R}x$ and for all $x, y \in V$ such that $x \neq y$, $x\mathcal{R}\ y$ if there exists two sequences $x_0 = x, \ldots, x_n = y$ and $y_0 = y, \ldots, y_p = x$ of vertices of T fulfilling: for all $i \in \{0, \ldots, n-1\}$, $x_i \to x_{i+1}$ and for all $j \in \{0, \ldots, p-1\}, y_j \to y_{j+1}$. The relation $\mathcal{R}$ is an equivalence relation whose classes are called the *strongly-connected components* of T. Say then that a tournament is strongly connected when it admits at most a single strongly connected component. For example, the tournament $C_3 = (\{1, 2, 3\}, \{(1, 2), (2, 3), (3, 1)\})$ is strongly connected. However, the two tournaments $D_1 = (\{1, 2, 3, 4\}, \{(1, 2), (2, 3), (3, 1), (1, 4), (2, 4), (3, 4)\})$ and $D_2 = D_1^*$, are non-strongly connected and admit two strongly connected components: $\{1, 2, 3\}$ and $\{4\}$. Each tournament isomorphic to C_3 (resp. to D_1 or D_2), is a *3-cycle* (resp. a *diamond*). Given a tournament $T = (V, A)$, a subset I of V is an *interval* [10] of T provided that for all x of $(V - I)$, we have either $x \to I$ or $I \to x$. Clearly, the empty set, the singletons of V and the set V are intervals of T, called *trivial intervals.* A tournament is then *indecomposable* if all its intervals are trivial. For instance, a diamond D has a unique non-trivial interval I with $D[I]$ is a 3-cycle. The vertex of D which is not in I, is said the center of D. A partition P of V is an *interval partition* of T if all its elements are intervals of T. It results that the elements of P may be considered as the vertices of a new tournament, the quotient $T/P = (P, A/P)$ of T by P, defined in the following way: for any $X \neq Y \in P$, $(X, Y) \in A/P$ if $(x, y) \in A$ for $x \in X$ and $y \in Y$. On the other hand, a subset X of V is a *strong interval* [8, 11] of T provided that X is an interval of T and for every interval Y of T, if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. In the remaining part of this work, for each tournament $T = (V, A)$ with $|V| \geq 2$, $P(T)$ denotes the family of maximal, strong intervals of T, under the inclusion, amongst the strong intervals of T distinct from V. For example, if T is not strongly connected, the partition $P(T)$ presents the family of the strongly connected components of T [8]. From the definition of a strong interval, note that $P(T)$ realizes an interval partition of T. Given a tournament $H = (\{1, \ldots, n\}, A)$ where $n \geq 1$, with every $i \in \{1, \ldots, n\}$ is associated a tournament $T_i = (V_i, A_i)$ with $|V_i| \geq 1$ such that the V_i's are mutually disjoint. The lexicographical sum of the T_i's over H is the tournament denoted by $H(T_1, \ldots, T_n)$ and defined on the union of V_i's as follows: given $u \in V_i$ and $v \in V_j$, where $i, j \in \{1, \ldots, n\}$, (u, v) is an arc of $H(T_1, \ldots, T_n)$ if either $i = j$ and $(u, v) \in A_i$ or $i \neq j$ and $(i, j) \in A$. We can also say that the tournament $H(T_1, \ldots, T_n)$ is obtained from the tournament H by dilating each vertex i of H by the tournament T_i.

3.2. Gallai's decomposition

At this stage, we present the Gallai's decomposition which consists in the following examination of the quotient $T/P(T)$.

Theorem 3.1 ([8, 11])**.** *Let T be a tournament of cardinality $|T| \geq 2$.*

(1) *The tournament T is not strongly connected if and only if $T/P(T)$ is a total order.*
(2) *The tournament T is strongly connected if and only if $T/P(T)$ is indecomposable and $|P(T)| \geq 3$.*

This definition plays an important role in this paper.

Definition 3.2. Let T be a tournament defined on a vertex set V with $|V| \geq 2$. We consider the partition $\widetilde{P}(T)$ of V defined as follows:

- If T is strongly connected, $\widetilde{P}(T) = P(T)$.
- If T is not strongly connected, $\widetilde{P}(T)$ is defined in the following way: For $A \subseteq V$, $A \in \widetilde{P}(T)$, if and only if either $A \in P(T)$ and $|A| \geq 2$, or A is a maximal union of consecutive vertices of the total order $T/P(T)$ which are singletons.

We make the following remark.

Remark 3.3. Let T be a non-strongly connected tournament with at least 2 vertices.

- T is a total order if and only if $|\widetilde{P}(T)| = 1$.
- If T is not a total order, $T/\widetilde{P}(T)$ is a total order: $X_1 < \cdots < X_k$ where $k \geq 2$. The set X_1 (resp. X_k) is called the first (resp. last) component of T.

Notation 3.4. Let a tournament T with at least 2 vertices and an integer $k \geq 1$. We denote by $\widetilde{P_k}(T)$ the set of elements of $\widetilde{P}(T)$ of cardinality k.

A. Boussaïri, G. Lopez, S. Thomassé, and P. Ille [8] established the following theorem called "the inversion theorem".

Theorem 3.5 ([8])**.** *Given an indecomposable tournament T with at least three vertices, the only tournaments which are $\{3\}$-hemimorphic to T are T and T^*.*

Remark 3.6 ([8]). Let T and $T^{'}$ be two $\{3\}$-hemimorphic tournaments with at least two vertices.

i) $P(T) = P(T^{'})$.

ii) T is strongly connected if and only if $T^{'}$ is strongly connected.

Moreover, if T is strongly connected, then either $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$ or $T^{'}/\widetilde{P}(T) = T^*/\widetilde{P}(T)$.

We can easily check that:

Remark 3.7. If T and $T^{'}$ are two (≤ 4)-hypomorphic non-strongly connected tournaments with at least two vertices, then $\widetilde{P}(T) = \widetilde{P}(T^{'})$ and $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$.

3.3. Tournament without diamonds

A tournament T is *called without diamonds* if none of its subtournaments is a diamond. For example, for each integer $h \geq 1$ the tournament T_{2h+1} defined on $\{0, \dots, 2h\}$ as follows: for all $i, j \in \{0, \dots, 2h\}$, $i \to j$, if there exists $k \in \{0, \dots, h\}$, such that $j = i + k$ modulo $(2h + 1)$, is indecomposable and without diamonds [12, 18].

The morphology of the tournaments without diamonds is described by (P. Ille and C. Gnanvo [12]) and (G. Lopez and C. Rauzy [18]). They obtained the following characterization.

Theorem 3.8 ([12, 18])**.** *A tournament T is without diamonds if and only if T is either a total order or a lexicographical sum of total orders over some T_{2h+1}, with $h \geq 1$.*

The following remark follows immediately from Theorem 3.8 and Remark 3.6.

Remark 3.9. Let T be a tournament without diamonds.
(1) T is (≤ 5)-selfdual.
(2) If T is not strongly connected, then T is a total order.
(3) Each subtournament of T is either strongly connected or a total order.
(4) If T is indecomposable and $|T| \geq 3$, then T is isomorphic to some T_{2h+1}, with $h \geq 1$.
(5) If T is not a total order and if $T^{'}$ is a tournament (≤ 3)-hypomorphic to T, then:
- There exists $h \geq 1$ such that $T/\widetilde{P}(T)$ is isomorphic to some T_{2h+1}.
- Either $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$ or $T^{'}/\widetilde{P}(T) = T^{*}/\widetilde{P}(T)$.
- T and $T^{'}$ are hereditarily isomorphic, if $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$, otherwise T^{*} and $T^{'}$ are hereditarily isomorphic.

4. Difference class

The definition below introduced by G. Lopez in 1972 presents an important tool in many reconstruction problems.

Definition 4.1 ([15, 16])**.** Let $T = (V, A)$ and $T^{'} = (V, A^{'})$ be two $\{2\}$-hemimorphic tournaments. The difference relation of T and $T^{'}$ is the equivalence relation denoted by $D_{T,T'}$ defined on V as follows: for all $x \in V$, $xD_{T,T'}x$ and for all $x, y \in V$, such that $x \neq y$, $xD_{T,T'}y$ if there exists a sequence $x_0 = x, x_1, \dots, x_n = y$ of some elements of V such that for all $i \in \{0, \dots, n-1\}$, $(x_i, x_{i+1}) \in A$ if and only if $(x_i, x_{i+1}) \notin A^{'}$. The equivalence classes of $D_{T,T'}$ are called the difference classes of T and $T^{'}$.

We use the following lemmas in the present work.

Lemma 4.2 ([7, 17]). *Let T and $T^{'}$ be two (≤ 3)-hemimorphic tournaments and C a class of the equivalence $D_{T,T'}$. Then*

(1) *C is an interval of T and $T^{'}$.*
(2) *The difference classes constitute an interval partition P of T and $T^{'}$ such that $T/P = T^{'}/P$.*

Lemma 4.3 ([18]). *Let T and $T^{'}$ be two (≤ 4)-hypomorphic tournaments and C a class of the equivalence $D_{T,T'}$. Then*

(1) *$T[C]$ is a tournament without diamonds.*
(2) *$T[C]$ is (≤ 5)-selfdual.*
(3) *$T^{'}[C]$ and $T^{*}[C]$ are hereditarily isomorphic.*
(4) *If there exists $X \subseteq C$ such that $T[X]$ is a 3-cycle, then $T^{'}[X] = T^{*}[X]$.*

Lemma 4.4 ([7]). *Let $T = (V, A)$ and $T^{'} = (V, A^{'})$ be two (≤ 5) hemimorphic tournaments, C a class of the equivalence $D_{T,T'}$ and $X \subseteq C$ such that $T[X]$ is a diamond. Then $T^{'}[X] \simeq T^{*}[X]$.*

After characterizing the difference classes under the hypothesis of the (≤ 4)-hypomorphy in [18], G. Lopez and C. Rauzy established in [19] the following result.

Theorem 4.5 ([19]). *The tournaments defined on $n \geq 7$ vertices are $\{4, n-1\}$-reconstructible.*

This corollary is an immediate consequence of the above theorem.

Corollary 4.6. *Given an integer $k \geq 1$ and two tournaments T and $T^{'}$ on $n \geq 6+k$ vertices, if T and $T^{'}$ are $\{4, n-k\}$-hypomorphic, then T and $T^{'}$ are isomorphic.*

5. Some preliminary results

In this section, we recall and establish some results which will be used in the proof of Theorem 2.12.

We begin with the following definition.

Definition 5.1 ([9]). Let $T = (V, A)$ be a tournament and X be a proper subset of V such that $|X| \geq 3$ and $T[X]$ is indecomposable. The following subsets of $V - X$ are defined.

- $\text{Ext}(X)$ is the set of $x \in V - X$ such that $T[X \cup \{x\}]$ is indecomposable.
- $\langle X \rangle$ is the set of $x \in V - X$ such that X is an interval of $T[X \cup \{x\}]$.
- For each $u \in X$, $X(u)$ is the set of $x \in V - X$ such that $\{u, x\}$ is an interval of $T[X \cup \{x\}]$.

The results below are due to Ehrenfeucht and Rozenberg.

Proposition 5.2 ([9]). *Given a tournament $T = (V, A)$ and a proper subset X of V such that $|X| \geq 3$ and $T[X]$ is indecomposable, the family $\{X(u), u \in X\} \cup \{\mathrm{Ext(X)}, \langle \mathrm{X} \rangle\}$ constitutes a partition of $V - X$ (Some elements of this family can be empty).*

Lemma 5.3 ([9]). *Let $T = (V, A)$ be an indecomposable tournament with $|V| \geq 5$. For every subset X of V such that $|X| \geq 3$, $|V - X| \geq 2$ and $T[X]$ is indecomposable, there exist two different elements x, y of $V - X$ such that $T[X \cup \{x, y\}]$ is indecomposable.*

As for every vertex x of an indecomposable tournament T with at least three vertices there exists a 3-cycle passing by x, then using the preceding lemma we deduce the following corollary.

Corollary 5.4 ([9]). *Let $T = (V, A)$ be an indecomposable tournament with $|V| \geq 5$ and $x \in V$.*

(1) *If $|V|$ is even, there exists $y \in V - \{x\}$ such that $T - \{y\}$ is indecomposable.*
(2) *If $|V|$ is odd, there exist $y \neq z \in V - \{x\}$ such that $T - \{y, z\}$ is indecomposable.*

The next two lammas are due to (M. Bouaziz and Y. Boudabbous) and J. W. Moon, respectively.

Lemma 5.5 ([3]). *Let $p \geq 2$ be an integer, R and $R^{'}$ be two tournaments defined on the same vertex set $\{1, \dots, p\}$, f be an isomorphism from R onto $R^{'}$ and H (resp. $H^{'}$) be a tournament defined on a vertex set which is disjoint from $\{1, \dots, p\}$. Given $i \in \{1, \dots, p\}$, and G (resp. $G^{'}$) be the tournament obtained from R (resp. $R^{'}$) by dilating the vertex i (resp. $f(i)$) by H (resp. $H^{'}$). Then*

(i) *$G \simeq G^{'}$ if and only if $H \simeq H^{'}$.*
(ii) *If $s_R(i) \neq s_{R^*}(i)$, $H^* \simeq H^{'}$ and H is not selfdual, then $G^* \not\simeq G^{'}$.*

Lemma 5.6 ([20]). *Let $T = (V, A)$ be a strongly connected tournament such that $|V| = n \geq 3$. Then, for all $k \in \{3, \dots, n\}$ and for all $x \in V$, there exists a subset X of V such that $x \in X$, $|X| = k$ and the subtournament $T[X]$ is strongly connected.*

We continue to establish the following 7 lemmas.

Lemma 5.7. *Let T and $T^{'}$ be two (≤ 4)-hypomorphic non-strongly connected tournaments such that $T/\widetilde{P}(T)$ is the total order $X_1 < \cdots < X_p$ where $p \geq 2$. If T and $T^{'}$ are $\{-1\}$-hemimorphic, then there exists $x \in X_1 \cup X_p$ fulfilling the following*:

(i) *$T - \{x\}$ and $T^{'} - \{x\}$ are isomorphic.*
(ii) *If $x \in X_1$ and $T[X_1]$ is a total order (resp. $x \in X_p$ and $T[X_p]$ is a total order), then $x = \min(X_1)$ (resp. $x = \max(X_p)$).*

Proof. By Remark 3.7, $\widetilde{P}(T^{'}) = \widetilde{P}(T)$ and $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$. In the following three cases, we will choose a vertex x of T such that by denoting Y (resp. Z) the first (resp. last) component of $T - x$ and $T^{'} - x$, we have: either $|Y| \neq |Z|$ or one of the subtournaments $T[Y]$ and $T[Z]$ is a total order and the other is strongly connected with at least three vertices. We then deduce that $T^{*} - x$ and $T^{'} - x$ are not isomorphic and then $T - x$ and $T^{'} - x$ are isomorphic.

- If both $T[X_1]$ and $T[X_p]$ are strongly connected such that $|X_1| \geq 3$ and $|X_p| \geq 3$, we consider $x \in X_1$, if $|X_1| \leq |X_p|$, otherwise $x \in X_p$.
- If both $T[X_1]$ and $T[X_p]$ are total orders, we consider $x = \min(X_1)$, if $|X_1| \leq |X_p|$, otherwise $x = \max(X_p)$.
- If one of $T[X_1]$, $T[X_p]$ is a total order and the other is strongly connected on at least 3 vertices. By considering X_p in the place of X_1, we may assume that $T[X_1]$ is a total order and $T[X_p]$ is strongly connected with $|X_p| \geq 3$. In this case if $|X_1| \geq 2$ we consider $x = \min(X_1)$. Suppose now that $|X_1| = 1$. If $|X_p| = 3$, we consider $x \in X_p$, otherwise using Lemma 5.6 we consider $x \in X_p$ such that $T[X_p - \{x\}]$ is strongly connected. □

Lemma 5.8. *Let T and $T^{'}$ be two non-strongly connected tournaments defined on the same vertex set and Q be an interval partition of T and $T^{'}$ such that T/Q and $T^{'}/Q$ are two equal total orders. If f is an isomorphism from T onto $T^{'}$, then for all $X \in Q$, $f(X) = X$.*

The proof of the above lemma is obvious.

Lemma 5.9. *Given two (≤ 5)-hemimorphic non-strongly connected tournaments T and $T^{'}$ with at least 2 vertices. Then $\widetilde{P}(T^{'}) = \widetilde{P}(T)$ and $(T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$ or $T^{'}/\widetilde{P}(T) = T^{*}/\widetilde{P}(T))$.*

Proof. Since T is a non-strongly connected tournament, then its Gallai's decomposition is presented as follows: $T = C(T_1, \ldots, T_k)$ such that $k \geq 2$, C is the total order $1 < 2 < \cdots < k$ and for all $i \in \{1, \ldots, k\}$, T_i is a strongly connected tournament defined on a vertex set S_i. If T is a total order, then the result is obvious, otherwise, we distinguish two cases.

- Case 1. If $k = 2$. Clearly, $P(T) = \widetilde{P}(T)$ and then by Remark 3.6, $\widetilde{P}(T) = \widetilde{P}(T^{'})$. Therefore, $T/\widetilde{P}(T) = T^{'}/\widetilde{P}(T)$ or $T^{*}/\widetilde{P}(T) = T^{'}/\widetilde{P}(T)$.
- Case 2. If $k \geq 3$. We deduce the result from the hemimorphy between the two tournaments $T[\{a_i, b_i, c_i, a_j, a_l\}]$ and $T^{'}[\{a_i, b_i, c_i, a_j, a_l\}]$ where i, j, l are three different elements of $\{1, \ldots, k\}$, $a_j \in S_j$, $a_l \in S_l$ and a_i, b_i, c_i are three elements of S_i such that $T[\{a_i, b_i, c_i\}]$ is a 3-cycle.

□

The corollary below follows directly from Corollary 2.9, Remark 3.6 and Lemma 5.9.

Corollary 5.10. *Given two $\{-5\}$-hemimorphic tournaments T and T' on at least 10 vertices, then $\widetilde{P}(T) = \widetilde{P}(T')$ and ($T/\widetilde{P}(T) = T'/\widetilde{P}(T)$ or $T^*/\widetilde{P}(T) = T'/\widetilde{P}(T)$).*

Lemma 5.11. *If two tournaments $T = (V, A)$ and $T' = (V, A')$ are (≤ 5)-hemimorphic, then T and T' are (≤ 6)-hemimorphic.*

Proof. Let $T = (V, A)$ and $T' = (V, A')$ be two (≤ 5)-hemimorphic tournaments. Consider a subset X of V such that $|X| = 6$ and showing that $T[X]$ and $T'[X]$ are hemimorphic. First, assume that both $D_{T[X],T'[X]}$ and $D_{T^*[X],T'[X]}$ have one class. It follows from Lemma 4.4 that $T[X]$ is without diamonds. So, $T'[X]$ and $T[X]$ are (≤ 4)-hypomorphic. Thus, $T[X]$ has no 3-cycle, by Lemma 4.3. Henceforth, $T[X]$ is a total order and then the result is obtained. Second, assume that $D_{T[X],T'[X]}$ has at least two classes. Let C be such a class. From Lemma 4.2, C is an interval of $T[X]$ and $T'[X]$. As $T[X]$ and $T'[X]$ are (≤ 5)-hemimorphic, it follows that $T[C]$ is without diamonds, by Lemma 4.4. So, $T[C]$ and $T'[C]$ are (≤ 4)-hypomorphic and then they are (≤ 5)-hypomorphic by Lemma 4.3. As $|X| = 6$, then for every class C of $D_{T[X],T'[X]}$, $|C| \leq 5$. Thus, for each class C of $D_{T[X],T'[X]}$, $T[C] \simeq T'[C]$ and then $T[X] \simeq T'[X]$. Finally, assume that $D_{T^*[X],T'[X]}$ has at least two classes. By replacing $T[X]$ by $T^*[X]$ in the second step, we obtain $T^*[X] \simeq T'[X]$. □

Lemma 5.12. *Let an integer $p \geq 1$ and a tournament $T = (V, A)$ such that $|V| \geq p + 2$. Then for all $x \in V$ there exists a subset B of $V - \{x\}$ such that $|B| = p$ and $s_{T-B}(x) \neq s_{T^*-B}(x)$.*

Proof. Let $x \in V$. Considering a set $N(x) \in \{N^+(x), N^-(x)\}$ such that $|N(x)| = \min\{|N^+(x)|, |N^-(x)|\}$. If $|N(x)| < p$ (resp. $|N(x)| \geq p$), we consider a subset B with p elements of $V - \{x\}$ such that $N(x) \subset B$ (resp. $B \subseteq N(x)$). We easily verify that $s_{T-B}(x) \neq s_{T^*-B}(x)$. □

Notation 5.13. We denote by $\mathcal{P}$ each tournament which is a lexicographical sum $C_3(T_1, T_2, T_3)$ where for all $i \in \{1, 2, 3\}$, T_i is a total order on i vertices.

Remark 5.14.

- The tournaments without diamonds on 6 vertices which are not hemimorphic to $\mathcal{P}$ are selfdual.
- Let R be a tournament hemimorphic to $\mathcal{P}$ and R' be a tournament (≤ 3)-hypomorphic to R. Then either $R \simeq R'$ or $R^* \simeq R'$.

Lemma 5.15. *Let T and T' be two (≤ 5)-hypomorphic tournaments defined on a vertex set V such that for all $X \subseteq V$, if $T[X]$ is hemimorphic to $\mathcal{P}$, then $T'[X]$ is isomorphic to $T[X]$. Then T and T' are (≤ 6)-hypomorphic.*

Proof. As T and T' are (≤ 5)-hypomorphic, it is sufficient to prove that T and T' are $\{6\}$-hypomorphic. Let $X \subseteq V$ such that $|X| = 6$. If $D_{T[X],T'[X]}$ has at least two classes, then each of these classes has at most five elements. Thus, since T and $T^{'}$ are (≤ 5)-hypomorphic, then for each class C of $D_{T[X],T^{'}[X]}$, $T[C] \simeq T^{'}[C]$. So $T[X] \simeq T^{'}[X]$. At present, assume that $D_{T[X],T^{'}[X]}$ has only one class. From Lemma 4.3, it follows that $T[X]$ is a tournament without diamonds and $T^{'}[X]$ is isomorphic to $T^{*}[X]$. If $T[X]$ is not hemimorphic to $\mathcal{P}$, then from Remark 5.14, $T[X] \simeq T^{'}[X]$, otherwise we conclude by the hypothesis. □

6. Proof of Theorem 2.12

Using Corollary 5.10, the following result enables us to deduce immediately Theorem 2.12.

Theorem 6.1. *Consider two $\{-5\}$-hemimorphic tournaments T and $T^{'}$ with at least 12 vertices.*
(i) *If $T/\widetilde{P}(T) = T^{'}/\widetilde{P}(T)$, then T and $T^{'}$ are hereditarily isomorphic.*
(ii) *If $T^{*}/\widetilde{P}(T) = T^{'}/\widetilde{P}(T)$, then T^{*} and $T^{'}$ are hereditarily isomorphic.*

Proof of Theorem 6.1

As by replacing the tournament T by T^{*} in (i), we immediately obtain the assertion (ii), we have to show in the sequel only the assertion (i). Consider two $\{-5\}$-hemimorphic tournaments T and $T^{'}$ defined on the same vertex set V on $n \geq 12$ elements such that $T/\widetilde{P}(T) = T^{'}/\widetilde{P}(T)$. Clearly, it is sufficient to prove that for all $X \in \widetilde{P}(T)$, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. As the result is obvious when T is a total order, we may assume that $|\widetilde{P}(T)| \geq 2$.

Lemma 6.2. *T and $T^{'}$ are (≤ 5)-hypomorphic.*

Proof. Notice that by Corollary 2.9, T and $T^{'}$ are (≤ 5)-hemimorphic. Since $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$, it is sufficient to prove that for all $X \in \widetilde{P}(T)$, $T[X]$ and $T^{'}[X]$ are (≤ 5)-hypomorphic. Let $X \in \widetilde{P}(T)$. The two tournaments $T[X]$ and $T^{'}[X]$ are (≤ 4)-hypomorphic. Indeed, since T and $T^{'}$ are (≤ 3)-hemimorphic, then T and $T^{'}$ are (≤ 3)-hypomorphic. So, it suffices to show that $T[X]$ and $T^{'}[X]$ are $\{4\}$-hypomorphic. Let $A \subseteq X$ such that $|A| = 4$ and suppose that $T[A]$ is not isomorphic to $T^{'}[A]$. The (≤ 3)-hypomorphy between $T[X]$ and $T^{'}[X]$ implies that $T[A]$ is necessarily a diamond and $T^{'}[A]$ is isomorphic to $T^{*}[A]$. Let $x \in V - X$. Clearly, the two tournaments $T[A \cup \{x\}]$ and $T^{'}[A \cup \{x\}]$ are not hemimorphic, which contradicts the (≤ 5)-hemimorphy between T and $T^{'}$. At present, from Lemma 4.2 it is enough to prove that $T[C]$ and $T^{'}[C]$ are (≤ 5)-hypomorphic for each class C of the equivalence $D_{T[X],T^{'}[X]}$. Let C_0 be such a class. Because $T[C_0]$ and $T^{'}[C_0]$ are (≤ 4)-hypomorphic, it follows from Lemma 4.3 that $T[C_0]$ is a (≤ 5)-selfdual tournament and $T^{'}[C_0]$ and $T^{*}[C_0]$ are hereditarily isomorphic. Thus, $T[C_0]$ and $T^{'}[C_0]$ are (≤ 5)-hypomorphic. □

Lemma 6.3. *For all $X \in \widetilde{P}(T)$, $T[X]$ and $T^{'}[X]$ are isomorphic.*

Proof. Since $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$ and $|\widetilde{P}(T)| \geq 2$, then the equivalence relation $D_{T,T'}$ has at least two classes and every class C is an interval of T and $T^{'}$ which is contained in an element X of $\widetilde{P}(T)$. Thus, every element of $\widetilde{P}(T)$ is an union of some classes of $D_{T,T'}$. So, from Lemma 4.2, it is sufficient to prove that $T[C]$ and $T^{'}[C]$ are isomorphic for every class C of $D_{T,T'}$. Let C be such a class. It results from Lemmas 4.3 and 6.2 that $T[C]$ is a tournament without diamonds. We distinguish the following three cases.

Case 1. If $|V - C| \geq 6$.
If the tournament $T[C]$ is a total order, then $T[C]$ and $T^{'}[C]$ are isomorphic, otherwise we consider the tournament $H = \mathcal{O}_2\ (H_1, H_2)$, such that $\mathcal{O}_2$ is the total order $1 < 2$, H_1 is a tournament on one vertex and $H_2 \simeq T[C]$. Let $(x, a, b, c) \in (V - C) \times C^3$ such that $T[\{a, b, c\}]$ is a 3-cycle. Without loss of generality we may assume that $x \to C$ in $T^{'}$ and T. As $n(T, H; \{x, a, b, c\}) \neq 0$, from Corollary 2.10, $n(T, H; \{x, a, b, c\}) = n(T^{'}, H; \{x, a, b, c\})$ and then $n(T^{'}, H; \{x, a, b, c\}) \neq 0$. So, there exists $Y \subset V$ such that $\{x, a, b, c\} \subset Y$ and $T^{'}[Y]$ is hemimorphic to H. Let $X = \{x\} \cup C$. We have obligatorily $Y = X$. Indeed, if there exists $y \in Y - X$, then we easily verify that x and y are two diamond's centers in $T^{'}[Y]$, whereas H has only one diamond's center. Thus, $T^{'}[\{x\} \cup C]$ and $T[\{x\} \cup C]$ are hemimorphic. As $x \to C$ in $T^{'}$ and T and by Remark 3.9, $T[C]$ and $T^{'}[C]$ are strongly connected, it results that $T^{'}[\{x\} \cup C]$ and $T[\{x\} \cup C]$ are isomorphic and then $T^{'}[C] \simeq T[C]$.

Case 2. If $|V - C| = 5$.
Let $(x, a, b, c, d) \in C \times (V - C)^4$ a uple of 5 different vertices. Since $T[C]$ is a tournament without diamonds, every subtournament of $T[C]$ is either strongly connected or a total order, by Remark 3.9. If the tournament $T[C - \{x\}]$ is a total order, then $T[C - \{x\}]$ and $T^{'}[C - \{x\}]$ are isomorphic, otherwise, as $T^{'}$ and T are $\{-5\}$-hemimorphic and $T[C - \{x\}]$ and $T^{'}[C - \{x\}]$ are strongly connected, we necessarily have $T - \{x, a, b, c, d\} \simeq T^{'} - \{x, a, b, c, d\}$. In particular, $T[C - \{x\}] \simeq T^{'}[C - \{x\}]$. Hence, $T[C]$ and $T^{'}[C]$ are $\{4, -1\}$-hypomorphic and then isomorphic by Theorem 4.5.

Case 3. If $1 \leq |V - C| \leq 4$.
Let $k = 6 - |V - C|$. Let I and J be two subsets of V such that $I \subset V - C$, $J \subset C$, $|J| = k$ and $|I| = 5 - k$. If $T[C - J]$ is a total order, then $T[C - J] \simeq T^{'}[C - J]$, otherwise, as $|V - (C \cup I)| = 1$, $|I \cup J| = 5$ and $T^{'}$ and T are $\{-5\}$-hemimorphic, we necessarily have $T - (I \cup J) \simeq T^{'} - (I \cup J)$. So, $T[C - J] \simeq T^{'}[C - J]$ and therefore, the two tournaments $T[C]$ and $T^{'}[C]$ are $\{4, -k\}$-hypomorphic. Thus, by Corollary 4.6, $T[C] \simeq T^{'}[C]$. $\square$

From Lemmas 6.2 and 6.3 we immediately deduce the following.

Corollary 6.4. *For each $X \in \widetilde{P}(T)$ such that $1 \leq |X| \leq 6$, $T^{'}[X]$ and $T[X]$ are hereditarily isomorphic.*

We directly obtain the proof of Theorem 6.1 from Propositions 6.6 and 6.11, where we discuss the two cases of Gallai's decomposition of the tournaments.

We need this notation.

Notation 6.5. For all $X \subset V$, if $T - X$ and $T^{'} - X$ are hemimorphic, we denote by f_X a hemimorphism between them.

Proposition 6.6. *If T is not strongly connected, then for all $X \in \widetilde{P}(T)$, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.*

Proof of Proposition 6.6

This proposition is an immediate consequence of the following three lemmas where, T is considered non-strongly connected with $\widetilde{P}(T) = \{X_1, X_2, \ldots, X_k\}$ such that $k \geq 2$ and $T/\widetilde{P}(T)$ is the total order: $X_1 < \cdots < X_k$.

Lemma 6.7. *If either $k \geq 3$ or ($k = 2$ and $\min(|X_1|, |X_2|) \geq 2$), then for all $X \in \widetilde{P}(T)$ such that $|X| \geq 10$, $T^{'}[X]$ and $T[X]$ are hereditarily isomorphic.*

Proof. Let $X \in \widetilde{P}(T)$ such that $|X| \geq 10$. From Theorem 2.6, it is sufficient to show that $T[X]$ and $T^{'}[X]$ are $\{-4\}$-hypomorphic. Given $A \subset X$ such that $|A| = 4$ and proving that $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. If $T[X - A]$ is a total order, the result is obvious. Let's assume in the remaining of this proof that $T[X - A]$ is not a total order. If $X \not\in \{X_1, X_k\}$, as $T - A$ and $T^{'} - A$ are (≤ 4)-isomorphic and $\{-1\}$-hemimorphic, then from Lemma 5.7, there exist $x \in V - X$ and an isomorphism $f_{(A\cup\{x\})}$ from $T - (A \cup \{x\})$ onto $T^{'} - (A \cup \{x\})$. Considering the set $Q = \{Y - (A \cup \{x\}); Y \in \widetilde{P}(T) - \{\{x\}\}\}$. It is clear that Q is an interval partition of $T - (A \cup \{x\})$ and $T^{'} - (A \cup \{x\})$ such that $(T - (A\cup\{x\}))/Q$ and $(T^{'} - (A\cup\{x\}))/Q$ are two equal total orders. Thus, from Lemma 5.8, we deduce that for all $Z \in Q$, $f_{(A\cup\{x\})}(Z) = Z$, in particular $f_{(A\cup\{x\})}(X - A) = X - A$ and thus $T[X - A] \simeq T^{'}[X - A]$. Assume now that $X \in \{X_1, X_k\}$. Clearly, X_1 (resp. X_k) is the first component of T and $T^{'}$ (resp. T^* and $(T^{'})^*$). So, when we show the result for only one of the two cases we deduce immediately the second one by coming back to the first and interchanging the considered two tournaments and their duals. For instance, assume in the remaining of this proof that $X = X_1$. As $T - A$ is not strongly connected, we denote by Y its first component. We may easily see that if $T[X - A]$ is strongly connected, $Y = X - A$, otherwise, Y is the first component of $T[X - A]$ and $T^{'}[X - A]$.

We need the following two facts.

Fact 1. If there are $j \in \{2, \ldots, k\}$ and $x \in X_j$, such that $f_{(A\cup\{x\})}$ is an isomorphism from $T - (A\cup\{x\})$ onto $T^{'} - (A\cup\{x\})$, then $T[X - A] \simeq T^{'}[X - A]$.

Indeed: It is clear that in $T-(A\cup\{x\})$ and $T^{'}-(A\cup\{x\})$, we have $(X-A)\to(V-(X\cup\{x\}))$. So, as $f_{(A\cup\{x\})}$ is an isomorphism from $T-(A\cup\{x\})$ onto $T^{'}-(A\cup\{x\})$, we get by Lemma 5.8 that $f_{(A\cup\{x\})}(X-A)=X-A$ and then $T[X-A]\simeq T^{'}[X-A]$.

Fact 2. If $T[X-A]$ is not strongly connected, $T[Y]\simeq T^{'}[Y]$ and there exists $x\in Y$ such that $f_{(A\cup\{x\})}$ is an isomorphism from $T-(A\cup\{x\})$ onto $T^{'}-(A\cup\{x\})$, then $T[X-A]\simeq T^{'}[X-A]$.

Indeed: It is clear that in $T-(A\cup\{x\})$ and $T^{'}-(A\cup\{x\})$, we have $(Y-\{x\})\to(X-(A\cup Y))\to(V-X)$. So, as $f_{(A\cup\{x\})}$ is an isomorphism from $T-(A\cup\{x\})$ onto $T^{'}-(A\cup\{x\})$, we get by Lemma 5.8 that $f_{(A\cup\{x\})}(X-(A\cup Y))=X-(A\cup Y)$ and then $T[X-(A\cup Y)]\simeq T^{'}[X-(A\cup Y)]$. Besides, since $T[Y]\simeq T^{'}[Y]$, we directly deduce that $T[X-A]\simeq T^{'}[X-A]$.

To complete the proof it remains to choose a vertex x of $T-A$ which requires an isomorphism from $T-(A\cup\{x\})$ and $T^{'}-(A\cup\{x\})$, using the cardinality and the strong connectivity type of their first and last components and then to conclude it suffices to apply either Fact 1 or Fact 2. The choice of x is determined as follows:

• If $T[X_k]$ is a total order and $T[Y]$ is strongly connected such that $|Y|\geq 3$, we consider $x\in X_{k-1}$ if $|X_k|=1$, otherwise $x\in X_k$.

• If $T[Y]$ and $T[X_k]$ are both total orders, we consider $x\in Z$ such that $Z\in\{X_k,Y\}$ and $|Z|=\min\{|X_k|,|Y|\}$.

• If $T[X_k]$ is strongly connected such that $|X_k|=3$ and $T[Y]$ is strongly connected such that $|Y|\geq 1$, we consider $x\in X_k$.

• If $T[X_k]$ is strongly connected such that $|X_k|=3$ and $T[Y]$ is a total order such that $|Y|\geq 2$, we consider $x\in Y$.

• If $T[X_k]$ is strongly connected such that $|X_k|\geq 4$ and $T[Y]$ is a total order, by using Lemma 5.6, we choose $x\in X_k$ such that $T[X_k-\{x\}]$ is strongly connected.

• If $T[X_k]$ is strongly connected such that $|X_k|\geq 4$ and $T[Y]$ is strongly connected such that $|Y|\geq 3$. In this case if $|Y|\neq|X_k|-1$, using Lemma 5.6, we consider $x\in X_k$ such that $T[X_k-\{x\}]$ is strongly connected. Otherwise, if there exists $x\in X_k$ such that $T^{'}[X_k-\{x\}]\not\simeq T^*[Y]$ or $T[X_k-\{x\}]\not\simeq(T^{'})^*[Y]$, then we conclude by Fact 1. Assume that for all $x\in X_k$, $T^{'}[X_k-\{x\}]\simeq T^*[Y]$ and $T[X_k-\{x\}]\simeq(T^{'})^*[Y]$. Hence, $T^{'}[X_k]$ and $T[X_k]$ are $\{-1\}$-monomorphic. In addition, as by Lemma 6.3 there exists an isomorphism g from $T[X_k]$ onto $T^{'}[X_k]$, then for all $x\in X_k$, $T[X_k-\{x\}]\simeq T^{'}[X_k-\{g(x)\}]\simeq T^{'}[X_k-\{x\}]$. Thus, $(T^{'})^*[Y]\simeq T^*[Y]$. So, $T[Y]\simeq T^{'}[Y]$. Consequently, if $T[X-A]$ is strongly connected the result is obvious, otherwise as in $T[X-A]$ and $T^{'}[X-A]$, we have $Y\to(X-(A\cup Y))$, it is enough to show that $T[X-(A\cup Y)]\simeq T^{'}[X-(A\cup Y)]$. Let $y\in Y$. As $|X_k|=|Y|+1$, it is clear that $f_{(A\cup\{y\})}$ is an isomorphism from $T-(A\cup\{y\})$ onto $T^{'}-(A\cup\{y\})$, and then $f_{(A\cup\{y\})}(X-(A\cup Y))=X-(A\cup Y)$. Thus, $T[X-(A\cup Y)]\simeq T^{'}[X-(A\cup Y)]$. □

Lemma 6.8. *If either $k \geq 3$ or ($k = 2$ and $\min\{|X_1|, |X_2|\} \geq 2$), then for all $X \in \widetilde{P}(T)$ such that $7 \leq |X| \leq 9$, $T^{'}[X]$ and $T[X]$ are hereditarily isomorphic.*

Proof. Let $X \in \widetilde{P}(T)$ such that $7 \leq |X| \leq 9$. As from Lemma 6.2, $T[X]$ and $T^{'}[X]$ are (≤ 5)-hypomorphic, using Corollary 2.1, it is sufficient to demonstrate that $T[X]$ and $T^{'}[X]$ are $\{6\}$-hypomorphic. We obtain immediately the proof from the following two cases.

Case 1. If $X \notin \{X_1, X_k\}$.
Let $p = |X|$ and $A \subset X$ such that $|A| = p-6$. As $|V| \geq 12$, there exists a subset B of $V - X$ such that $|B| = 10-p$, $X_k - B \neq \emptyset$ and $X_1 - B \neq \emptyset$. It is clear that $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are $\{-1\}$-hemimorphic and (≤ 4)-hypomorphic. So from Lemma 5.7, there exists $x \in V - (X \cup B)$ such that $T - (A \cup B \cup \{x\})$ and $T^{'} - (A \cup B \cup \{x\})$ are isomorphic. Thus, by Lemma 5.8, we deduce that $T[X - A]$ and $T^{'}[X - A]$ are isomorphic.

Case 2. If $X \in \{X_1, X_k\}$.
First, assume that $X = X_1$. Let $p = |X|$, $A \subset X$ such that $|A| = p - 6$ and suppose that $T[X-A]$ is hemimorphic to $\mathcal{P}$. By Lemma 5.15 it suffices to show that $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. If $|X_k| \leq 11 - p$, we consider a subset $B \subset V - X$ such that $|B| = 11 - p$ and $X_k - B \neq \emptyset$. We directly verify that $X - A$ is the first component of $T - (A \cup B)$ and $T^{'} - (A \cup B)$ and their last component Z is either strongly connected included in $X_k - B$ with $|Z| < |X - A|$ or it is a total order. So, we directly deduce that $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are necessarily isomorphic. Thus, by Lemma 5.8, it results that $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. Assume now that $|X_k| > 11 - p$. If $|X_k| \neq 17 - p$, we consider a subset C of X_k verifying $|C| = 11 - p$ and if $T[X_k]$ is strongly connected and $|X_k - C| \geq 3$, $T[X_k - C]$ is strongly connected. In that case, by seeing either the strong connectivity or the cardinality of the first and the last components of $T - (A \cup C)$ and $T^{'} - (A \cup C)$ we deduce that $T - (A \cup C)$ and $T^{'} - (A \cup C)$ are isomorphic and consequently by Lemma 5.8, $T[X-A]$ and $T^{'}[X-A]$ are isomorphic. Otherwise, if there exists $B \subset X_k$ such that $|B| = 11 - p$ and $T^{'}[X_k - B] \not\simeq T^*[X - A]$ or $T[X_k - B] \not\simeq (T^{'})^*[X - A]$, then $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are necessarily isomorphic. Thus, by Lemma 5.8, $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. At present, assume that for all subset C of X_k such that $|C| = 11 - p$, $T[X_k - C] \simeq (T^{'})^*[X - A]$ and $T^*[X-A] \simeq T^{'}[X_k - C]$. So, $T^{'}[X_k]$ and $T[X_k]$ are $\{-(11-p)\}$-monomorphic. As Lemma 6.3 said that $T^{'}[X_k] \simeq T[X_k]$, it results that $T[X-A]$ and $T^{'}[X-A]$ are isomorphic. Thus, if $X = X_1$, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. Second, assume that $X = X_k$. By applying the first step on the tournaments T^* and $(T^{'})^*$, we deduce that $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. □

This notation is needed.

Notation 6.9. Let R be a tournament defined on a vertex set $V^{'}$, $X \in \widetilde{P}(R)$, an element x of X and a subset B of $V^{'} - X$ with at most 5 elements. We denote by $V^{'}_{(x,B)}$ the set $(V^{'} - (X \cup B)) \cup \{x\}$ and $R_{(x,B)}$ the subtournament $R[V^{'}_{(x,B)}]$.

Lemma 6.10. *If $k = 2$ and $\widetilde{P}(T) = \{\{a\}, X\}$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.*

Proof. First, assume that $T^{'}[X]/\widetilde{P}(T[X]) = T^{*}[X]/\widetilde{P}(T[X])$. As $T[X]$ and $T^{'}[X]$ are (≤ 4)-hypomorphic, then $T[X]/\widetilde{P}(T[X])$ is necessarily a tournament without diamonds. For the same reason for all $Y \in \widetilde{P}(T[X])$, $T[Y]$ has no a 3-cycle; which implies that for all $Y \in \widetilde{P}(T[X])$, $T[Y]$ is a total order. So, $T[X]$ is without diamonds. Let $A \subset X$ such that $|A| = 5$. As from Remark 3.9, each subtournament of $T[X]$ is either a total order or strongly connected, we have to distinguish these two situations. If $T[X - A]$ is strongly connected, we obligatorily have $T^{'} - A \simeq T - A$ and then $T^{'}[X - A] \simeq T[X - A]$. Otherwise, $T[X-A]$ is necessarily a total order and therefore $T^{'}[X-A] \simeq T[X-A]$. Thus, $T[X]$ and $T^{'}[X]$ are $\{-5\}$-hypomorphic. Hence, from Theorem 2.6, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

At present, due to Remark 3.6 one may assume that

$$T[X]/\widetilde{P}(T[X]) = T^{'}[X]/\widetilde{P}(T[X]).$$

In this case, to obtain the result, we have to show that for all $Y \in \widetilde{P}(T[X])$, $T[Y]$ and $T^{'}[Y]$ are hereditarily isomorphic. For this, because Corollary 2.10 presents an important tool in the proof, we have first to establish the following fact in which we study the cases where we can not apply it.

Fact 1. For all $Y \in \widetilde{P}(T[X])$, such that $|Y| > n - |\widetilde{P}(T[X])| - 5$, $T[Y]$ and $T^{'}[Y]$ are hereditarily isomorphic.

Indeed: Let $Y \in \widetilde{P}(T[X])$, such that $|Y| > n - |\widetilde{P}(T[X])| - 5$. First, assume that $6 \leq |Y| \leq 10$. Let $p = |Y|$, $x \in Y$ and $A \subset Y$ such that $|A| = p - 6$. As $|Y| > n - |\widetilde{P}(T[X]) - 5$, then it is clear that for all $Z \in \widetilde{P}(T[X]) - \{Y\}$, $1 \leq |Z| \leq 5$ and for all subset B of $V - Y$, such that $|B| = 11 - p$, $|V_{(x,B)}| \geq 2$. So, by applying Lemma 5.12 to the tournament $\mathcal{T} = T_{(x,\emptyset)}$, there exists $C \subset V - Y$, such that $|C| = 11 - p$ and $s_{\mathcal{T}-C}(x) \neq s_{(\mathcal{T})^{*}-C}(x)$. Suppose that $T[Y - A]$ is hemimorphic to $\mathcal{P}$ and $T^{'}[Y - A] \not\simeq T[Y - A]$. As from Lemma 6.2, T and $T^{'}$ are (≤ 5)-hypomorphic, then by Remark 5.14, $T^{'}[Y - A] \simeq T^{*}[Y - A]$ and $T[Y - A] \not\simeq T^{*}[Y - A]$. We may easily see that there exists an isomorphism g from $T_{(x,C)} = \mathcal{T} - C$ onto $T^{'}_{(x,C)}$, such that $g(x) = x$. In addition, as the tournament $T-(A\cup C)$ (resp. $T^{'}-(A\cup C)$) is obtained from $T_{(x,C)}$ (resp. $T^{'}_{(x,C)}$) by dilating the vertex x by $T[Y-A]$ (resp. $T^{'}[Y-A]$), then Lemma 5.5 said that $T - (A \cup C)$ and $T^{'} - (A \cup C)$ are not hemimorphic; contradiction. Thus, from Lemma 5.15, it follows that $T[Y]$ and $T^{'}[Y]$ are (≤ 6)-hypomorphic and then we apply Corollary 2.1. At present, assume that $|Y| \geq 11$. Consider $A \subset Y$ such that $|A| = 5$. Since $T[X - A]$ is strongly connected on at least 6 vertices, then the tournaments $T - A$ and $T^{'} - A$ are necessarily isomorphic and thus $T[X-A]$ and $T^{'}[X-A]$ are isomorphic. Besides, as for all $Z \in \widetilde{P}(T[X]) - \{Y\}$, $1 \leq |Z| \leq 5$, then $T[Y-A] \simeq T^{'}[Y-A]$. So, $T[Y]$ and $T^{'}[Y]$ are hereditarily isomorphic, by Theorem 2.6.

Now, using Corollary 2.10 and Fact 1, the following three facts permit us to complete the proof.

Fact 2. For all $Y \in \widetilde{P}(T[X])$ (resp. such that $7 \leq |Y| \leq 9$), $T[Y]$ and $T^{'}[Y]$ are isomorphic (resp. hereditarily isomorphic).

Indeed: Consider an element Y of $\widetilde{P}(T[X])$ (resp. an element Y of $\widetilde{P}(T[X])$ such that $p = |Y| \in \{7, 8, 9\}$ and a subset $A \subset Y$ such that $|A| = p - 6$). Given the tournament K obtained from $T[X]/\widetilde{P}(T[X])$ by dilating Y by $T[Y]$ (resp. by $T[Y - A]$) and the tournament H obtained from $T/\widetilde{P}(T)$ by dilating X by K. By Fact 1, we can assume that $|H| \leq n - 5$. Let $y_1 \neq y_2 \in Y$ (resp. $y_1 \neq y_2 \in Y - A$). From Corollary 2.10, it follows that $n(T, H; \{y_1, y_2\}) = n(T^{'}, H; \{y_1, y_2\})$ (resp. $n(T - A, H; \{y_1, y_2\}) = n(T^{'} - A, H; \{y_1, y_2\})$). As $n(T, H; \{y_1, y_2\}) \neq 0$ (resp. $n(T - A, H; \{y_1, y_2\}) \neq 0$), then there exists a subset F of V (resp. of $V - A$) such that $\{y_1, y_2\} \subset F$ and $T^{'}[F]$ is hemimorphic to H. We can verify that $Y \subset F$ (resp. $Y - A \subset F$) for all $Z \in \widetilde{P}(T[X]) - \{Y\}$, $|Z \cap F| = 1$ and $a \in F$. Thus, $T^{'}[F]$ and H are necessarily isomorphic and then $T[Y] \simeq T^{'}[Y]$ (resp. $T[Y - A] \simeq T^{'}[Y - A]$ and then we conclude by Lemma 6.2 and Corollary 2.1).

Fact 3. For all $Y \in \widetilde{P}(T[X])$ such that $|Y| = 10$, $T[Y]$ and $T^{'}[Y]$ are hereditarily isomorphic.

Indeed: Let $Y \in \widetilde{P}(T[X])$ such that $|Y| = 10$. Consider a subset $A \subset Y$ such that $|A| = 4$. From Fact 1, we may assume that there exists $Z \in \widetilde{P}(T[X]) - \{Y\}$ such that $|Z| \geq 2$. Let $x \in Z$. It is clear that $T - (A \cup \{x\})$ and $T^{'} - (A \cup \{x\})$ are necessarily isomorphic. Hence $T[X - (A \cup \{x\})]$ and $T^{'}[X - (A \cup \{x\})]$ are isomorphic. In addition, by using Fact 2, it follows that for all $K \in \widetilde{P}_6(T[X - (A \cup \{x\})]) - \{Y - A\}$, $T^{'}[K] \simeq T[K]$. So, $T[Y - A]$ and $T^{'}[Y - A]$ are isomorphic and then we apply Theorem 2.6.

Fact 4. For all $Y \in \widetilde{P}(T[X])$, such that $|Y| \geq 11$, $T[Y]$ and $T^{'}[Y]$ are hereditarily isomorphic.

Indeed: Let $Y \in \widetilde{P}(T[X])$ such that $|Y| \geq 11$ and a subset $A \subset Y$ such that $|A| = 5$. As $T[X - A]$ is strongly connected on at least 6 vertices, then the tournaments $T - A$ and $T^{'} - A$ are necessarily isomorphic. So, $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. Besides, from Fact 2, we have for all $Z \in \widetilde{P}_{|Y-A|}((T[X])) - \{Y\}$, $T[Z]$ and $T^{'}[Z]$ are isomorphic. Thus, we directly deduce that $T[Y - A]$ and $T^{'}[Y - A]$ are isomorphic and then we conclude by Theorem 2.6. □

Proposition 6.11. *If T is strongly connected, then for all $X \in \widetilde{P}(T)$, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.*

Proof of Proposition 6.11

We proceed by induction on the number n of vertices of the tournaments T and $T^{'}$.

First, assume that $n = 12$. Let $X \in \widetilde{P}(T)$. If $|X| \leq 6$, by Corollary 6.4, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. Otherwise, if $T[X]$ has no subtournaments hemimorphic to $\mathcal{P}$, as T and $T^{'}$ are (≤ 5)-hypomorphic by Lemma 6.2, then according to Lemma 5.15, $T[X]$ and $T^{'}[X]$ are (≤ 6)-hypomorphic. Thus we apply Corollary 2.1. At present, assume that there is $Y \subseteq X$ such that $T[Y]$ is hemimorphic to $\mathcal{P}$. Consider $y \in V - X$ such that $X \to y$ in T and $T^{'}$. Because $Y \to y$ in T and $T^{'}$ and the subtournaments $T[Y \cup \{y\}]$ and $T^{'}[Y \cup \{y\}]$ are hemimorphic (since $|Y \cup \{y\}| = n - 5$), then $T[Y]$ and $T^{'}[Y]$ are necessarily isomorphic. Thus we conclude by Lemma 5.15 and Corollary 2.1.

Second, assume that $n \geq 13$ and that for all integer p with $12 \leq p < n$, if R and $R^{'}$ are two $\{-5\}$-hemimorphic strongly connected tournaments on at least p vertices such that $R/\widetilde{P}(R) = R^{'}/\widetilde{P}(R)$, then for all $Z \in \widetilde{P}(R)$, $T[Z]$ and $T^{'}[Z]$ are hereditarily isomorphic.

Using the following five lemmas, one may immediately deduce the proof of Proposition 6.11.

Lemma 6.12. *Given $X \in \widetilde{P}(T)$, if for all $Y \in \widetilde{P}(T) - \{X\}$, $|Y| \leq 6$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.*

Proof. Let $X \in \widetilde{P}(T)$ such that for all $Y \in \widetilde{P}(T) - \{X\}$, $|Y| \leq 6$ and let $x \in X$. From Corollary 6.4, we may assume that $|X| \geq 7$. We obtain immediately the proof from the following three facts.

Fact 1. If $7 \leq |X| \leq 10$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

Indeed: Using Corollary 2.1 and Lemma 6.2, it is sufficient to demonstrate that $T[X]$ and $T^{'}[X]$ are $\{6\}$-hypomorphic. Denoting $p = |X|$ and let $A \subset X$ such that $|A| = p - 6$. By Lemma 5.15, we should demonstrate that if $T[X - A]$ is hemimorphic to $\mathcal{P}$, then $T^{'}[X - A]$ is isomorphic to $T[X - A]$. Suppose that $T[X - A]$ is hemimorphic to $\mathcal{P}$ and $T^{'}[X - A]$ is isomorphic to $T^{*}[X - A]$. Since $|V_{(x,\emptyset)}| \geq 13 - p$, then from Lemma 5.12, there exists $B \subset V - X$, such that $|B| = 11 - p$ and $s_{T(x,B)}(x) \neq s_{T^{*}(x,B)}(x)$. Furthermore, as for all $Y \in \widetilde{P}(T) - \{X\}$, $T[Y]$ and $T^{'}[Y]$ are hereditarily isomorphic and $T^{'}/\widetilde{P}(T) = T/\widetilde{P}(T)$, then we directly verify that there exists an isomorphism g from $T_{(x,B)}$ onto $T^{'}_{(x,B)}$ such that $g(x) = x$. Since $T - (A \cup B)$ (resp. $T^{'} - (A \cup B)$) is obtained from the tournament $T_{(x,B)}$ (resp. $T^{'}_{(x,B)}$) by dilating the vertex x by $T[X - A]$ (resp. $T^{'}[X - A]$), then from Lemma 5.5, $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are not hemimorphic; which contradicts the fact that $T^{'}$ and T are $\{-5\}$-hemimorphic.

Fact 2. If $|X| \geq 11$, and there exists $Y \in \widetilde{P}(T) - \{X\}$ such that $|Y| \geq 2$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

Indeed: From Theorem 2.6, it is sufficient to show that $T[X]$ and $T^{'}[X]$ are $\{-4\}$-hypomorphic. Let $A \subset X$ such that $|A| = 4$, $Y \in \widetilde{P}(T) - \{X\}$, such that $|Y| \geq 2$ and $y \in Y$. It is clear that $T - (A \cup \{y\})$ and $T^{'} - (A \cup \{y\})$ are strongly connected hemimorphic tournaments such that $\widetilde{P}(T - (A \cup \{y\})) = \widetilde{P}(T^{'} - (A \cup \{y\})) = (\widetilde{P}(T) - \{X, Y\}) \cup \{X - A, Y - \{y\}\}$. As $|X - A| \geq 7$ and for all $Z \in \widetilde{P}(T - (A \cup \{y\})) - \{X - A\}$, $|Z| \leq 6$, we directly verify that $T[X-A]$ and $T^{'}[X-A]$ are hemimorphic. Suppose that $T^{'}[X-A] \simeq T^{*}[X-A]$ and $T[X - A]$ is not selfdual. As $|V_{(x,\emptyset)}| \geq 3$, then from Lemma 5.12, there exists $b \in V - X$ such that $s_{T(x,\{b\})}(x) \neq s_{T^{*}(x,\{b\})}(x)$. Hence, there exists an isomorphism g from $T_{(x,\{b\})}$ onto $T^{'}_{(x,\{b\})}$ such that $g(x) = x$. Since $T - (A \cup \{b\})$ (resp. $T^{'} - (A \cup \{b\})$) is obtained from the tournament $T_{(x,\{b\})}$ (resp. $T^{'}_{(x,\{b\})}$) by dilating the vertex x by $T[X - A]$ (resp. $T^{'}[X - A]$), then from Lemma 5.5, $T - (A \cup \{b\})$ and $T^{'} - (A \cup \{b\})$ are not hemimorphic; contradiction.

Fact 3. If $|X| \geq 11$, and for all $Y \in \widetilde{P}(T) - \{X\}$, $|Y| = 1$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. Indeed:

Case 1. $s_{T(x,\emptyset)}(x) \neq s_{T^{*}(x,\emptyset)}(x)$.
Using Theorem 2.6, it is enough to show that $T[X]$ and $T^{'}[X]$ are $\{-5\}$-hypomorphic. Consider $A \subset X$ such that $|A| = 5$. It is clear that $T - A$ and $T^{'} - A$ are strongly connected such that $\widetilde{P}(T - A) = \widetilde{P}(T^{'} - A) = (\widetilde{P}(T) - \{X\}) \cup \{X - A\}$. As $|X - A| \geq 6$ and for all $Z \in \widetilde{P}(T) - \{X\}$, $|Z| = 1$, then $f_A(X - A) = X - A$. Furthermore, as $s_{T(x,\emptyset)}(x) \neq s_{T^{*}(x,\emptyset)}(x)$, f_A is necessarily an isomorphism and then $T[X - A] \simeq T^{'}[X - A]$.

Case 2. If $s_{T(x,\emptyset)}(x) = s_{T^{*}(x,\emptyset)}(x)$.
Clearly, $|\widetilde{P}(T)|$ is odd. From Theorem 2.6 it is sufficient to prove that $T[X]$ and $T^{'}[X]$ are $\{-4\}$-hypomorphic. Let $A \subset X$ such that $|A| = 4$. If $|T/\widetilde{P}(T)| \geq 5$, as $T/\widetilde{P}(T)$ is indecomposable, then from Corollary 5.4 there exist two distinct elements $Y, Z \in \widetilde{P}(T) - \{X\}$ such that $(T/\widetilde{P}(T)) - \{Y, Z\}$ is indecomposable. Let $V^{'} = P(T) - \{Y, Z\}$. Necessarily, there exists $\alpha \in \{Y, Z\}$, such that $\alpha \notin V^{'}(X)$. For instance, $Y \notin V^{'}(X)$. From Proposition 5.2, we have to distinguish these three situations.

- $Y \in \langle V^{'} \rangle$. In that case we obtain either $Y \to V - (A \cup Y \cup Z)$ or $V - (A \cup Y \cup Z) \to Y$ in $T - (A \cup Z)$ and $T^{'} - (A \cup Z)$. We may also verify that $T - (A \cup Y \cup Z)$ and $T^{'} - (A \cup Y \cup Z)$ are strongly connected non singleton tournaments. So, $f_{(A \cup Z)}$ is necessarily an isomorphism from $T - (A \cup Z)$ onto $T^{'} - (A \cup Z)$ and in particular $f_{(A \cup Z)}(V - (A \cup Y \cup Z)) = V - (A \cup Y \cup Z)$. Furthermore, as $X - A$ is the only element of $\widetilde{P}(T - (A \cup Y \cup Z)) = \widetilde{P}(T^{'} - (A \cup Y \cup Z))$ which is not a singleton, then $f_{(A\cup Z)}(X-A)=X-A$. Thus $T[X-A]$ and $T^{'}[X-A]$ are isomorphic.

- $Y\in \mathrm{Ext}(V^{'})$. Clearly, $X-A$ is the only element of $\widetilde{P}(T-(A\cup Z))=\widetilde{P}(T^{'}-(A\cup Z))$ which is not a singleton. So $f_{(A\cup Z)}(X-A)=X-A$. As moreover $s_{T(x,Z)}(x)\neq s_{T^*(x,Z)}(x)$, $f_{(A\cup Z)}$ is necessarily an isomorphism from $T-(A\cup Z)$ onto $T^{'}-(A\cup Z)$. So we directly deduce that $T[X-A]$ and $T^{'}[X-A]$ are isomorphic.
- $Y\in V^{'}(l)$, where $l\in V^{'}-\{X\}$. One may directly verify that $\widetilde{P}(T-(A\cup Z))=\widetilde{P}(T^{'}-(A\cup Z))$ admits only two non-singleton elements, one of them is on two elements denoted by W and contains Y and the other is $X-A$. Because $|X-A|\geq 7$, it results that $f_{(A\cup Z)}(X-A)=X-A$ and $f_{(A\cup Z)}(W)=W$. Consequently, $f_{(A\cup Z)}$ is necessarily an isomorphism from $T-(A\cup Z)$ onto $T^{'}-(A\cup Z)$ and then $T[X-A]$ and $T^{'}[X-A]$ are isomorphic.

Now assume that $|T/\widetilde{P}(T)|=3$. We denote $\{Y,Z\}=T/\widetilde{P}(T)-\{X\}$. If there exists $l\in\{Y,Z\}$ such that $T-(A\cup l)\simeq T^{'}-(A\cup l)$, the result is obtained, otherwise, $T^{'}[X-A]$ and $T[X-A]$ are obligatorily non-strongly connected tournaments and non total orders. Since T and $T^{'}$ are (≤ 4) hypomorphic, it follows from Remark 3.7 that $\widetilde{P}(T[X-A])=\widetilde{P}(T^{'}[X-A])=\{J_1,\dots,J_p\}$ such that $p\geq 2$ and $T[X-A]/\widetilde{P}(T[X-A])=T^{'}[X-A]/\widetilde{P}(T[X-A])$ with: $J_1<\dots<J_p$. Since $T^*-(A\cup Y)\simeq T^{'}-(A\cup Y)$ and $T^*-(A\cup Z)\simeq T^{'}-(A\cup Z)$, then $T[J_1]$ and $T^{'}[J_p]$ are total orders. Thus, $1+|J_1|=|J_p|$ and $1+|J_p|=|J_1|$; contradiction. □

Lemma 6.13. *For all $X\in\widetilde{P}(T)$ such that $|X|\geq 12$, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.*

Proof. Let $X\in\widetilde{P}(T)$ such that $|X|\geq 12$ and $x\in X$. We immediately deduce the proof from the following two facts.

Fact 1. $T^{'}[X]$ and $T[X]$ are $\{-5\}$-hemimorphic.

Indeed: Let $A\subset X$ such that $|A|=5$ and proving that $T[X-A]$ and $T^{'}[X-A]$ are hemimorphic. Clearly, $T-A$ and $T^{'}-A$ are strongly connected tournaments such that $\widetilde{P}(T^{'}-A)=\widetilde{P}(T-A)=(\widetilde{P}(T)-\{X\})\cup\{X-A\}$. As from Lemma 6.3, for all $Z\in\widetilde{P}_{|X-A|}(T-A)-\{X-A\}$, $T[Z]$ and $T^{'}[Z]$ are hemimorphic, then the hemimorphy between $T-A$ and $T^{'}-A$ requires the hemimorphy between $T[X-A]$ and $T^{'}[X-A]$.

Fact 2. $T^{'}[X]$ and $T[X]$ are hereditarily isomorphic.

Indeed: If $T[X]$ is a total order, the result is obvious. Otherwise, if $T[X]$ is not strongly connected, as by Remark 3.7, $T[X]/\widetilde{P}(T[X])=T^{'}[X]/\widetilde{P}(T[X])$, then Proposition 6.6 permits us to conclude. Now assume that $T[X]$ is strongly connected. If $T[X]/\widetilde{P}(T[X]) = T^{'}[X]/\widetilde{P}(T[X])$, we apply the induction hypothesis. Otherwise, as proven in the beginning of Lemma 6.10, the subtournament $T[X]$ is without diamonds.

First, assume that $s_{T_{(x,\emptyset)}}(x) \neq s_{T^*_{(x,\emptyset)}}(x)$. From Theorem 2.6, it is sufficient, to prove that $T[X]$ and $T^{'}[X]$ are $\{-5\}$-hypomorphic. Given $A \subset X$ such that $|A| = 5$ and proving that $T[X-A] \simeq T^{'}[X-A]$. Because $T[X]$ is a tournament without diamonds and $T^*[X]/\widetilde{P}(T[X]) = T^{'}[X]/\widetilde{P}(T[X])$, from Remark 3.9 it follows that $T^{'}[X]$ and $T^*[X]$ are hereditarily isomorphic. So, to obtain the result it suffices to show that $T[X-A]$ is selfdual. Suppose by contradiction that it is not selfdual. As for all $Z \in \widetilde{P}(T)-\{X\}$, $T[Z]$ and $T^{'}[Z]$ are isomorphic, there exists an isomorphism g from $T_{(x,\emptyset)}$ onto $T^{'}_{(x,\emptyset)}$ such that $g(x) = x$. Since $T-A$ (resp. $T^{'}-A$) is obtained from the tournament $T_{(x,\emptyset)}$ (resp. $T^{'}_{(x,\emptyset)}$) by dilating the vertex x by $T[X-A]$ (resp. $T^{'}[X-A]$), then from Lemma 5.5, $T-A$ and $T^{'}-A$ are not hemimorphic; contradiction.

Second, assume that $s_{T_{(x,\emptyset)}}(x) = s_{T^*_{(x,\emptyset)}}(x)$. Using Theorem 2.6 and Remark 3.9, it is enough to show that $T[X]$ is $\{-4\}$-selfdual. Suppose by contradiction that there is $B \subset X$ such that $|B| = 4$ and $T[X-B]$ is not selfdual. According to Lemma 6.12, we may assume that there exists $Y \in \widetilde{P}(T)-\{X\}$ such that $|Y| \geq 7$. Let $M \in \widetilde{P}(T)-\{X\}$ such that $|M| = \min\{|Z|; Z \in \widetilde{P}(T)-\{X\}$ and $|Z| \geq 2\}$. If there exists $m \in M$ such that $T^{'}[M-\{m\}] \simeq T[M-\{m\}]$, we may easily verify that $s_{T_{(x,\{m\})}}(x) \neq s_{T^*_{(x,\{m\})}}(x)$ and then we obtain a contradiction by Lemma 5.5. Otherwise, as T and $T^{'}$ are (≤ 5)-hypomorphic, then $|M| \geq 7$. Considering $m \in M$ and envisaging the following three cases.

Case 1. If $|X-B| > |M|-1$.
As $T^{'}-(B\cup\{m\})$ and $T-(B\cup\{m\})$ are strongly connected with $\widetilde{P}_{|M|-1}(T^{'}-(B\cup\{m\})) = \widetilde{P}_{|M|-1}(T-(B\cup\{m\})) = \{M-\{m\}\}$. Thus, $f_{(B\cup\{m\})}(M-\{m\}) = M-\{m\}$. Since $T[M-\{m\}] \not\simeq T^{'}[M-\{m\}]$, then $f_{(B\cup\{m\})}$ is necessarily an isomorphism from $T^{'}-(B\cup\{m\})$ onto $T^*-(B\cup\{m\})$. It results that $s_{T_{(m,B)}}(m) = s_{T^*_{(m,B)}}(m)$. Let $a \neq b \in M$ and $C \subset X$ such that $|C| = 3$. It is clear that $T-(C\cup\{a,b\})$ and $T^{'}-(C\cup\{a,b\})$ are strongly connected with $\widetilde{P}_{|M|-2}(T-(C\cup\{a,b\})) = \widetilde{P}_{|M|-2}(T^{'}-(C\cup\{a,b\})) = \{M-\{a,b\}\}$. So, $f_{(C\cup\{a,b\})}(M-\{a,b\}) = M-\{a,b\}$. As $s_{T_{(m,C)}}(m) \neq s_{T^*_{(m,C)}}(m)$, then $f_{(C\cup\{a,b\})}$ is necessarily an isomorphism from $T-(C\cup\{a,b\})$ onto $T^{'}-(C\cup\{a,b\})$ and in particular $T[M-\{a,b\}] \simeq T^{'}[M-\{a,b\}]$. As from Lemma 6.3, for all $Z \in \widetilde{P}(T-(C\cup\{a,b\}))-\{X-C, M-\{a,b\}\}$, $T[Z]$ and $T^{'}[Z]$ are isomorphic, it results that $T[X-C] \simeq T^{'}[X-C]$ and then $T[X-C] \simeq T^*[X-C]$. Hence, $T[X]$ is $\{-3\}$-selfdual. Thus, it follows from Lemma 2.7 that $T[X]$ and $T^*[X]$ are hereditarily isomorphic; which contradicts the fact that $T[X-B]$ is not selfdual.

Case 2. If $|X - B| < |M| - 1$.
It is clear that $T^{'} - (B \cup \{m\})$ and $T - (B \cup \{m\})$ are strongly connected with $\widetilde{P}_{|X-B|}(T^{'} - (B \cup \{m\})) = \widetilde{P}_{|X-B|}(T - (B \cup \{m\})) = \{X - B\}$ and $\widetilde{P}_{|M|-1}(T^{'} - (B \cup \{m\})) = \widetilde{P}_{|M|-1}(T - (B \cup \{m\})) = \{M - \{m\}\}$. So, $f_{(B\cup\{m\})}(X - B) = X - B$ and $f_{(B\cup\{m\})}(M - \{m\}) = M - \{m\}$. Thus, $f_{(B\cup\{m\})}$ is necessarily an isomorphism from $T - (B \cup \{m\})$ onto $T^{'} - (B \cup \{m\})$; which contradicts the fact that $T^{'}[M - \{m\}] \not\simeq T[M - \{m\}]$.

Case 3. If $|X - B| = |M| - 1$.
Consider an element y of M. Clearly, $T^{'} - (B \cup \{y\})$ and $T - (B \cup \{y\})$ are strongly connected with $\widetilde{P}_{|M|-1}(T^{'} - (B \cup \{y\})) = \widetilde{P}_{|M|-1}(T - (B \cup \{y\})) = \{X - B, M - \{y\}\}$. Moreover, as $T^{'}[M - \{y\}] \not\simeq T[M - \{y\}]$, then $f_{(B\cup\{y\})}$ is an isomorphism from $T^{'} - (B \cup \{y\})$ onto $T^{*} - (B \cup \{y\})$ such that $f_{(B\cup\{y\})}(X - B) = M - \{y\}$ and $f_{(B\cup\{y\})}(M - \{y\}) = X - B$. So, $T^{'}[X - B] \simeq T^{*}[M - \{y\}]$ and $T^{'}[M - \{y\}] \simeq T^{*}[X - B]$. Thus, $T^{'}[M]$ and $T[M]$ are $\{-1\}$-monomorphic. Since $T^{'}[X - B] \simeq T^{*}[X - B]$, then for all $z \in M$, $T^{'}[M - \{z\}] \simeq T^{*}[M - \{z\}]$ and thus $T^{'}[M]$ and $T^{*}[M]$ are $\{-1\}$-hypomorphic. Furthermore, as $T[X - B]$ is a tournament without diamonds, $T[M]$ is also without diamonds. Using Remark 3.9 and Lemma 6.2, it results that $T^{'}[M]$ and $T^{*}[M]$ are $\{4\}$-hypomorphic. Hence, $T^{'}[M]$ and $T^{*}[M]$ are $\{4, -1\}$-hypomorphic and then they are isomorphic by Theorem 4.5. Besides, as $T^{'}[M] \simeq T[M]$, by Lemma 6.3, then $T[M]$ is selfdual. From the $\{-1\}$-monomorphy of $T[M]$, it follows that for all $z \in M$, $T[M - \{z\}]$ is selfdual and then $T^{'}[M - \{z\}] \simeq T[M - \{z\}]$; which is absurd. $\square$

Lemma 6.14. *For all $X \in \widetilde{P}(T)$ such that $|X| = 7$, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.*

Proof. Let $X \in \widetilde{P}(T)$ such that $|X| = 7$ and $x \in X$. By Lemma 6.12, we may assume, in the sequel of this proof, that there exists $Y \in \widetilde{P}(T) - \{X\}$ such that $|Y| \geq 7$. The following two facts allow us to directly obtain the proof.

Fact 1. If $|X| \leq n - |P(T)| - 9$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

Indeed: Let $B \subseteq Y$ such that $|B| = 7$, $a \neq b \in X - \{x\}$, $c \neq d \in B$ and consider the tournament H obtained from $T/\widetilde{P}(T)$ by dilating X by $T[X - \{x\}]$ and Y by $T[B]$. Since $T - x$ and $T^{'} - x$ are $\{-4\}$-hemimorphic and $|H| \leq n - 5$, it follows from Corollary 2.10 that $n(T - x, H; \{a, b, c, d\}) = n(T^{'} - x, H; \{a, b, c, d\})$. As $n(T - x, H; \{a, b, c, d\}) \neq 0$, then there exists a subset K of $V - \{x\}$ such that $a, b, c, d \in K$ and $T^{'}[K]$ is hemimorphic to H. As $K \cap (X - \{x\})$ and $K \cap Y$ are non trivial intervals of $T^{'}[K]$, we directly verify that: $X - \{x\} \subset K$, $|K \cap Y| = 7$ and for all $Z \in \widetilde{P}(T) - \{X, Y\}$, $|K \cap Z| = 1$. Thus, $T^{'}[K]$ and H are necessarily isomorphic. So, $T[X - \{x\}]$ and $T^{'}[X - \{x\}]$ are isomorphic and then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic, by Lemma 6.2 and Corollary 2.1.

Fact 2. If $|X| \geq n - |P(T)| - 8$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

Indeed: Clearly, for all $Z \in \widetilde{P}(T) - \{X, Y\}$, $1 \leq |Z| \leq 5$.

- If $|Y| \neq 10$. One can verify that there is a subset A of $V - X$ such that $|A| = 4$ and satisfying the following: for all $Z \in \widetilde{P}(T) - \{X\}$, $Z - A \neq \emptyset$, $\widetilde{P}_6(T - (A \cup \{x\})) = \{X - \{x\}\}$ and there is an integer $k \geq 2$ fulfilling $k \neq 6$ and $|\widetilde{P}_k(T - (A \cup \{x\}))| = 1$. Clearly, $T - (A \cup \{x\})$ and $T^{'} - (A \cup \{x\})$ are necessarily isomorphic. Hence $T[X - \{x\}]$ and $T^{'}[X - \{x\}]$ are isomorphic and thus they are hereditarily isomorphic, by Lemma 6.2 and Corollary 2.1.
- If $|Y| = 10$. Consider a subset B of Y such that $|B| = 3$ and a subset A of V such that $|A| = 2$ and verifying: $A \subset X$ if $\widetilde{P}_5(T) = \emptyset$, otherwise there is $Z \in \widetilde{P}_5(T)$ such that $|A \cap Z| = |A \cap X| = 1$. Clearly, $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are strongly connected with $\widetilde{P}_7(T - (A \cup B)) = \widetilde{P}_7(T^{'} - (A \cup B)) = \{Y - B\}$ and there is $k \in \{5, 6\}$ such that $\widetilde{P}_k(T^{'} - (A \cup B)) = \widetilde{P}_k(T - (A \cup B)) = \{X - A\}$. Therefore, we deduce that $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are isomorphic and in particular $T[Y - B] \simeq T^{'}[Y - B]$. So, $T[Y]$ and $T^{'}[Y]$ are $\{-3\}$-hypomorphic. At present, suppose that $T[X - \{x\}]$ is hemimorphic to $\mathcal{P}$ and $T^{'}[X - \{x\}] \simeq T^{*}[X - \{x\}]$. Let C be a subset of Y such that $|C| = 3$. As $|V - (X \cup C)| \geq 2$, it follows from Lemma 5.12 that there exists $z \in V - (X \cup C)$ such that $s_{T(x, C \cup \{z\})}(x) \neq s_{T^{*}(x, C \cup \{z\})}(x)$. By Lemma 5.5, we can deduce that the tournaments $T - (C \cup \{a, z\})$ and $T^{'} - (C \cup \{a, z\})$ are not hemimorphic; contradiction. So, $T^{'}[X - \{x\}] \simeq T[X - \{x\}]$ and then from Lemma 5.15 and Corollary 2.1, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. □

Lemma 6.15. *For all* $X \in \widetilde{P}(T)$ *such that* $8 \leq |X| \leq 10$*,* $T[X]$ *and* $T^{'}[X]$ *are hereditarily isomorphic.*

Proof. Let $X \in \widetilde{P}(T)$ such that $8 \leq |X| \leq 10$ and $x \in X$. For each value $p = |X|$, we can assume that for all $Y \in \widetilde{P}(T)$ such that $1 \leq |Y| \leq p - 1$, $T[Y]$ and $T^{'}[Y]$ are hereditarily isomorphic. So, to demonstrate the lemma for $p = |X|$, we should show the result firstly for $p = 8$, secondly for $p = 9$ and finally for $p = 10$. Consider the following sets:

$$E_1 = \bigcup_{k=1}^{p-1} \widetilde{P}_k(T), E_2 = (\bigcup_{k \geq p} \widetilde{P}_k(T)) - \{X\}, F_1 = \bigcup_{Y \in E_1} Y \text{ and } F_2 = \bigcup_{Y \in E_2} Y.$$

Let $A \subset X$ such that $|A| = p - 6$. The following three facts permit us to deduce immediately the proof.

Fact 1. If either ($E_2 = \emptyset$) or ($E_1 \neq \emptyset$ and there exists $B \subseteq F_1$ such that $|B| = 11 - p$ and $s_{T(x,B)} \neq s_{T^{*}(x,B)}$), then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

Indeed: By Lemma 6.12, we may assume that there exists $Y \in \widetilde{P}(T) - \{X\}$ such that $|Y| \geq 7$. If $E_2 = \emptyset$, it follows from Lemma 5.12 that there exits $C \subset V - X = F_1$ such that $|C| = 11 - p$ and $s_{T(x,C)}(x) \neq s_{T^*(x,C)}(x)$. Hence, in the two situations of the hypothesis of the fact there exists $B \subset F_1$ such that $|B| = 11 - p$ and $s_{T(x,B)}(x) \neq s_{T^*(x,B)}(x)$. Suppose that $T[X - A]$ is hemimorphic to $\mathcal{P}$ and $T^{'}[X - A]$ is isomorphic to $T^*[X - A]$. As for all $Z \in \widetilde{P}(T) - \{X\}$ (resp. $Z \in E_1$), $T^{'}[Z]$ and $T[Z]$ are isomorphic (resp. hereditarily isomorphic), then there exists an isomorphism g from $T_{(x,B)}$ onto $T^{'}_{(x,B)}$ such that $g(x) = x$. In addition, the tournament $T - (A \cup B)$ (resp. $T^{'} - (A \cup B)$) is obtained from $T_{(x,B)}$ (resp. $T^{'}_{(x,B)}$) by dilating the vertex x by $T[X - A]$ (resp. $T^{'}[X - A]$). Then from Lemma 5.5, $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are not hemimorphic; contradiction. Thus, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic, by Lemma 5.15 and Corollary 2.1.

Fact 2. If $E_2 \neq \emptyset$ and either ($|F_1| \leq 2$) or ($|F_1| \geq 3$ and for all $Y \in E_1$, $1 \leq |Y| \leq 11 - p$), then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

Indeed: Consider an element Z of E_2 such that $|Z| = k = \min\{|Y|; Y \in E_2\}$.

- If either $p \in \{9, 10\}$ or ($p = 8$ and $k \neq 9$). Consider a subset C of Z such that $|C| = 11 - p$. Clearly, $T - (A \cup C)$ and $T^{'} - (A \cup C)$ are strongly connected tournaments with $\widetilde{P}_6(T - (A \cup C)) = \widetilde{P}_6(T^{'} - (A \cup C) = \{X - A\}$ and $\widetilde{P}_{k-(11-p)}(T - (A \cup C)) = \widetilde{P}_{k-(11-p)}(T^{'} - (A \cup C) = \{Z - C\}$. So, $f_{(A \cup C)}(X - A) = X - A$ and $f_{(A \cup C)}(Z - C) = Z - C$. Hence, $f_{(A \cup C)}$ is necessarily an isomorphism and in particular $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. Consequently, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic, by Lemma 6.2 and Corollary 2.1.
- If ($p = 8$ and $k = 9$). If there is a subset C of Z such that $|C| = 3$ and $T^{'}[Z - C] \not\simeq T^*[X - A]$, then it results that $T^{'} - (A \cup C) \simeq T - (A \cup C)$ and then $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. Otherwise, the tournament $T^{'}[Z]$ is $\{-3\}$-monomorphic and then by the combinatorial lemma of Pouzet, it is $\{3\}$-monomorphic. Besides, as $|Z| = 9$ and every tournament on four vertices contains at least a total order on three vertices, then $T^{'}[Z]$ is a total order. Thus, $T[X - A]$ is a total order and then $T[X - A]$ and $T^{'}[X - A]$ are isomorphic. According to Lemma 6.2 and Corollary 2.1, $T^{'}[X]$ and $T[X]$ are hereditarily isomorphic.

Fact 3. If $E_2 \neq \emptyset$, $|F_1| \geq 3$ and there is $Y \in E_1$ such that $|Y| > 11 - p$, then $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.

Indeed: By Fact 1, we may assume that for all $B \subseteq F_1$, such that $|B| = 11 - p$, $s_{T(x,B)}(x) = s_{T^*(x,B)}(x)$. So, we may directly verify that F_1 is an interval of $T[X \cup F_1]$. Let B be a subset of Y such that $|B| = 11 - p$. Clearly, $T - (A \cup B)$ and $T^{'} - (A \cup B)$ are strongly connected and $\widetilde{P}(T - (A \cup B)) = \widetilde{P}(T^{'} - (A \cup B)) = E_2 \cup \{X - A, Y - B\} \cup (E_1 - \{Y\})$. So, $f_{(A \cup B)}(\{X - A, Y - B\} \cup (E_1 - \{Y\})) = \{X - A, Y - B\} \cup (E_1 - \{Y\})$ and in particular $T^{'}[(F_1 \cup X) - (A \cup B)]$ and

$T[(F_1 \cup X)-(A \cup B)]$ are hemimorphic. Suppose that $T[X-A]$ is hemimorphic to $\mathcal{P}$ and $T^{'}[X-A]$ is isomorphic to $T^*[X-A]$. As, for all $Z \in E_1$, $T[Z]$ and $T^{'}[Z]$ are hereditarily isomorphic and F_1 and X are two intervals of $T[F_1 \cup X]$ and $T^{'}[F_1 \cup X]$, then by Lemma 5.5, we may easily verify that $T^{'}[(F_1 \cup X)-(A \cup B)]$ and $T[(F_1 \cup X)-(A \cup B)]$ are not hemimorphic; contradiction. So, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. □

Lemma 6.16. *For all $X \in \widetilde{P}(T)$ such that $|X| = 11$, $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic.*

Proof. Notice that from the preceding lemmas it follows that for all $Z \in \widetilde{P}(T)$ such that $|Z| \neq 11$, $T[Z]$ and $T^{'}[Z]$ are hereditarily isomorphic. Let $X \in \widetilde{P}(T)$ such that $|X| = 11$ and $x \in X$. If $s_{T(x,\emptyset)} \neq s_{T^*(x,\emptyset)}$, we consider a subset $A \subset X$ such that $|A| = 5$. As for all $Y \in \widetilde{P}(T) - \{X\}$, $T[Y]$ and $T^{'}[Y]$ are isomorphic, it follows from Lemmas 5.5 and 5.15, that $T^{'}[X-A]$ and $T[X-A]$ are isomorphic and thus they are hereditarily isomorphic, by Corollary 2.1. Otherwise, we consider a subset $A \subset X$ such that $|A| = 4$. From Lemma 6.12, we may assume that there exists $Y \in \widetilde{P}(T) - \{X\}$ such that $|Y| \geq 7$. First, assume that there exists $Z \in \widetilde{P}(T) - \{X\}$, such that $|Z| \geq 2$ and $|Z| \neq 11$. Let $y \in Z$. As $T - (A \cup \{y\})$ and $T^{'} - (A \cup \{y\})$ are strongly connected, $X - A \in \widetilde{P}_7(T - (A \cup \{y\})) = \widetilde{P}_7(T^{'} - (A \cup \{y\}))$ and for all $Y \in \widetilde{P}_7(T - (A \cup \{y\})) - \{X - A\}$, $T[Y]$ and $T^{'}[Y]$ are hemimorphic, then the hemimorphy between $T - (A \cup \{y\})$ and $T^{'} - (A \cup \{y\})$ requires the hemimorphy between $T^{'}[X - A]$ and $T[X - A]$. Assume that $T^{'}[X - A]$ is isomorphic to $T^*[X - A]$ and $T^*[X - A] \not\simeq T[X - A]$. It is clear that $s_{T(x,\{y\})}(x) \neq s_{T^*(x,\{y\})}(x)$. So, as $T[Z - \{y\}] \simeq T^{'}[Z - \{y\}]$ and for all $M \in \widetilde{P}(T) - \{X, Z\}$, $T^{'}[M] \simeq T[M]$, then we conclude by Lemma 5.5 and Theorem 2.6. At present, assume that for all $Y \in \widetilde{P}(T) - (\{X\} \cup \widetilde{P}_1(T))$, $|Y| = 11$. Consider $Z \in \widetilde{P}_{11}(T) - \{X\}$ and $z \in Z$. As $\widetilde{P}_7(T - (A \cup \{z\})) = \{X - A\}$ and $s_{T(x,\{z\})}(x) \neq s_{T^*(x,\{z\})}(x)$, then $T^{'} - (A \cup \{z\})$ and $T - (A \cup \{z\})$ are necessarily isomorphic. Thus, $T^{'}[X - A]$ and $T[X - A]$ are isomorphic and then Theorem 2.6 said that $T[X]$ and $T^{'}[X]$ are hereditarily isomorphic. □

References

[1] J. A. Bondy, *A graph reconstructor's manual*, Surveys in combinatorics, 1991 (Guildford, 1991), 221–252, London Math. Soc. Lecture Note Ser., 166, Cambridge Univ. Press, Cambridge, 1991.

[2] J. A. Bondy and R. L. Hemminger, *Graph reconstruction—a survey*, J. Graph Theory **1** (1977), no. 3, 227–268.

[3] M. Bouaziz and Y. Boudabbous, *La demi-isomorphie et les tournois fortement connexes finis*, C. R. Math. Acad. Sci. Paris **335** (2002), no. 2, 105–110.

[4] H. Bouchaala and Y. Boudabbous, *La $\{-k\}$-autodualité des sommes lexicographiques finies de tournois suivant un 3-cycle ou un tournoi critique*, Ars Combin. **81** (2006), 33–64.

[5] Y. Boudabbous, *Isomorphie héréditaire et $\{-4\}$-hypomorphie pour les tournois*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 15-16, 841–844.

[6] Y. Boudabbous and J. Dammak, *Sur la $(-k)$-demi-reconstructibilité des tournois finis*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 9, 1037–1040.

[7] Y. Boudabbous and G. Lopez, *La relation différence et l'anti-isomorphie*, Math. Logic Quart. **41** (1995), no. 2, 268–280.

[8] A. Boussaïri, P. Ille, G. Lopez, and S. Thomassé, *The C_3-structure of the tournaments*, Discrete Math. **277** (2004), no. 1-3, 29–43.

[9] A. Ehrenfeucht and G. Rozenberg, *Primitivity is hereditary for 2-structures*, Theoret. Comput. Sci. **70** (1990), no. 3, 343–358.

[10] R. Fraïssé, *Abritement entre relations et spécialement entre chaînes*, Symposia Mathematica, Vol. V (INDAM, Rome, 1969/70) pp. 203–251 Academic Press, London, 1971.

[11] T. Gallai, *Transitiv orientierbare Graphen*, Acta Math. Acad. Sci. Hungar **18** (1967), 25–66.

[12] C. Gnanvo and P. Ille, *La reconstructibilité des tournois*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 14, 577–580.

[13] J. G. Hagendorf and G. Lopez, *La demi-reconstructibilité des relations binaires d'au moins 13 elements*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 1, 7–12.

[14] P. Ille, *La reconstructibilité des relations binaires*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 15, 635–638.

[15] G. Lopez, *Deux résultats concernant la détermination d'une relation par les types d'isomorphie de ses restrictions*, C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A1525–A1528.

[16] ———, *Sur la détermination d'une relation par les types d'isomorphie de ses restrictions*, C. R. Acad. Sci. Paris Sér. A-B **275** (1972), A951–A953.

[17] ———, *L'indéformabilité des relations et multirelations binaires*, Z. Math. Logik Grundlag. Math. **24** (1978), no. 4, 303–317.

[18] G. Lopez and C. Rauzy, *Reconstruction of binary relations from their restrictions of cardinality $2,3,4$ and $(n-1)$. I*, Z. Math. Logik Grundlag. Math. **38** (1992), no. 1, 27–37.

[19] ———, *Reconstruction of binary relations from their restrictions of cardinality $2,3,4$ and $(n-1)$. II*, Z. Math. Logik Grundlag. Math. **38** (1992), no. 2, 157–168.

[20] J. W. Moon, *Topic on Tournaments, Holt, Rinehart and Winston*, New York, 1968.

[21] M. Pouzet, *Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations*, Math. Z. **150** (1976), no. 2, 117–134.

[22] ———, *Relations non reconstructibles par leurs restrictions*, J. Combin. Theory Ser. B **26** (1979), no. 1, 22–34.

[23] K. B. Reid and C. Thomassen, *Strongly self-complementary and hereditarily isomorphic tournaments*, Monatsh. Math. **81** (1976), no. 4, 291–304.

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