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REGULARITY OF SOLUTIONS OF ABSTRACT QUASILINEAR DELAY INTEGRODIFFERENTIAL EQUATIONS

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Abstract. We prove the existence and uniqueness of classical solutions for a quasilinear delay integrodifferential equation in Banach spaces. The result is established by using the semigroup theory and the Banach fixed point theorem.

1. Introduction

Abstract quasilinear integrodifferential equations arise in many areas of science such as population dynamics, mathematical physics, heat conduction theory of material with memory etc. For this reason, this type of equations have received much attention in recent years. The literature related to quasilinear differential and integrodifferential equations is very extensive. A general theory of quasilinear evolution equations has been developed by Kato [14, 15]. Using the method of semigroup, existence and uniqueness of mild and classical solutions of quasilinear evolution equations have been discussed by Pazy [20]. The problem of existence of solutions of quasilinear evolution equations in Banach spaces has been studied by several authors [2, 6, 15, 16, 17, 18]. Pazy [20] considered the following quasilinear equation of the form

$$\begin{aligned} u'(t) + A(t,u)u(t) &= 0, \quad 0 < t \le T, \\ u(0) &= u_0, \end{aligned}$$

and discussed the mild and classical solutions by using the fixed point argument. The existence of classical solution has been studied to the nonhomogeneous quasilinear evolution equation

$$\begin{aligned} u'(t) + A(t,u)u(t) &= f(t,u), \quad 0 < t \le T, \\ u(0) &= u_0, \end{aligned}$$

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by Furuya [11], Kato [13] and Yagi [21]. Bahuguna [1] proved the existence, uniqueness and continuous dependence of a strong solution of quasilinear integrodifferential equation of the form

$$\begin{aligned} u'(t) + A(t,u)u(t) &= \int_0^t a(t-s)k(s,u(s))ds + f(t), \quad 0 \le t \le T, \\ u(0) &= u_0, \end{aligned}$$

by using the method of lines (also known as Rothe's method) and the techniques of Crandall and Souganidis [10]. He also established a local classical solution for the same equation in [2]. Oka and Tanaka [19] investigated the existence of classical solution of quasilinear integrodifferential equation of the hyperbolic type

$$\begin{aligned} u'(t) &= A(t,u)u(t) + \int_0^t \mathcal{F}(t,s,u(s))u(s)ds, \quad 0 \le t \le T, \\ u(0) &= u_0, \end{aligned}$$

in a pair of Banach spaces (Y, X) such that Y is continuously imbedded in X. Oka [18] proved the existence of classical solution of abstract quasilinear integrodifferential equations. Balachandran and Uchiyama [5] discussed the existence and uniqueness of local mild and classical solutions of quasilinear integrodifferential equations. Recently Balachandran and Park [3] studied the existence of solutions of quasilinear integrodifferential evolution equations by using the Schauder fixed point theorem.

In this paper we study the following quasilinear delay integrodifferential equation with time varying delay of the form

$$\begin{aligned} &u'(t) + A(t,u)u(t) \\ &= K(u)(t) + f\Big(t, u(t), u(\beta(t), \int_0^t b(t-s)h(s,u(s),u(\gamma(s)))ds\Big), \end{aligned} \tag{1}$$

$$u(0) = u_0, \ t \in [0,T] = I, \tag{2}$$

where $A(t,u)$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0 \in X$, $f : I \times X \times X \times X \to X$, $h : I \times X \times X \to X$ are functions and K is the nonlinear Volterra operator

$$K(u)(t) = \int_0^t a(t-s)k(s,u(s),u(\alpha(s)))ds.$$

Here $a, b : I \to I$ are real-valued continuous functions, k: $I \times X \times X \to X$, $\alpha, \beta, \ \gamma : I \to I$ are absolutely continuous functions. The equations considered in [1, 11, 13, 20, 21] are particular cases of the equation (1)-(2) and generalize the results of [3, 4, 12, 14, 15, 20].

2. Preliminaries

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach space Z, the norm of Z is denoted by $\|\cdot\|$ or $\|\cdot\|_Z$. The space of all bounded linear operators from X to Y is denoted by $B(X,Y)$ and $B(X,X)$ is written as $B(X)$. We recall some definitions and known facts from Pazy [20].

Definition 2.1. Let S be a linear operator in X and let Y be a subspace of X. The operator $\tilde{S}$ defined by $D(\tilde{S}) = \{x \in D(S) \cap Y : Sx \in Y\}$ and $\tilde{S}x = Sx$ for $x \in D(\tilde{S})$ is called the part of S in Y.

Definition 2.2. Let B be a subset of X and, for every $0 \le t \le T$ and $b \in B$, let $A(t,b)$ be the infinitesimal generator of a C_0 semigroup $S_{t,b}(s), s \ge 0$, on X. The family of operators $\{A(t,b), (t,b) \in I \times B\}$, is stable if there are constants $M \ge 1$ and ω such that

$$\rho(A(t,b)) \supset (\omega, \infty) \quad \text{for } (t,b) \in I \times B,$$

$$\|\prod_{j=1}^{k} R(\lambda : A(t_j, b_j))\| \;\le\; M(\lambda - \omega)^{-k}$$

for $\lambda > \omega,\ 0 \le t_1 \le t_2 \le \cdots \le t_k \le T,\ b_j \in B,\ 1 \le j \le k.$

The stability of $\{A(t,b)\}, (t,b) \in I \times B$ implies (see [20]) that

$$\|\prod_{j=1}^{k} S_{t_j,b_j}(s_j)\| \le M \exp\left\{\omega \sum_{j=1}^{k} s_j\right\}, \quad s_j \ge 0$$

$0 \le t_1 \le t_2 \le \cdots \le t_k \le T, \quad b_j \in B,\ 1 \le j \le k.\ \ k = 1, 2, \ldots.$

Definition 2.3. Let $S_{t,b}(s), s \ge 0$, be the C_0-semigroup generated by $A(t,b)$, $(t,b) \in I \times B$. A subspace Y of X is called $A(t,b)$-admissible if Y is an invariant subspace of $S_{t,b}(s)$ and the restriction of $S_{t,b}(s)$ to Y is a C_0-semigroup in Y.

Let $B \subset X$ be a subset of X such that, for every $(t,b) \in I \times B$, $A(t,b)$ is the infinitesimal generator of a C_0-semigroup $S_{t,b}(s), s \ge 0$ on X. We make the following assumptions:

- (H1) The family $\{A(t,b)\}, (t,b) \in I \times B$, is stable.
- (H2) Y is $A(t,b)$-admissible for $(t,b) \in I \times B$ and the family $\{\tilde{A}(t,b)\}, (t,b) \in I \times B$ of parts $\tilde{A}(t,b)$ of $A(t,b)$ in Y is stable in Y.
- (H3) For $(t,b) \in I \times B$, $D(A(t,b)) \supset Y$, $A(t,b)$ is a bounded linear operator from Y to X and $t \to A(t,b)$ is continuous in the $B(Y,X)$ norm $\|\cdot\|$ for every $b \in B$.
- (H4) There is a positive constant N such that

  $$\|A(t,b_1) - A(t,b_2)\|_{Y \to X} \;\le N\|b_1 - b_2\|_X$$

  holds for every $b_1, b_2 \in B$ and $0 \le t \le T$.

Let B be a subset of X and $\{A(t,b),(t,b) \in I \times B\}$, be a family of operators satisfying the conditions (H1)-(H4). If $u \in C(I:X)$ has values in B, then there is a unique evolution system $U_u(t,s), 0 \le s \le t \le T$, in X satisfying (see Theorem 5.3.1 and Lemma 6.4.2 in [20] pp. 135, 201–202).

(i) $\|U_u(t,s)\| \le K_1 e^{\omega(t-s)}$ and $0 \le s \le t \le T$, where K_1 and ω are stability constants.

(ii) $\frac{\partial^+}{\partial t} U_u(t,s)y = A(s,u(s))U_u(t,s)y$ for $y \in Y$, and $0 \le s \le t \le T$.

(iii) $\frac{\partial}{\partial s} U_u(t,s)y = -U_u(t,s)A(s,u(s))y$ for $y \in Y$, and $0 \le s \le t \le T$.

(H5) For every $u \in C(I:X)$ satisfying $u(t) \in B$ for $0 \le t \le T$, we have

$$U_u(t,s)Y \subset Y, \quad 0 \le s \le t \le T,$$

where $U_u(t,s)$ is strongly continuous in Y for $0 \le s \le t \le T$.

(H6) X and Y are reflexive Banach spaces and there exist an isometry between them.

(H7) For every $(t,b_1,b_2,b_3) \in I \times B \times B$, $f(t,b_1,b_2,b_3) \in Y$.

(H8) The real-valued function a and b are continuous on I and there exist positive constants a_T and b_T such that $|a(t)| \le a_T$ and $|b(t)| \le b_T$ for $t \in I$.

(H9) α, β, γ: $I \to I$ are absolutely continuous and there exist constants $\delta_i > 0$, $i = 1,2,3$ such that $\alpha'(t) \ge \delta_1$, $\beta'(t) \ge \delta_2$ and $\gamma'(t) \ge \delta_3$ respectively for $0 < t \le T$.

(H10) The nonlinear map $k : I \times X \times X \to X$ satisfies

$$\int_0^t \|k(s,x_1,y_1) - k(s,x_2,y_2)\| ds \le N_A[\|x_1(t) - x_2(t)\| + \|y_1(t) - y_2(t)\|]$$

for a.e. $t \in I$, where N_A and N_0 are positive constants and

$$N_0 = \max \int_0^t \|k(s,0,0)\| ds.$$

For the conditions (H11) and (H12), let Z be taken as both X and Y.

(H11) $f : I \times Z \times Z \times Z \to Z$ is continuous and there exist constants $F_A > 0$ and $F_0 > 0$ such that

$$\begin{aligned} &\|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)\|_Z \\ &\le F_A(\|u_1 - u_2\|_Z + \|v_1 - v_2\|_Z + \|w_1 - w_2\|_Z), \\ F_0 = &\max_{\tau \in I} \|f(t,0,0,0)\|_Z. \end{aligned}$$

(H12) $h : I \times Z \times Z \to Z$ is continuous and there exist constants $H_A > 0$ and $H_0 > 0$ such that

$$\int_0^t \|h(s,u_1,v_1) - h(s,u_2,v_2)\|_Z \, ds \le H_A\Big(\|u_1(t) - u_2(t)\|_Z + \|v_1(t) - v_2(t)\|\Big),$$

$$H_0 = \max\{\int_0^t \|h(s,0,0)\|_Z ds\}.$$

Further, there exist a positive constant K_0 such that for every $u, v \in C(I;X)$ with values in B and every $y \in Y$, we have

$$\|U_u(t,s)y - U_v(t,s)y\|_X \leq K_0\|y\|_Y \int_s^t \|u(\tau) - v(\tau)\|_X \, d\tau.$$

For details of the above mentioned results, we refer to Theorem 6.4.3 and Lemma 6.4.4 in Pazy [20].

To prove our main result we need the following theorems.

Theorem 2.1 (Theorem 5.5.2 [20]). *Let $A(t),\ 0 \leq t \leq T$, be the infinitesimal generator of a C_0 semigroup $S_t(s),\ s \leq 0$ on X. If the family satisfying the conditions* (H$_1$)-(H$_3$) [20, Page 135], *then there exists a unique evolution system $U(t,s),\ 0 \leq t \leq s \leq T$, in X satisfying*

(E1) $\|U(t,s)\| \leq Me^{\omega(t-s)}$ *and* $0 \leq s \leq t \leq T$,

(E2) $\frac{\partial^+}{\partial t}U(t,s)v\Big|_{t=s} = A(s)v$ *for* $v \in Y$, *and* $0 \leq s \leq T$,

(E3) $\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v$ *for* $v \in Y$, *and* $0 \leq s \leq t \leq T$,

where the derivative from the right in (E2) *and the derivative in* (E3) *are in the strong sense in X.*

Theorem 2.2 (Theorem 5.5.2 [20]). *Let $A(t)_{t\in[0,T]}$ satisfy the conditions of Theorem 2.1 and let $U(t,s),\ 0 \leq s \leq t \leq T$ be the evolution system given in Theorem 2.1. If*

(E4) *$U(t,s)Y \supset Y$ for $0 \leq s \leq t \leq T$.*

(E5) *For $v \in Y$, $U(t,s)v$ is continuous in Y for $0 \leq s \leq t \leq T$, then for every $v \in Y$, $U(t,s)v$ is the unique Y-valued solution of the initial value problem*

$$\frac{du(t)}{dt} = A(t)u(t) \quad \text{for} \quad 0 \leq s < t \leq T, \tag{3}$$

$$u(s) = v. \tag{4}$$

Theorem 2.3 (Theorem 5.5.2 [20]). *Let $\{A(t)\}_{t\in[0,T]}$ satisfy the condition of Theorem 2.2. If $f \in C([s,T] : Y)$, then for every $v \in Y$ the initial value problem*

$$\frac{du(t)}{dt} = A(t)u(t) + f(t) \quad \text{for} \quad 0 \leq s < t \leq T, \tag{5}$$

$$u(s) = v, \tag{6}$$

possesses a unique Y-valued solution u given by

$$u(t) = U(t,s)v + \int_s^t U(t,\eta)f(\eta)d\eta. \tag{7}$$

3. Existence of solutions

By a mild solution of (1)-(2), we mean a function $u \in C(I : X)$ and $u_0 \in X$ satisfying the integral equation

$$u(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s)\Big[K(u)(s) + f\Big(s, u(s), u(\beta(s)), \int_0^s b(s-\tau)h(\tau, u(\tau), u(\gamma(\tau)))d\tau\Big)\Big]ds. \tag{8}$$

A function $u \in C(I : X)$ such that $u(t) \in D(A(t, u(t)))$ for $t \in (0, T], u \in C^1((0, T] : X)$ and satisfies (1)-(2) in X is called a classical solution of (1)-(2) on $[0, T]$.

Theorem 3.1. *Let $u_0 \in Y$ and the family $A(t, b)$ of linear operators for $t \in I = [0, T]$ and $b \in B = \{u \in Y : \|u\|_Y \le r\}$, $r > 0$, satisfy the assumptions* (H1)-(H12) *and $A(t, b)u_0 \in Y$ with*

$$\|A(t, b)u_0\|_Y \le C_A,\ C_A > 0,$$

for all $(t, b) \in I \times B$. Then there is a positive constant T_0 such that the quasilinear problem (1)-(2) has a unique classical solution $u \in C([0, T_0] : Y) \cap C^1((0, T_0] : X)$.

Proof. First we prove the existence of a unique local mild solution for (1) and (2). The assumption (H5) implies that there exists a constant $K_1 > 0$ such that

$$\|U_u(t, s)\|_{B(Y)} \le K_1$$

for $s \le t,\ s,\ t \in I$ and every $u \in C(I : X)$. Choose

$$T_0 = \min\left\{T, \frac{1}{2K_1C_A}, \frac{1}{2T_1}, \frac{1}{2T_2}\right\},$$

where

$$T_1 = K_1\Big\{r\Big[a_TN_A(1 + 1/\delta_1) + F_A(1 + 1/\delta_2) + b_TF_AH_A(1 + 1/\delta_3)\Big] + a_TN_0 + b_TF_AH_0 + F_0\Big\}$$

and

$$T_2 = K_0\|u_0\|_Y + K_0T\Big[r[a_TN_A(1 + 1/\delta_1) + F_A(1 + 1/\delta_2) + b_TF_AH_A(1 + 1/\delta_3)] + a_TN_0 + b_TF_AH_0 + F_0\Big] + K_1\Big[a_TN_A(1 + 1/\delta_1) + F_A(1 + 1/\delta_2) + b_TF_AH_A(1 + 1/\delta_3)\Big].$$

Let S be the subset of $C([0, T_0] : X)$ defined by

$$S = \{u : u \in C([0, T_0] : X),\ \|u(t)\|_Y \le r,\ u(0) = u_0 \ \text{ for } 0 \le t \le T_0\}.$$

Define a mapping $P : S \to S$ given by

$$(Pu)(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s)\Big[K(u)(s) + f(s,u(s),u(\beta(s)), \int_0^s b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau)))d\tau)\Big]ds.$$

We claim that P maps S into S. Clearly $Pu(0) = u_0$ and $Pu(t) \in Y$ for $0 \le t \le T_0$ and it remains to show that $\|Pu(t)\| \le r$ in Y. Integrating (iii) in X from 0 to t, we find

$$U_u(t,0)u_0 - u_0 = -\int_0^t U_u(t,\tau)A(\tau,u(\tau))u_0 \, d\tau$$

and hence

$$\|U_u(t,0)u_0 - u_0\| \le K_1 C_A T_0.$$

Also we have

$$\begin{aligned}
&\|Pu(t)\| \\
&= \|U_u(t,0)u_0 - u_0\| + \int_0^t \|U_u(t,s)\Big[\int_0^s a(s-\tau)k(\tau,u(\tau),u(\alpha(\tau)))d\tau \\
&\quad + f\Big(s,u(s),u(\beta(s)),\int_0^s b(s-\tau)h(\tau,u(\tau)u(\gamma(\tau)))d\tau\Big)\Big]\|ds \\
&\le K_1C_AT_0 + K_1\Big\{\int_0^t\int_0^s \|a(s-\tau)\|\Big[\|k(\tau,u(\tau),u(\alpha(s))) - k(\tau,0,0)\| \\
&\quad + \|k(\tau,0,0)\|\Big]d\tau ds + \int_0^t\Big[\|f(s,u(s),u(\beta(s)),\int_0^s b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau)))d\tau) \\
&\quad - f(s,0,0,0)\| + \|f(s,0,0,0)\|\Big]ds\Big\}.
\end{aligned}$$

Using the assumptions (H8)-(H12), we get

$$\begin{aligned}
&\|Pu(t)\| \\
&\le K_1\{C_AT_0 + a_TN_A\int_0^t(\|u(s)\| + \|u(\alpha(s))\|)ds + a_TN_0T_0 \\
&\quad + F_A\int_0^t(\|u(s)\| + \|u(\beta(s))\| + \|b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau))d\tau\|)ds + F_0T_0\} \\
&\le K_1\{C_AT_0 + a_TN_ArT_0 + a_TN_A/\delta_1\int_{\alpha(0)}^{\alpha(t)}\|u(s)\|ds + a_TN_0T_0 + F_ArT_0 \\
&\quad + F_A/\delta_2\int_{\beta(0)}^{\beta(t)}\|u(s)\|ds + b_TF_AH_ArT_0 + b_TF_AH_A/\delta_3\int_{\gamma(0)}^{\gamma(t)}\|u(s)\|ds \\
&\quad + b_TF_AH_0T_0 + F_0T_0\} \\
&\le K_1T_0\Big\{C_A + r\Big[a_TN_A(1+1/\delta_1) + F_A(1+1/\delta_2) + b_TF_AH_A(1+1/\delta_3)\Big] \\
&\quad + a_TN_0 + b_TF_AH_0 + F_0\Big\}.
\end{aligned}$$

From the assumption, we get $\|Pu(t)\| \leq r$. Therefore P maps S into itself. Moreover, if $u, v \in S$, then

$$\begin{aligned}
&\|Pu(t) - Pv(t)\| \\
\leq\ &\|U_u(t,0)u_0 - U_v(t,0)u_0\| \\
&+ \int_0^t \|U_u(t,s)\Big[K(u)(s) + f(s,u(s),u(\beta(s))), \int_0^s b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau)))d\tau)\Big] \\
&-U_v(t,s)\Big[K(v)(s) + f(s,v(s),v(\beta(s))), \int_0^s b(s-\tau)h(\tau,v(\tau),v(\gamma(\tau)))d\tau)\Big]\|ds \\
\leq\ &\|U_u(t,0)u_0 - U_v(t,0)u_0\| \\
&+ \int_0^t \|U_u(t,s)\Big[K(u)(s) + f(s,u(s),u(\beta(s))), \int_0^s b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau)))d\tau)\Big] \\
&-U_v(t,s)\Big[K(u)(s) + f(s,u(s),u(\beta(s))), \int_0^s b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau)))d\tau)\Big]\|ds \\
&-U_v(t,s)\Big[K(v)(s) + f(s,v(s),v(\beta(s))), \int_0^s b(s-\tau)h(\tau,v(\tau),v(\gamma(\tau)))d\tau)\Big]\|ds.
\end{aligned}$$

From our assumption, we have

$$\begin{aligned}
I_1 \leq\ &K_0\|u_0\|_Y T_0 \max_{\tau\in I}\|u(\tau) - v(\tau)\|, \\
I_2 \leq\ &\int_0^t \|U_u(t,s) - U_v(t,s)\|\Big[\|K(u)(s)\| \\
&\quad + \|f(s,u(s),u(\beta(s))), \int_0^s b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau)))d\tau)\|\Big]ds \\
\leq\ &K_0TT_0\Big\{r[a_TN_A(1+1/\delta_1) + F_A(1+1/\delta_2) + b_TF_AH_A(1+1/\delta_3)] \\
&\quad + a_TN_0 + b_TF_AH_0 + F_0\Big\}\max_{\tau\in I}\|u(\tau) - v(\tau)\|, \\
I_3 \leq\ &\int_0^t \|U_v(t,s)\|\Big[\|K(u)(s) - K(v)(s)\| \\
&\quad + \|f\Big(s,u(s),u(\beta(s))), \int_0^s b(s-\tau)h(\tau,u(\tau),u(\gamma(\tau)))d\tau) \\
&\quad - f(s,v(s),v(\beta(s))), \int_0^s b(s-\tau)h(\tau,v(\tau),v(\gamma(\tau)))d\tau\Big)\|\Big]ds \\
\leq\ &K_1T_0\Big\{a_TN_A(1+1/\delta_1) + F_A(1+1/\delta_2) + b_TF_AH_A(1+1/\delta_3)\Big\} \\
&\qquad \times \max_{\tau\in I}\|u(\tau) - v(\tau)\|.
\end{aligned}$$

From these inequalities it follows that, for any $t \in I$,

$$\|Pu(t) - Pv(t)\| \leq \frac{1}{2}\max_{\tau\in I}\|u(\tau) - v(\tau)\|,$$

so that P is a contraction on S. From the contraction mapping theorem, it follows that P has a unique fixed point $u \in S$ which is the mild solution of (1) and (2) on $[0, T_0]$.

Now we consider the evolution equation

$$v'(t) + B(t)v(t) = l(t), \quad t \in [0, T_0], \tag{9}$$

$$v(0) = u_0, \tag{10}$$

where $B(t) = A(t, u(t))$ and $l(t) = K(u)(t) + f(t, u(t), u(\beta(t)), \int_0^t b(t-s)h(s, u(s), u(\gamma(s))ds$, $t \in [0, T_0]$, and u is the unique fixed point of P in S. We note that $B(t)$ satisfies (H_1)-(H_3) of [20] (Section 5.5.3) and $l \in C(I : Y)$. Theorem 5.5.2 in [20] implies that there exists a unique function $v \in C(I : Y)$ such that $v \in C^1((0, T_0] : X)$ satisfying (1) and (2) in X and v is given by

$$v(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, s)[K(u)(s) + f(s, u(s), u(\beta(s)), \int_0^s b(s-\tau)(\tau, u(\tau), u(\gamma(\tau)))d\tau]ds,$$

where $U_u(t, s)$ is the evolution system generated by the family $\{A(t, u(t))\}, t \in I$, of the linear operators in X. The uniqueness of v implies that $v = u$ on $t \in [0, T_0]$. Hence u is a unique local classical solution of (1)-(2) and $u \in C([0, T_0] : Y) \cap C^1((0, T_0] : X)$. □

4. Nonlocal Cauchy problem

The nonlocal Cauchy problem for semilinear evolution equations in Banach space was studied first by Byszewski [7, 8, 9] where he established the existence and uniqueness of mild and classical solutions. The nonlocal conditions were motivated by physical problems and their importance is discussed in [7, 8, 9]. Balachandran et al [3, 4, 5, 6, 12] studied the nonlocal Cauchy problem for various type of quasilinear integrodifferential equations. Consider the nonlocal condition of the form

$$u(0) + g(u) = u_0, \ t \in [0, T] = I \tag{11}$$

for the quasilinear integrodifferential equation (1).

Assume the following conditions:

(H13) $g : C(I : B) \to Y$ is Lipschitz continuous in X and bounded in Y, that is, there exist constants $G > 0$ and $G_1 > 0$ such that

$$\|g(u)\|_Y \leq G,$$
$$\|g(u) - g(v)\|_Y \leq G_1 \max_{t \in I} \|u(t) - v(t)\|_X.$$

(H14) There exists a positive constant $r > 0$ such that

$$K_1\Big\{\|u_0\|_Y + G + T_0\Big[r[a_T N_A(1 + 1/\delta_1)$$

$$+F_A(1+1/\delta_2)+b_TF_AH_A(1+1/\delta_3)]+a_TN_0+b_TF_AH_0+F_0\Big]\Big\}\le r,$$

$$q=\Big\{K_0T_0\Big[\|u_0\|_Y+G\Big]+K_1G_1+K_0TT_0\Big[r[a_TN_A(1+1/\delta_1)+F_A(1+1/\delta_2)$$
$$+b_TF_AH_A(1+1/\delta_3)]+a_TN_0+b_TF_AH_0+F_0\Big]+K_1T_0\Big[a_TN_A(1+1/\delta_1)$$
$$+F_A(1+1/\delta_2)+b_TF_AH_A(1+1/\delta_3)\Big]\Big\}<1.$$

By a mild solution of (1) and (11), we mean a function $u \in C(I : X)$ and $u_0 \in X$ satisfying the integral equation

$$u(t)=U_u(t,0)u_0-U_u(t,0)g(u)+\int_0^t U_u(t,s)\Big[K(u)(s)$$
$$+f\Big(s,u(s),u(\beta(s)),\int_0^s b(t-s)h(\tau,u(\tau),u(\gamma(\tau)))d\tau\Big)ds\Big]. \tag{12}$$

A function $u \in C(I;X)$ such that $u(t) \in D(A(t,u(t)))$ for $t \in (0,T], u \in C^1((0,T] : X)$ and satisfies (1) and (11) in X is called a classical solution of (1) and (11) on $[0,T]$.

Theorem 4.1. *Let $u_0 \in Y$ and $B = \{u \in Y : \|u\|_Y \le r\}$, $r > 0$. If the assumptions* (H1)-(H14) *are satisfied, then there is a positive constant T_0 such that the quasilinear problem* (1) *and* (11) *has a unique classical solution $u \in C([0,T_0] : Y) \cap C^1((0,T_0] : X)$.*

Proof. First we prove the existence of a unique mild solution for (1) and (11). The assumption (H5) implies that there exists a constant $K_1 > 0$ such that

$$\|U_u(t,s)\|_{B(Y)} \le K_1$$

for $s \le t,\ s,\ t \in I$ and every $u \in C(I;X)$. Let S be the subset of $C([0,T_0] : X)$ defined by

$$S=\{u : u \in C([0,T_0] : X),\ \|u(t)\| \le r \ \text{ for } 0 \le t \le T_0\}.$$

Define a mapping $Q : S \to S$ by

$$Qu(t)=U_u(t,0)u_0-U_u(t,0)g(u)+\int_0^t U_u(t,s)\Big[K(u)(s)$$
$$+f\Big(s,u(s),u(\beta(s)),\int_0^s b(t-s)h(\tau,u(\tau),u(\gamma(\tau)))d\tau\Big)\Big]ds.$$

We claim that Q maps S into S. For $u \in S$, we have

$$\|Qu(t)\|$$
$$=\|U_u(t,0)u_0-U_u(t,0)g(u)\|$$
$$+\int_0^t \|U_u(t,s)\Big[K(u)(s)+f(s,u(s),u(\beta(s)),\int_0^s b(t-s)h(\tau,u(\tau)u(\gamma(\tau)))d\tau\Big]ds\|$$
$$\le K_1\Big\{\|u_0\|_Y+G+T_0\Big[r[a_TN_A(1+1/\delta_1)+F_A(1+1/\delta_2)+b_TF_AH_A(1+1/\delta_3)]$$

$$+a_T N_0 + b_T F_A H_0 + F_0\Big]\Big\}.$$

From the assumption (H14), one gets $\|Qu(t)\| \le r$. Therefore Q maps S into itself. Moreover, if $u, v \in S$, then

$$\begin{aligned}
&\|Qu(t) - Qv(t)\| \\
\le\ &\|U_u(t,0)u_0 - U_v(t,0)u_0\| + \|U_u(t,0)g(u) - U_v(t,0)g(v)\| \\
&+ \int_0^t \|U_u(t,s)\Big[K(u)(s) + f\Big(s, u(s), u(\beta(s)), \int_0^s b(t-s)h(\tau, u(\tau), u(\gamma(\tau)))d\tau\Big)\Big] \\
&-U_v(t,s)\Big[K(v)(s) + f\Big(s, v(s), v(\beta(s)), \int_0^s b(t-s)h(\tau, v(\tau), v((\gamma(\tau)))d\tau\Big)\Big]\|ds \\
\le\ &\Big\{K_0T_0\Big[\|u_0\|_Y + G\Big] + K_1G_1 + K_0TT_0\Big[r[a_T N_A(1+1/\delta_1) + F_A(1+1/\delta_2) \\
&+ b_T F_A H_A(1+1/\delta_3)] + a_T N_0 + b_T F_A H_0 + F_0\Big] + K_1T_0\Big[a_T N_A(1+1/\delta_1) \\
&+ F_A(1+1/\delta_2) + b_T F_A H_A(1+1/\delta_3)\Big]\Big\} \max_{\tau\in I}\|u(\tau) - v(\tau)\| \\
=\ &q \max_{\tau\in I}\|u(\tau) - v(\tau)\|,
\end{aligned}$$

where $0 < q < 1$. From these inequalities, it follows that, for any $t \in I$,

$$\|Qu(t) - Qv(t)\| \le q \max_{\tau\in I}\|u(\tau) - v(\tau)\|,$$

so that Q is a contraction on S. Hence Q has a unique fixed point $u \in S$ such that $Qu(t) = u(t)$ which is the mild solution of (1) and (11).

Now we consider the evolution equation

$$w'(t) + B(t)w(t) = v(t), \quad t \in [0, T_0], \tag{13}$$

$$w(0) = u_0 - g(u), \tag{14}$$

where $B(t) = A(t, u(t))$ and $v(t) = K(u)(t) + f(t, u(t), u(\beta(t)), \int_0^t b(t-s)h(s, u(s), u(\gamma(s))ds$, $t \in [0, T_0]$ and u is the unique fixed point of Q in S. We note that $B(t)$ satisfies (H_1)-(H_3) of [20] (Section 5.5.3) and $v \in C(I : Y)$. Theorem 5.5.2 in [20] implies that there exists a unique function $w \in C(I : Y)$ such that $w \in C^1((0, T_0] : X)$ satisfying (1) and (11) in X and w is given by

$$\begin{aligned}
w(t) = U_u(t,0)u_0 - U_u(t,0)g(u) + \int_0^t U_u(t,s)[K(u)(s) \\
+ f(s, u(s), u(\beta(s)), \int_0^s b(s-\tau)(\tau, u(\tau), u(\gamma(\tau)))d\tau]ds,
\end{aligned}$$

where $U_u(t, s)$ is the evolution system generated by the family $\{A(t, u(t))\}, t \in I$, of the linear operators in X. The uniqueness of w implies that $w = u$ on $t \in [0, T_0]$. Hence u is a unique classical solution of (1) and (11) and $u \in C([0, T_0] : Y) \cap C^1((0, T_0] : X)$. □

5. Conclusion

The present paper contains results concerning the existence and uniqueness of classical solutions for a quasilinear delay integrodifferential equation in Banach spaces. The result shows that the Banach fixed point theorem can effectively used to study the regularity of solutions for abstract quasilinear delay integrodifferential equation. Under suitable assumptions we have also proved the classical solutions for quasilinear integrodifferential equation with time varying delay and nonlocal condition.

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