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A VERY SIMPLE CHARACTERIZATION OF GROMOV HYPERBOLICITY FOR A SPECIAL KIND OF DENJOY DOMAINS

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Abstract. In this paper we provide characterizations for the Gromov hyperbolicity of some particular Denjoy domains and besides some sufficient conditions to guarantee or discard the hyperbolicity of some others. The conditions obtained involve just the lengths of some special simple closed geodesics in the domain. These results, on the one hand, show the extraordinary complexity of determining the hyperbolicity of a domain and, on the other hand, allow us to construct easily a large variety of both hyperbolic and non-hyperbolic domains.

1. Introduction

In the 1980s Mikhail Gromov introduced a notion of abstract hyperbolic spaces, which have thereafter been studied and developed by many authors. Initially, the research was mainly centered on hyperbolic group theory, but lately researchers have shown an increasing interest in more direct studies of spaces endowed with metrics used in geometric function theory.

To understand the connections between graphs and potential theory on Riemannian manifolds (see e.g. [4], [12], [14], [27], [28], [29], [30], [40], [41], [46], [47]), Gromov hyperbolic spaces are a useful tool. Besides, the concept of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [13], [16], [18], [19] and the references therein).

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One of the primary questions is naturally whether a metric space (X, d) is hyperbolic in the sense of Gromov or not. The most classical examples, mentioned in every textbook on this topic, are metric trees, the classical Poincaré hyperbolic metric developed in the unit disk and, more generally, simply connected complete Riemannian manifolds with sectional curvature $K \leq -k^2 < 0$.

However, it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Klein-Hilbert metric (see [10, 31]) is Gromov hyperbolic (under particular conditions on the domain of definition); the Gehring-Osgood j-metric (see [21]) is Gromov hyperbolic; and the Vuorinen j-metric (see [21]) is not Gromov hyperbolic except in the punctured space. Also, in [32] the hyperbolicity of the conformal modulus metric μ and the related so-called Ferrand metric λ^*, is studied.

The quasihyperbolic metric has also recently been a topic of interest regarding the question of Gromov hyperbolicity. In [11], Bonk, Heinonen and Koskela found necessary and sufficient conditions for when a planar domain D endowed with the quasihyperbolic metric is Gromov hyperbolic. This was extended by Balogh and Buckley, [6]: they found two different necessary and sufficient conditions which work in Euclidean spaces of all dimensions and also in metric spaces under some conditions.

Since the Poincaré metric is also the metric giving rise to what is commonly known as the hyperbolic metric when speaking about open domains in the complex plane or in Riemann surfaces, it could be expected that there is a connection between the notions of hyperbolicity. For simply connected subdomains Ω of the complex plane, it follows directly from the Riemann mapping theorem that Ω is, in fact, Gromov hyperbolic. However, as soon as simple connectedness is omitted, there is no immediate answer to whether the space Ω is hyperbolic or not. The question has lately been studied in [3], [22]–[26], [34]–[45] and [47].

In the current paper our main aim is to study the Gromov hyperbolicity of Denjoy domains, that is to say, plane domains Ω with $\partial\Omega \subset \mathbb{R}$, with the Poincaré metric. This kind of surfaces are becoming more and more important in Geometric Theory of Functions, since, on the one hand, they are a very general type of Riemann surfaces, and, on the other hand, they are more manageable due to its symmetry. For instance, Garnett and Jones have proved in [15] the Corona Theorem for Denjoy domains, and in [2], [42] the authors have got characterizations of Denjoy domains which satisfy a linear isoperimetric inequality.

The Gromov hyperbolicity of Denjoy domains with the Poincaré and quasihyperbolic metrics has been studied previously in [22], [23] and [24] in terms of the Euclidean size of the boundary of the Denjoy domain. The same topic, just for the Poincaré metric, has been dealt with in [3] and [38], but from a geometric point of view.

In this paper we provide new and easily applicable geometric criteria (involving the lengths of some kind of closed geodesics) in order to guarantee or discard Gromov hyperbolicity of some Denjoy domains.

First of all we find a very simple characterization of Gromov hyperbolicity for a special kind of Denjoy domains (see Theorem 3.1). If we enlarge slightly this class, the situation becomes extraordinarily complicated (see Theorem 3.9 for sufficient conditions on non-hyperbolicity and Theorem 3.10 for a sufficient condition on hyperbolicity); these results allow to construct easily a large variety of both hyperbolic and non-hyperbolic domains. Again, by imposing certain regularity condition on these domains, we get a characterization (see Theorem 4.3).

Theorem 4.3 reduces the study of such a complicated topic as hyperbolicity to something as simple as checking the boundedness or the limit of some sequences. The complexity of the statement of Theorem 4.3 shows the difficulty of the problem, but it turns out to be really easy to apply in practice.

Notations. We denote by X a geodesic metric space. By d_X and L_X we shall denote, respectively, the distance and the length in the metric of X. From now on, when there is no possible confusion, we will not write the subindex X. We denote by $\Re z$ and $\Im z$ the real and imaginary part of z, respectively. We denote by Ω a Denjoy domain with its Poincaré metric. Finally, we denote by c and c_i, positive constants which can assume different values in different theorems.

2. Background

We denote by $\mathbb{H}$ the upper half plane, $\{z \in \mathbb{C} : \ \Im z > 0\}$ and by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : \ |z| < 1\}$. Recall that a domain $\Omega \subset \mathbb{C}$ is said to be of *non-exceptional* if it has at least two finite boundary points. The universal cover of such domain is the unit disk $\mathbb{D}$. In Ω we can define the *Poincaré metric*, i.e., the metric obtained by projecting the metric $ds = 2|dz|/(1 - |z|^2)$ of the unit disk by any universal covering map $\pi : \mathbb{D} \longrightarrow \Omega$. Equivalently, we can project the metric $ds = |dz|/\Im z$ of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$. Therefore, any simply connected subset of Ω is isometric to a subset of $\mathbb{D}$. With this metric, Ω is a geodesically complete Riemannian manifold with constant curvature -1; in particular, Ω is a geodesic metric space (see Definition 2.1 below). The Poincaré metric is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

A *Denjoy domain* $\Omega \subset \mathbb{C}$ is a domain whose boundary is contained in the real axis. Since $\Omega \cap \mathbb{R}$ is an open set contained in $\mathbb{R}$, it is the union of pairwise disjoint open intervals; as each interval contains a rational number, this union is countable. Hence, we can write $\Omega \cap \mathbb{R} = \cup_{n \in \Lambda}(a_n, b_n)$, where Λ is a countable index set, $\{(a_n, b_n)\}_{n \in \Lambda}$ are pairwise disjoint, and it is possible to have $a_{n_1} = -\infty$ for some $n_1 \in \Lambda$ and/or $b_{n_2} = \infty$ for some $n_2 \in \Lambda$.

In order to study Gromov hyperbolicity, we consider the case where Λ is countably infinite, since if Λ is finite, then Ω is Gromov hyperbolic by [22, Proposition 3.6]or [42, Proposition 3.2].

As we mentioned in the introduction of this paper, Denjoy domains are becoming more and more interesting in Geometric Function Theory (see e.g. [1], [2], [15], [17], [42]). In particular, they contain the class of flute surfaces (see, e.g. [7], [8]); these ones are important since they are the simplest examples of infinite ends, and besides, in a flute surface it is possible to give a fairly precise description of the ending geometry (see, e.g. [20]).

We collect some basic facts on Gromov hyperbolicity (see e.g. [13] and/or [16] for deeper background).

Definition 2.1. If $\gamma : [a,b] \longrightarrow X$ is a continuous curve in a metric space (X,d), the *length* of γ is

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

We say that γ is a *geodesic* if it is an isometry, i.e., $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a,b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by xy any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but convenient as well).

Definition 2.2. If X is a geodesic metric space and J is a polygon whose sides are $J_1, J_2, \dots, J_n$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \le \delta$. We say that a polygon is *geodesic* if all of its sides are geodesics. The space X is *δ-thin* (or *δ-hyperbolic*) if every geodesic triangle in X is δ-thin.

Remark 2.3. If X is δ-thin, it is easy to check that every geodesic polygon with n sides is $(n-2)\delta$-thin. In particular, since $\mathbb{H}$ is δ_0-thin, with $\delta_0 := \log(1+\sqrt{2})$ (see, e.g. [5, p. 130]), every geodesic hexagon in $\mathbb{H}$ is $4\delta_0$-thin; more generally, every simply connected geodesic hexagon in a Riemann surface endowed with its Poincaré metric is $4\delta_0$-thin, since it is isometric to a geodesic hexagon in $\mathbb{H}$.

Example 2.4.

(1) Every bounded metric space X is $(diam X)$-hyperbolic (see e.g. [16, p. 29]).
(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [16, p. 52]).
(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [16, p. 29]).

In order to prove our results, we will need the following theorems.

Theorem 2.5 ([3, Theorem 5.1])**.** *Let $\Omega \subset \mathbb{C}$ be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, and for each $n \ge 1$, consider a fixed geodesic γ_n joining (a_0, b_0) with (a_n, b_n). Then, Ω is δ-hyperbolic if and only if there exists a constant c such that $d_\Omega(z, \mathbb{R}) \le c$ for every $z \in \cup_n \gamma_n$.*

Furthermore, if Ω is δ-hyperbolic, then c is a constant which only depends on δ. If $d_\Omega(z, \mathbb{R}) \le c$ for every $z \in \cup_n \gamma_n$, then Ω is δ-hyperbolic, with δ a constant which only depends on c.

Remark 2.6. The hypothesis $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$ is not restrictive at all: the case $\Omega \cap \mathbb{R} = \cup_{n=0}^{N}(a_n, b_n)$ is always hyperbolic, since Ω is of finite type (see e.g. [22, Proposition 3.6] or [43, Proposition 3.2]).

Definition 2.7. A *train* is a Denjoy domain $\Omega \subset \mathbb{C}$ with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, such that $-\infty \le a_0$ and $b_n \le a_{n+1}$ for every n. A *flute surface* is a train with $b_n = a_{n+1}$ for every n.

We say that a curve in a train Ω is a *fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_0, b_0) and (a_n, b_n) for some $n > 0$; we denote by γ_n^* the fundamental geodesic corresponding to n and $2l_n := L_\Omega(\gamma_n)$. A curve in a train Ω is a *second fundamental geodesic* if it is a simple closed geodesic which just intersects $\mathbb{R}$ in (a_n, b_n) and (a_{n+1}, b_{n+1}) for some $n \ge 0$; we denote by σ_n the second fundamental geodesic corresponding to n and $2r_n := L_\Omega(\sigma_n)$. If $b_n = a_{n+1}$, we define σ_n as the puncture at this point and $r_n = 0$.

A *fundamental Y-piece* in a train Ω is the generalized Y-piece in Ω bounded by $\gamma_n, \gamma_{n+1}, \sigma_n$ for some $n > 0$; we denote by Y_n the fundamental Y-piece corresponding to n. A *fundamental hexagon* in a train Ω is the intersection $H_n := Y_n^+ = Y_n \cap \{z \in \mathbb{C} : \Im z \ge 0\}$ for some $n > 0$. We denote by α_n the length of the opposite side to σ_n^+ in H_n.

We will need the following results.

Theorem 2.8 ([3, Theorem 5.17])**.** *Let us consider a train Ω and a subsequence $\{n_k\}_k$ of natural numbers verifying $\lim_{k\to\infty} l_{n_k+1} = \lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+1} = \infty$ and $l_{n_k}, l_{n_k+2} \le l_{n_k+1} + c$ for every k. Then Ω is not hyperbolic.*

Definition 2.9. Given a train Ω and a point $z \in \Omega$, we define the *height* of z as $h(z) := d_\Omega(z, (a_0, b_0))$.

Lemma 2.10 ([3, Lemma 5.7])**.** *Let us consider a train Ω. We have $d_\Omega(z, w) \ge |h(z) - h(w)|$ for every $z, w \in \Omega$.*

Lemma 2.11 ([38, Proposition 3.9 and Remark])**.** *Let us consider the fundamental hexagon H_n and the point z in $\gamma_{n+1}^* \cap H_n$ with $h(z) = h$, $l_{n+1} \ge h \ge l_n$ and $l_{n+1} \ge l_0$ for some constant $l_0 > 0$. Then*

$$d_\Omega(z, (a_n, b_n)) \asymp e^{-h+l_n} + e^{-(l_{n+1}-h-r_n)_+} + (r_n + h - l_{n+1})_+ \,.$$

Furthermore, the constants in the inequalities only depend on l_0.

Lemma 2.12 ([3, Lemma 5.8]). *Let us consider a train Ω. If $l_0 \le l_{n+2} \le l_{n+1}$ and $r_{n+1} \le c_1$ for some fixed n, then $d_\Omega(z,\mathbb{R}) \le d_\Omega(z,(a_{n+2},b_{n+2})) \le c_2$ for every $z \in \gamma^*_{n+1}$ with $h(z) \in [l_{n+2}, l_{n+1}]$, where c_2 only depends on c_1 and l_0.*

3. Main results

We start this section with a characterization of hyperbolicity for some Denjoy domains in terms of the lengths of fundamental geodesics.

Theorem 3.1. *Let Ω be a Denjoy domain. If there exist $\{n_k\}_k$ such that $1 \le n_{k+1} - n_k \le 2$, and $\sup_k l_{n_k} < \infty$, then Ω is hyperbolic if and only if there exists C such that*

$$\min\{l_{n_k+1}, r_{n_k}, r_{n_k+1}\} \le C \qquad \text{for every } k \text{ with } n_{k+1} - n_k = 2. \tag{3.1}$$

Proof. By hypothesis, the lengths l_{n_k} of the fundamental geodesics $\gamma^*_{n_k}$ satisfy $l_{n_k} \le c_0$ for some constant c_0.

Assume first that (3.1) holds.

Let us choose an arbitrary fundamental geodesic γ^*_m. On the one hand, if $m \in \{n_k\}_k$, then $d_\Omega(z,\mathbb{R}) \le l_m \le c_0$ for every $z \in \gamma^*_m$. On the other hand, if $m \notin \{n_k\}_k$ there exists some k such that $m-1 = n_k$ and $n_{k+1} - n_k = 2$. Notice that if $l_m \le C$, then $d_\Omega(z,\mathbb{R}) \le l_m \le C$ for every $z \in \gamma^*_m$. However, if $l_m > C$, then, by hypothesis, $\min\{r_{m-1}, r_m\} \le C$. Without loss of generality we can assume that $r_m \le C$. In particular, the fundamental hexagon H_m is $4\delta_0$-thin (see Remark 2.3), and then for any fixed $z \in \gamma^*_m$ there exists z_0 belonging to one of the other five sides of H_m with $d_\Omega(z, z_0) \le 4\delta_0$. Let us analyze now what happens depending on what side of the fundamental hexagon H_m the point z_0 belongs to:

Firstly, in the case that $z_0 \in (a_0,b_0) \cup (a_m,b_m) \cup (a_{m+1},b_{m+1})$, then $d_\Omega(z,\mathbb{R}) \le 4\delta_0$.

Secondly, if $z_0 \in \sigma_m$, then $d_\Omega(z_0,\mathbb{R}) \le d_\Omega(z_0,(a_m,b_m) \cup (a_{m+1},b_{m+1})) \le r_m \le C$. Therefore, $d_\Omega(z,\mathbb{R}) \le 4\delta_0 + C$.

Finally, if $z_0 \in \gamma^*_{m+1} = \gamma^*_{n_{k+1}}$, then $d_\Omega(z,\mathbb{R}) \le d_\Omega(z_0,(a_0,b_0)) \le l_{n_{k+1}} \le c_0$. Hence, $d_\Omega(z,\mathbb{R}) \le 4\delta_0 + c_0$.

Consequently, taking into account every possible case, $d_\Omega(z,\mathbb{R}) \le 4\delta_0 + \max\{c_0, C\}$. Applying now Theorem 2.5 we conclude that Ω is hyperbolic.

In order to prove the converse, let us assume now that

$$\sup_k \Big\{ \min\{l_{n_k+1}, r_{n_k}, r_{n_k+1}\} : n_{k+1} - n_k = 2 \Big\} = \infty.$$

It means that there exists a subsequence $\{m_j\}_j \subseteq \{n_k\}_k$ such that

$$\lim_{j\to\infty} \min\{l_{m_j+1}, r_{m_j}, r_{m_j+1}\} = \infty.$$

That is to say, $\lim_{j\to\infty} l_{m_j+1} = \lim_{j\to\infty} r_{m_j} = \lim_{j\to\infty} r_{m_j+1} = \infty$. Let us fix now m_j; there exists k such that $m_j = n_k$ and therefore $m_j + 2 = n_{k+1}$. Then, $l_{m_j}, l_{m_j+2} \le c_0$ and, obviously, $l_{m_j}, l_{m_j+2} \le l_{m_j+1} + c_0$. Theorem 2.8 gives the final argument to finish the proof. □

Next, a sufficient condition for hyperbolicity.

Theorem 3.2. *Let Ω be a Denjoy domain. If there exists C such that $r_n \le C$ for every n and there exist N and $\{n_k\}_k$ such that $1 \le n_{k+1} - n_k \le N$ and $\sup_k l_{n_k} < \infty$, then Ω is hyperbolic.*

Proof. Notice that there are, at most, $N-1$ fundamental geodesics with unbounded length between every two consecutive elements in the subsequence $\{\gamma^*_{n_k}\}_k$, and that by hypothesis, the lengths l_{n_k} of the fundamental geodesics $\gamma^*_{n_k}$ in the subsequence verify that $l_{n_k} \le c_0$ for some constant c_0.

In order to apply Theorem 2.5, we have to show that $d_\Omega(z, \mathbb{R}) \le c$ for some constant c and every $z \in \cup_m \gamma^*_m$.

We claim first that $d_\Omega(z, \mathbb{R}) \le c_j$ for some constant c_j and every $z \in \gamma^*_{n_k+j}$, with $0 \le j < n_{k+1} - n_k$. We will prove this claim by induction on j, with $0 \le j < N$.

Notice that, if $j = 0$, then z belongs to $\gamma^*_{n_k}$, one of the fundamental geodesics that verify $l_{n_k} \le c_0$, and then $d_\Omega(z, \mathbb{R}) \le d_\Omega(z, (a_0, b_0)) \le l_{n_k} \le c_0$. Let us assume now that the claim is true for $j-1$ and let us prove it for j.

We know that the fundamental hexagon H with two of its sides coincident with the fundamental geodesic $\gamma^*_{n_k+j}$ and the previous one, $\gamma^*_{n_k+j-1}$, is $4\delta_0$-thin, (see Remark 2.3). It means that for any fixed $z \in \gamma^*_{n_k+j}$ there exists z_0 belonging to one of the other five sides of H with $d_\Omega(z, z_0) \le 4\delta_0$. Repeating the argument in the proof of Theorem 3.1 and analyzing what happens depending on what side of the fundamental hexagon H the point z_0 belongs to, we can easily see that the worst case scenario occurs when $z_0 \in \gamma^*_{n_k+j-1}$ and in that case, by inductive assumption, $d_\Omega(z, \mathbb{R}) \le 4\delta_0 + c_{j-1}$. Otherwise, $d_\Omega(z, \mathbb{R}) \le 4\delta_0 + C$. That is to say, for every $z \in \gamma^*_{n_k+j}$, we have that $d_\Omega(z, \mathbb{R}) \le 4\delta_0 + \max\{c_{j-1}, C\}$. Iterating the argument $j-1$ additional times (and taking into account that $0 \le j < N$) on the fundamental hexagons located to the left of H, we can conclude that $d_\Omega(z, \mathbb{R}) \le 4(N-1)\delta_0 + \max\{c_0, C\}$ for every $z \in \gamma^*_{n_k+j}$ with $0 \le j < N$. Consequently, $d_\Omega(z, \mathbb{R}) \le 4(N-1)\delta_0 + \max\{c_0, C\}$ for every $z \in \cup_m \gamma^*_m$.

Theorem 2.5 gives the final argument to finish the proof. □

Theorem 3.9 gives sufficient conditions for non-hyperbolicity. In order to simplify its proof, we will split it into several lemmas.

Lemma 3.3. *Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ of natural numbers and a constant c with $l_{n_k}, l_{n_k+3} \le c$ for every k. Assume also that $r_{n_k}, l_{n_k+1} - l_{n_k+2} \le c$ for every k and $\lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \infty$. Then Ω is not hyperbolic.*

Proof. The conclusion is straightforward applying Theorem 2.8. □

Lemma 3.4. *Let us consider a train Ω and some fixed n. We take $z \in \gamma^*_n$ with $h(z) = l_n - s$. Then*

$$d_\Omega(z, (a_0, b_0)) = l_n - s, \quad d_\Omega(z, (a_{n+1}, b_{n+1})) \ge r_n - s,$$

$$d_\Omega(z,\gamma^*_{n+1}) \ge \operatorname{Arcsinh} e^{\frac12 (r_n+l_n-l_{n+1}-2s)}.$$

Proof. It is straightforward that $d_\Omega(z_n,(a_0,b_0)) = h(z_n) = l_n - s$. We also have that $r_n = d_\Omega((a_n,b_n),(a_{n+1},b_{n+1})) \le s + d_\Omega(z_n,(a_n,b_n))$, and then

$$d_\Omega(z_n,(a_n,b_n)) \ge r_n - s\,.$$

Standard hyperbolic trigonometry (see e.g. [9, p. 161]) in H_n (see Definition 2.7 in order to recall the definitions of $H_n, \alpha_n, \dots$) gives

$$\cosh\alpha_n = \frac{\cosh r_n + \cosh l_n \cosh l_{n+1}}{\sinh l_n \sinh l_{n+1}} \ge \frac{\frac12\, e^{r_n}}{\frac12\, e^{l_n}\frac12\, e^{l_{n+1}}} + 1 = 1 + 2e^{r_n-l_n-l_{n+1}}.$$

Then, we have

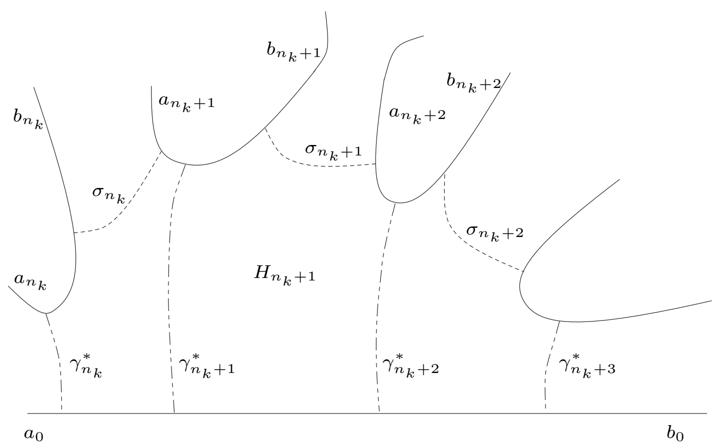
$$\frac12 \sinh\alpha_n \ge \sinh\frac{\alpha_n}{2} = \sqrt{\frac{\cosh\alpha_n - 1}{2}} \ge e^{\frac12 (r_n-l_n-l_{n+1})}.$$

Standard hyperbolic trigonometry for right-angled quadrilaterals (see e.g. [9, p. 157]) gives

$$\begin{aligned}\sinh d_\Omega(z_n,\gamma^*_{n+1}) &= \sinh\alpha_n \cosh(l_n - s)\\ &\ge 2e^{\frac12 (r_n-l_n-l_{n+1})}\frac12\, e^{l_n-s} = e^{\frac12 (r_n+l_n-l_{n+1}-2s)}.\end{aligned}$$

□

Lemma 3.5. *Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ of natural numbers and a constant c with $l_{n_k}, l_{n_k+3} \le c$ for every k. Assume also that $r_{n_k}, l_{n_k+1} + l_{n_k+2} - r_{n_k+1} \le c$ for every k and $\lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \lim_{k\to\infty}(l_{n_k+1} - l_{n_k+2}) = \infty$. Then Ω is not hyperbolic.*



Proof. Consider $z_k \in \gamma^*_{n_k+2}$ with height $h(z_k) = l_{n_k+2} - s_k$, where $s_k := \log(\min\{l_{n_k+2}, r_{n_k+2}\})$.

It is direct that $d_\Omega(z_k, (a_{n_k+2}, b_{n_k+2})) = s_k \longrightarrow \infty$, if $k \longrightarrow \infty$.

Lemma 3.4 implies the following facts:

$$\begin{aligned} d_\Omega(z_k, (a_0, b_0)) &\geq l_{n_k+2} - \log l_{n_k+2} \longrightarrow \infty\,, \\ d_\Omega(z_k, (a_{n_k+3}, b_{n_k+3})) &\geq r_{n_k+2} - \log r_{n_k+2} \longrightarrow \infty\,, \\ d_\Omega(z_k, \gamma_{n_k+3}) &\geq \operatorname{Arcsinh} e^{\frac{1}{2}(r_{n_k+2}+l_{n_k+2}-l_{n_k+3}-2\log r_{n_k+2})} \\ &\geq \operatorname{Arcsinh} e^{\frac{1}{2}(r_{n_k+2}+l_{n_k+2}-c-2\log r_{n_k+2})} \longrightarrow \infty\,, \end{aligned}$$

if $k \to \infty$. Note that the geodesic joining z_k and $\cup_{n>n_k+3}(a_n, b_n)$ must intersect γ_{n_k+3} (see the figure above); hence,

$$d_\Omega(z_k, \cup_{n>n_k+3}(a_n, b_n)) \geq d_\Omega(z_k, \gamma_{n_k+3}) \longrightarrow \infty\,,$$

if $k \to \infty$. Then $\lim_{k\to\infty} d_\Omega(z_k, \cup_{n\geq n_k+2}(a_n, b_n) \cup (a_0, b_0)) = \infty$.

Since every $w \in (a_{n_k+1}, b_{n_k+1})$ verifies $h(w) := d_\Omega(w, (a_0, b_0)) \geq l_{n_k+1}$, we obtain by Lemma 2.10

$$d_\Omega(z_k, (a_{n_k+1}, b_{n_k+1})) \geq h(w) - h(z_k) \geq l_{n_k+1} - l_{n_k+2} + s_k \geq s_k \longrightarrow \infty\,,$$

if $k \to \infty$.

Let us consider now the fundamental hexagon H_{n_k+1} (see Definition 2.7). Standard hyperbolic trigonometry (see e.g. [9, p. 161]) gives

$$\cosh \alpha_{n_k+1} = \frac{\cosh r_{n_k+1} + \cosh l_{n_k+1} \cosh l_{n_k+2}}{\sinh l_{n_k+1} \sinh l_{n_k+2}} \approx 1 + 2\, e^{r_{n_k+1} - l_{n_k+1} - l_{n_k+2}},$$

if $k \to \infty$. Since $l_{n_k+1} + l_{n_k+2} - r_{n_k+1} \leq c$, we deduce that $\alpha_{n_k+1} \geq c_0$ for some positive constant c_0. Standard hyperbolic trigonometry (see e.g. [9], p. 157) gives

$$\begin{aligned} \sinh d_\Omega(z_k, \gamma_{n_k+1}) &= \sinh \alpha_{n_k+1} \cosh h(z_k) \\ &\geq \sinh c_0 \cosh(l_{n_k+2} - \log l_{n_k+2}) \longrightarrow \infty\,, \end{aligned}$$

if $k \to \infty$. Therefore,

$$d_\Omega(z_k, \cup_{n=1}^{n_k}(a_n, b_n)) \geq d_\Omega(z_k, \gamma_{n_k+1}) \longrightarrow \infty\,,$$

if $k \to \infty$, and we obtain $\lim_{k\to\infty} d_\Omega(z_k, \cup_{n=1}^{n_k+1}(a_n, b_n)) = \infty$.

Consequently, $\lim_{k\to\infty} d_\Omega(z_k, \mathbb{R}) = \infty$. Hence, Ω is not hyperbolic by Theorem 2.5. □

Lemma 3.6. *Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ of natural numbers and a constant c with $l_{n_k}, l_{n_k+3} \leq c$ for every k. Assume that $r_{n_k} \leq c$ for every k and $\lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \lim_{k\to\infty}(l_{n_k+1} - l_{n_k+2}) = \lim_{k\to\infty}(l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \lim_{k\to\infty}(l_{n_k+2} - l_{n_k+1} + r_{n_k+1}) = \infty$. Then Ω is not hyperbolic.*

Proof. Consider $z_k \in \gamma^*_{n_k+2}$ with height $h(z_k) = l_{n_k+2} - s_k$, where

$$s_k := \log(\min\{l_{n_k+2}, r_{n_k+2}, l_{n_k+2} - l_{n_k+1} + r_{n_k+1}\}) .$$

Repeating exactly the argument in the proof of Lemma 3.5, we conclude that $\lim_{k\to\infty} d_\Omega(z_k, \cup_{n\geq n_k+1}(a_n, b_n) \cup (a_0, b_0)) = \infty$.

Let us consider now the fundamental hexagon H_{n_k+1} (see Definition 2.7). Standard hyperbolic trigonometry (see e.g. [9, p. 161]) gives

$$\cosh \alpha_{n_k+1} = \frac{\cosh r_{n_k+1} + \cosh l_{n_k+1} \cosh l_{n_k+2}}{\sinh l_{n_k+1} \sinh l_{n_k+2}} \approx 1 + 2\, e^{r_{n_k+1} - l_{n_k+1} - l_{n_k+2}},$$

if $k \to \infty$. Since $\lim_{k\to\infty}(l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \infty$, we deduce that

$$\cosh \alpha_{n_k+1} \approx 1 + \frac{1}{2}\, \alpha^2_{n_k+1} \approx 1 + 2\, e^{r_{n_k+1} - l_{n_k+1} - l_{n_k+2}},$$

$$\alpha_{n_k+1} \approx 2\, e^{\frac{1}{2}(-l_{n_k+1} - l_{n_k+2} + r_{n_k+1})},$$

if $k \longrightarrow \infty$. Standard hyperbolic trigonometry (see e.g. [9, p. 157]) gives

$$\begin{aligned} \sinh d_\Omega(z_k, \gamma_{n_k+1}) &= \sinh \alpha_{n_k+1} \cosh h(z_k) \\ &\geq c\, 2\, e^{\frac{1}{2}(-l_{n_k+1} - l_{n_k+2} + r_{n_k+1})} \frac{1}{2}\, e^{l_{n_k+2} - s_k} \\ &\geq c\, e^{\frac{1}{2}(l_{n_k+2} - l_{n_k+1} + r_{n_k+1}) - \log(l_{n_k+2} - l_{n_k+1} + r_{n_k+1})} \longrightarrow \infty \,, \end{aligned}$$

if $k \to \infty$. Therefore,

$$d_\Omega(z_k, \cup_{n=1}^{n_k}(a_n, b_n)) \geq d_\Omega(z_k, \gamma_{n_k+1}) \longrightarrow \infty \,,$$

if $k \to \infty$, and we obtain $\lim_{k\to\infty} d_\Omega(z_k, \mathbb{R}) = \infty$. Hence, Ω is not hyperbolic by Theorem 2.5. □

Lemma 3.7. *Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ of natural numbers and a constant c with $l_{n_k}, l_{n_k+3} \leq c$ for every k. Assume that $r_{n_k+1}, l_{n_k+2} - l_{n_k+1} \leq c$ for every k, and $\lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \lim_{k\to\infty}(l_{n_k+2} - l_{n_k+1} + r_{n_k}) = \infty$. Then Ω is not hyperbolic.*

Note that, under these hypotheses, we have $\lim_{k\to\infty} l_{n_k+1} = \infty$.

Proof. Consider $z_k \in \gamma^*_{n_k+2}$ with height $h(z_k) = l_{n_k+2} - s_k$, where

$$s_k := \log(\min\{l_{n_k+2}, r_{n_k+2}, l_{n_k+2} - l_{n_k+1} + r_{n_k}\}) .$$

Repeating exactly the argument in the proof of Lemma 3.5, we conclude that $\lim_{k\to\infty} d_\Omega(z_k, \cup_{n\geq n_k+2}(a_n, b_n) \cup (a_0, b_0)) = \infty$.

If k is large enough, when w is the nearest point in (a_{n_k+1}, b_{n_k+1}) to z_k, by Lemma 2.10,

$$d_\Omega(z_k, (a_{n_k+1}, b_{n_k+1})) \geq h(w) - h(z_k) \geq l_{n_k+1} - l_{n_k+2} + s_k \geq s_k - c \longrightarrow \infty \,,$$

if $k \to \infty$.

Seeking for a contradiction let us assume that Ω is hyperbolic. Then, by Theorem 2.5, there exists a constant c_0 such that $d_\Omega(z_k, \mathbb{R}) \leq c_0$ for every k. Since $\lim_{k\to\infty} h(z_k) = \infty$ and $l_{n_k} \leq c$, then $d_\Omega(z_k, (a_{n_k}, b_{n_k})) \leq c_0$ for k large enough.

Let us denote by w_k the point in (a_{n_k}, b_{n_k}) with $d_\Omega(z_k,(a_{n_k}, b_{n_k})) = d_\Omega(z_k, w_k)$. Consider the point $z'_k \in \gamma^*_{n_k+1}$ with $d_\Omega(z_k, w_k) = d_\Omega(z_k, z'_k) + d_\Omega(z'_k, w_k) \le c_0$. By Lemma 2.10, we conclude that $|h(z'_k) - h(z_k)| \le d_\Omega(z_k, z'_k) \le c_0$. Since

$$\begin{aligned} r_{n_k} = d_\Omega((a_{n_k}, b_{n_k}),(a_{n_k+1}, b_{n_k+1})) &\le d_\Omega(z'_k, w_k) + l_{n_k+1} - h(z'_k) \\ &\le c_0 + l_{n_k+1} - h(z_k) + c_0. \end{aligned}$$

We conclude that $r_{n_k} - l_{n_k+1} + l_{n_k+2} - s_k \le 2c_0$. Then,

$$l_{n_k+2} - l_{n_k+1} + r_{n_k} - \log(l_{n_k+2} - l_{n_k+1} + r_{n_k}) \le 2c_0\,,$$

and this contradicts the assumption $\lim_{k\to\infty}(l_{n_k+2} - l_{n_k+1} + r_{n_k}) = \infty$. Therefore Ω is not hyperbolic. □

Lemma 3.8. *Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ of natural numbers and a constant c with $l_{n_k}, l_{n_k+3} \le c$ for every k. Assume also that $\lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} \max\{l_{n_k+1}, l_{n_k+2}\} = \infty$. Then Ω is not hyperbolic.*

Proof. We define a sequence of natural numbers $\{m_k\}_k$ in the following way

$$m_k := \begin{cases} n_k + 1 & \text{if } l_{n_k+1} \ge l_{n_k+2}, \\ n_k + 2 & \text{if } l_{n_k+2} \ge l_{n_k+1}, \end{cases}$$

with m_1 large enough so that $l_{m_k} \ge c$ for every k. Then, $\lim_{k\to\infty} l_{m_k} = \infty, l_{m_k} \ge l_{m_k-1}, l_{m_k+1}$ and $\lim_{k\to\infty} r_{m_k-1} = \lim_{k\to\infty} r_{m_k} = \infty$. Applying Theorem 2.8 to $\{m_k\}_k$, we conclude that Ω is not hyperbolic. □

In the two following theorems the items labeled with (j') cover the symmetric case to the one listed exactly one position above (i.e., the one labeled with (j)).

Lemmas 3.3-3.8 give directly the following result for non-hyperbolicity.

Theorem 3.9. *Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ of natural numbers and a constant c with $l_{n_k}, l_{n_k+3} \le c$ for every k. Then Ω is not hyperbolic if we have either*:

(a) $\sup_k r_{n_k} < \infty$,

$$\lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \infty \quad \textit{and}$$
$$\sup_k (l_{n_k+1} - l_{n_k+2}) < \infty.$$

(a') $\sup_k r_{n_k+2} < \infty$,

$$\lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} l_{n_k+1} = \infty \quad \textit{and}$$
$$\sup_k (l_{n_k+2} - l_{n_k+1}) < \infty.$$

(b) $\sup_k r_{n_k} < \infty$,

$$\lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \lim_{k\to\infty} (l_{n_k+1} - l_{n_k+2}) = \infty \quad \textit{and}$$
$$\sup_k (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) < \infty.$$

(b') $\sup_k r_{n_k+2} < \infty$,

$$\lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} l_{n_k+1} = \lim_{k\to\infty} (l_{n_k+2} - l_{n_k+1}) = \infty \quad \textit{and}$$
$$\sup_k (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) < \infty.$$

(c) $\sup_k r_{n_k} < \infty$ *and*

$$\begin{aligned}\lim_{k\to\infty} r_{n_k+1} &= \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \lim_{k\to\infty} (l_{n_k+1} - l_{n_k+2}) \\ &= \lim_{k\to\infty} (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \lim_{k\to\infty} (l_{n_k+2} - l_{n_k+1} + r_{n_k+1}) \\ &= \infty.\end{aligned}$$

(c') $\sup_k r_{n_k+2} < \infty$ *and*

$$\begin{aligned}\lim_{k\to\infty} r_{n_k} &= \lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} l_{n_k+1} = \lim_{k\to\infty} (l_{n_k+2} - l_{n_k+1}) \\ &= \lim_{k\to\infty} (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \lim_{k\to\infty} (l_{n_k+1} - l_{n_k+2} + r_{n_k+1}) = \infty.\end{aligned}$$

(d) $\sup_k r_{n_k+1} < \infty$, $\sup_k (l_{n_k+2} - l_{n_k+1}) < \infty$ *and*

$$\lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+2} = \lim_{k\to\infty} (l_{n_k+2} - l_{n_k+1} + r_{n_k}) = \infty.$$

(d') $\sup_k r_{n_k+1} < \infty$, $\sup_k (l_{n_k+1} - l_{n_k+2}) < \infty$ *and*

$$\lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+2} = \lim_{k\to\infty} l_{n_k+1} = \lim_{k\to\infty} (l_{n_k+1} - l_{n_k+2} + r_{n_k+2}) = \infty.$$

(e) $\lim_{k\to\infty} r_{n_k} = \lim_{k\to\infty} r_{n_k+1} = \lim_{k\to\infty} r_{n_k+2}$

$$= \lim_{k\to\infty} \max\{l_{n_k+1}, l_{n_k+2}\} = \infty.$$

We prove now the following sufficient condition for hyperbolicity.

Theorem 3.10. *Let Ω be a train such that there exists a subsequence $S := \{n_k\}_k$ with $n_{k+1} - n_k = 3$ and $\sup_k l_{n_k} < \infty$. Then Ω is hyperbolic if there exist a constant C and a partition $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ of $\mathbb{N}$ such that each A_j is either empty or infinite and all of the following conditions hold*:

(a) $\min\big\{ \max\{r_{n_k}, r_{n_k+1}\}, \max\{r_{n_k}, r_{n_k+2}\}, \max\{r_{n_k+1}, r_{n_k+2}\}\big\} \le C$ *for every $k \in A_1$.*

(b) $\min\big\{ \max\{r_{n_k}, l_{n_k+2}\}, \max\{r_{n_k+2}, l_{n_k+1}\}\big\} \le C$ *for every $k \in A_2$.*

(c) $\max\big\{r_{n_k}, l_{n_k+2} - l_{n_k+1} + r_{n_k+1}\big\} \le C$ *for every $k \in A_3$, and*

$$\begin{aligned}\lim_{k\to\infty, k\in A_3} r_{n_k+1} &= \lim_{k\to\infty, k\in A_3} l_{n_k+2} = \lim_{k\to\infty, k\in A_3} (l_{n_k+1} - l_{n_k+2}) \\ &= \lim_{k\to\infty, k\in A_3} (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \infty.\end{aligned}$$

(c') $\max\big\{r_{n_k+2}, l_{n_k+1} - l_{n_k+2} + r_{n_k+1}\big\} \le C$ *for every $k \in A_4$, and*

$$\begin{aligned}\lim_{k\to\infty, k\in A_4} r_{n_k+1} &= \lim_{k\to\infty, k\in A_4} l_{n_k+1} = \lim_{k\to\infty, k\in A_4} (l_{n_k+2} - l_{n_k+1}) \\ &= \lim_{k\to\infty, k\in A_4} (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \infty.\end{aligned}$$

(d) $\max\big\{r_{n_k+1},\, \min\{l_{n_k+1},\, l_{n_k+2}\}\big\} \le C$ *for every* $k \in A_5$.

(e) $\max\big\{r_{n_k+1},\, l_{n_k+2} - l_{n_k+1} + r_{n_k}\big\} \le C$ *for every* $k \in A_6$, *and*

$$\lim_{k\to\infty, k\in A_6} l_{n_k+1} = \lim_{k\to\infty, k\in A_6} l_{n_k+2} = \infty.$$

(e') $\max\big\{r_{n_k+1},\, l_{n_k+1} - l_{n_k+2} + r_{n_k+2}\big\} \le C$ *for every* $k \in A_7$, *and*

$$\lim_{k\to\infty, k\in A_7} l_{n_k+1} = \lim_{k\to\infty, k\in A_7} l_{n_k+2} = \infty.$$

(f) $\max\big\{l_{n_k+1},\, l_{n_k+2}\big\} \le C$ *for every* $k \in A_8$.

Proof. First of all notice that, by hypothesis, the lengths l_{n_k} of the fundamental geodesics $\gamma^*_{n_k}$ belonging to the subsequence S satisfy $l_{n_k} \le c$ for some constant c. Therefore, if $m \in \{n_k\}_k$, then $d_\Omega(z, \mathbb{R}) \le l_m \le c$ for every $z \in \gamma^*_m$. It means that we just have to care about $d_\Omega(z, \mathbb{R})$ for every $z \in \{\gamma^*_m\}_{m \notin S}$.

Assume first that $k \in A_1$. Let us consider $z \in \gamma^*_{n_k+1} \cup \gamma^*_{n_k+2}$ for some fixed $k \in A_1$. Let us assume that for instance, $r_{n_k}, r_{n_k+1} \le C$, since the reasoning in any of the remaining cases in (a) is similar. If that is the situation, as we proved in Theorem 3.1, the fundamental hexagons H_{n_k} and H_{n_k+1} are $4\delta_0$-thin and it means (repeating the same argument in that theorem) that $d_\Omega(z, \mathbb{R}) \le 8\delta_0 + C$ for every $z \in \gamma^*_{n_k+1} \cup \gamma^*_{n_k+2}$.

Assume that $k \in A_2$. By symmetry, we can assume that $r_{n_k}, l_{n_k+2} \le C$. Since the fundamental hexagon H_{n_k} is $4\delta_0$-thin, we know that $d_\Omega(z, \mathbb{R}) \le 4\delta_0 + C$ for every $z \in \gamma^*_{n_k+1}$. We also have $d_\Omega(z, \mathbb{R}) \le l_{n_k+2} \le C$ for every $z \in \gamma^*_{n_k+2}$.

Consider $k \in A_3$. Note that the fundamental hexagon H_{n_k} is $4\delta_0$-thin and it means that $d_\Omega(z, \mathbb{R}) \le 4\delta_0 + C$ for every $z \in \cup_{k\in A_3} \gamma^*_{n_k+1}$.

Consider $z \in \gamma^*_{n_k+2}$ with height $h(z) = h \in [0, l_{n_k+2}]$.

Let us consider now the fundamental hexagon H_{n_k+1}. Standard hyperbolic trigonometry (see e.g. [9, p. 161]) gives

$$\cosh \alpha_{n_k+1} = \frac{\cosh r_{n_k+1} + \cosh l_{n_k+1} \cosh l_{n_k+2}}{\sinh l_{n_k+1} \sinh l_{n_k+2}} \approx 1 + 2\, e^{r_{n_k+1} - l_{n_k+1} - l_{n_k+2}},$$

if $k \to \infty, k \in A_3$. Since $\lim_{k\to\infty, k\in A_3}(l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \infty$, we deduce that

$$\cosh \alpha_{n_k+1} \approx 1 + \frac{1}{2}\, \alpha^2_{n_k+1} \approx 1 + 2\, e^{r_{n_k+1} - l_{n_k+1} - l_{n_k+2}},$$
$$\alpha_{n_k+1} \approx 2\, e^{-\frac{1}{2}(l_{n_k+1} + l_{n_k+2} - r_{n_k+1})}.$$

Standard hyperbolic trigonometry (see e.g. [9, p. 157]) gives

$$\begin{aligned} \sinh d_\Omega(z_k, \gamma_{n_k+1}) &= \sinh \alpha_{n_k+1} \cosh h \\ &\le c\, e^{-\frac{1}{2}(l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) + h} \\ &\le c\, e^{-\frac{1}{2}(l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) + l_{n_k+2}} \end{aligned}$$

$$\leq c\, e^{\frac{1}{2}(l_{n_k+2}-l_{n_k+1}+r_{n_k+1})} \leq c\, e^{C},$$

if $k \in A_3$. Therefore,

$$d_\Omega(z,\mathbb{R}) \leq 4\delta_0 + C + d_\Omega(z,\gamma_{n_k+1}) \leq 4\delta_0 + C + \operatorname{Arcsinh}(c\, e^{C})$$

for every $z \in \cup_{k\in A_3}\gamma^*_{n_k+2}$.

If $k \in A_4$, then we can use a symmetric argument to the previous one.

Assume now that $k \in A_5$. We have $r_{n_k+1} \leq C$ and $\min\{l_{n_k+1},\, l_{n_k+2}\} \leq C$. By symmetry, we can assume that $l_{n_k+1} \leq C$. Then we have $d_\Omega(z,\mathbb{R}) \leq l_{n_k+1} \leq C$ for every $z \in \gamma^*_{n_k+1}$.

Since the fundamental hexagon H_{n_k+1} is $4\delta_0$-thin, we have that $d_\Omega(z,\mathbb{R}) \leq 4\delta_0 + C$ for every $z \in \gamma^*_{n_k+2}$.

Let us consider $k \in A_6$.

Let $z \in \gamma^*_{n_k+1}$ with height $h(z) = h \in [0, l_{n_k+1}]$. If $h \in [0, l_{n_k}]$, then $d_\Omega(z,\mathbb{R}) \leq d_\Omega(z,(a_0,b_0)) = h \leq l_{n_k} \leq c$.

Assume first that $l_{n_k+2} \leq l_{n_k+1}$. If $h \in [l_{n_k}, l_{n_k+2}]$, by Lemma 2.11,

$$\begin{aligned} d_\Omega(z,(a_{n_k},b_{n_k})) &\asymp e^{-h+l_{n_k}} + e^{-(l_{n_k+1}-h-r_{n_k})_+} + (r_{n_k}+h-l_{n_k+1})_+ \\ &\leq 1 + 1 + (r_{n_k}+l_{n_k+2}-l_{n_k+1})_+ \leq 2 + C\,. \end{aligned}$$

If $h \in [l_{n_k+2}, l_{n_k+1}]$, then Lemma 2.12 gives that $d_\Omega(z,(a_{n_k+2},b_{n_k+2})) \leq c_1$, where c_1 only depends on C and $\min_{k\in A_6} l_{n_k+2} > 0$.

Assume now that $l_{n_k+1} < l_{n_k+2}$. If $h \in [l_{n_k}, l_{n_k+1}]$, by Lemma 2.11,

$$\begin{aligned} d_\Omega(z,(a_{n_k},b_{n_k})) &\asymp e^{-h+l_{n_k}} + e^{-(l_{n_k+1}-h-r_{n_k})_+} + (r_{n_k}+h-l_{n_k+1})_+ \\ &\leq 1 + 1 + (r_{n_k}+l_{n_k+2}-l_{n_k+1})_+ \leq 2 + C\,. \end{aligned}$$

Then we also have $d_\Omega(z,(a_{n_k+2},b_{n_k+2})) \leq c_1$.

Hence, $d_\Omega(z,\mathbb{R}) \leq c_2$ for every $z \in \gamma^*_{n_k+1}$.

If $z \in \gamma^*_{n_k+2}$, then $d_\Omega(z,\mathbb{R}) \leq 4\delta_0 + \max\{C, c_2\}$, since the fundamental hexagon H_{n_k+1} is $4\delta_0$-thin.

If $k \in A_7$, then we can use a symmetric argument to the previous one.

Finally, let us consider $k \in A_8$. For every $z \in \gamma^*_{n_k+1} \cup \gamma^*_{n_k+2}$ we have that $d_\Omega(z,\mathbb{R}) \leq d_\Omega(z,(a_0,b_0)) \leq \max\{l_{n_k+1}, l_{n_k+2}\} \leq C$.

This finishes the proof. □

4. A characterization for regular domains

In this section we take advantage of the results proved in the previous section to deduce a characterization of hyperbolicity for some especially regular Denjoy domains.

Theorem 4.3 shows that the hypotheses in Theorems 3.9 and 3.10 are quite close to be complementary.

Definition 4.1. We say that a sequence (or subsequence) is *regular* if either it is upper bounded or its limit is infinity. Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ with $1 \le n_{k+1} - n_k \le 3$ for every k and $\sup_k l_{n_k} < \infty$. We denote by S the subsequence of every n_k verifying $n_{k+1} - n_k = 3$. We say that Ω is *regular* if the following subsequences (with $n_k \in S$) are regular: $\{r_{n_k}\}$, $\{r_{n_k+1}\}$, $\{r_{n_k+2}\}$, $\{l_{n_k+1}\}$, $\{l_{n_k+2}\}$, $\{l_{n_k+1} - l_{n_k+2}\}$, $\{l_{n_k+2} - l_{n_k+1}\}$, $\{l_{n_k+2} - l_{n_k+1} + r_{n_k+1}\}$, $\{l_{n_k+1} - l_{n_k+2} + r_{n_k+1}\}$, $\{l_{n_k+1} + l_{n_k+2} - r_{n_k+1}\}$, $\{l_{n_k+2} - l_{n_k+1} + r_{n_k}\}$, $\{l_{n_k+1} - l_{n_k+2} + r_{n_k+2}\}$.

Remark 4.2. (1) The maximum and the minimum of several regular sequences is also a regular sequence.

(2) Note that any rational function evaluated in the variables $n^{a_1}, \dots, n^{a_j}$, $e^{b_1 n}, \dots, e^{b_j n}, (\log n)^{c_1}, \dots, (\log n)^{c_j}$, is regular. In particular, if $\{r_{n_k}\}, \{r_{n_k+1}\}$, $\{r_{n_k+2}\}, \{l_{n_k+1}\}, \{l_{n_k+2}\}$ are rational functions in these variables, then the domain is regular.

Theorem 4.3. *Let Ω be a train such that there exists a subsequence $\{n_k\}_k$ with $1 \le n_{k+1} - n_k \le 3$ for every k and $\sup_k l_{n_k} < \infty$. Denote by S the subsequence of every n_k verifying $n_{k+1} - n_k = 3$, and assume also that Ω is regular. Then Ω is hyperbolic if and only if there exists a constant C such that $\min\{l_{n_k+1}, r_{n_k}, r_{n_k+1}\} \le C$ for every k with $n_{k+1} - n_k = 2$, and we have any of the following conditions*:

(a) $\min\big\{ \max\{r_{n_k}, r_{n_k+1}\}, \max\{r_{n_k}, r_{n_k+2}\}, \max\{r_{n_k+1}, r_{n_k+2}\}\big\} \le C$ *for every* $n_k \in S$.

(b) $\min\big\{ \max\{r_{n_k},\, l_{n_k+2}\},\, \max\{r_{n_k+2},\, l_{n_k+1}\}\big\} \le C$ *for every* $n_k \in S$.

(c) $\max\big\{r_{n_k},\, l_{n_k+2} - l_{n_k+1} + r_{n_k+1}\big\} \le C$ *for every* $n_k \in S$, *and*

$$\lim_{k\to\infty, n_k\in S} r_{n_k+1} = \lim_{k\to\infty, n_k\in S} l_{n_k+2} = \lim_{k\to\infty, n_k\in S} (l_{n_k+1} - l_{n_k+2}) = \lim_{k\to\infty, n_k\in S} (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \infty.$$

(c') $\max\big\{r_{n_k+2},\, l_{n_k+1} - l_{n_k+2} + r_{n_k+1}\big\} \le C$ *for every* $n_k \in S$, *and*

$$\lim_{k\to\infty, n_k\in S} r_{n_k+1} = \lim_{k\to\infty, n_k\in S} l_{n_k+1} = \lim_{k\to\infty, n_k\in S} (l_{n_k+2} - l_{n_k+1}) = \lim_{k\to\infty, n_k\in S} (l_{n_k+1} + l_{n_k+2} - r_{n_k+1}) = \infty.$$

(d) $\max\big\{r_{n_k+1},\, \min\{l_{n_k+1},\, l_{n_k+2}\}\big\} \le C$ *for every* $n_k \in S$.

(e) $\max\big\{r_{n_k+1},\, l_{n_k+2} - l_{n_k+1} + r_{n_k}\big\} \le C$ *for every* $n_k \in S$, *and*

$$\lim_{k\to\infty, n_k\in S} l_{n_k+1} = \lim_{k\to\infty, n_k\in S} l_{n_k+2} = \infty.$$

(e') $\max\big\{r_{n_k+1},\, l_{n_k+1} - l_{n_k+2} + r_{n_k+2}\big\} \le C$ *for every* $n_k \in S$, *and*

$$\lim_{k\to\infty, n_k\in S} l_{n_k+1} = \lim_{k\to\infty, n_k\in S} l_{n_k+2} = \infty.$$

(f) $\max\big\{l_{n_k+1},\, l_{n_k+2}\big\} \le C$ *for every* $n_k \in S$.

Proof. First of all notice that, by hypothesis, the lengths l_{n_k} of the fundamental geodesics $\gamma_{n_k}^*$ belonging to the subsequence $\{n_k\}_k$ satisfy $l_{n_k} \leq c$ for some constant c. Therefore, if $m \in \{n_k\}_k$, then $d_\Omega(z,\mathbb{R}) \leq l_m \leq c$ for every $z \in \gamma_m^*$. It means that we just have to care about $d_\Omega(z,\mathbb{R})$ for every $z \in \{\gamma_m^*\}$ with $m \notin \{n_k\}_k$.

By Theorem 3.1 we know that $d_\Omega(z,\mathbb{R}) \leq c_1$ for every $z \in \gamma_{n_k+1}^*$ with $n_{k+1} - n_k = 2$ if and only if $\min\{l_{n_k+1}, r_{n_k}, r_{n_k+1}\} \leq C$ for every k with $n_{k+1} - n_k = 2$. Then we just have to deal with $d_\Omega(z,\mathbb{R})$ for every $z \in \gamma_m^*$ with $m \in S$. Therefore, in what follows, we will assume that for all the subsequences involved, the index runs on the values k with $n_k \in S$.

By hypothesis we know that every subsequence appearing in the items (a)-(f) either is upper bounded or has infinite limit.

First, let us assume that none of the subsequences $\{r_{n_k}\}_k, \{r_{n_k+1}\}_k, \{r_{n_k+2}\}_k$ is bounded. In this situation, if $\max\{l_{n_k+1}, l_{n_k+2}\} = \infty$, we conclude that Ω is not hyperbolic by Theorem 3.9(e). Otherwise, Ω is hyperbolic by Theorem 3.10(f).

Second, let us assume either that two of the subsequences $\{r_{n_k}\}_k, \{r_{n_k+1}\}_k$, $\{r_{n_k+2}\}_k$ are bounded and the other is not bounded, or all three sequences $\{r_{n_k}\}_k, \{r_{n_k+1}\}_k, \{r_{n_k+2}\}_k$ are bounded. Then, by Theorem 3.10(a), Ω is hyperbolic.

Hence, from now on, we will assume that only one of the previously mentioned subsequences is bounded. If this is the situation, we have to distinguish two different cases: the first one occurs when either $\{r_{n_k}\}_k$ or $\{r_{n_k+2}\}_k$ is bounded, and the second one, when $\{r_{n_k+1}\}_k$ is the bounded subsequence. Let us deal with the first case, but we will just analyze the first possibility, i.e., when the bounded sequence is $\{r_{n_k}\}_k$, since the remaining case is symmetric. Notice that, if $l_{n_k+2} \leq C$ for every k, then Ω is hyperbolic by Theorem 3.10(b). If this is not the case, i.e., if $\lim_{k\to\infty} l_{n_k+2} = \infty$, then we will again distinguish two possibilities:

(1) If $l_{n_k+1} - l_{n_k+2} \leq C$ for every k, then by Theorem 3.9(a), Ω is not hyperbolic.

(2) Otherwise, i.e., if $\lim_{k\to\infty}(l_{n_k+1} - l_{n_k+2}) = \infty$, we have to consider two options:

   (a) If $l_{n_k+1} + l_{n_k+2} - r_{n_k+1} \leq C$ for every k, then, by Theorem 3.9(b), Ω is not hyperbolic.

   (b) If the expression mentioned above tends to infinity with k, then we have two alternatives:

      (i) If $l_{n_k+2} - l_{n_k+1} + r_{n_k+1} \leq C$ for every k, then Ω is hyperbolic by Theorem 3.10(c).

      (ii) If the expression mentioned above tends to infinity with k, then Ω is not hyperbolic by Theorem 3.9(c).

This covers all the possibilities when the subsequence $\{r_{n_k}\}_k$ (or $\{r_{n_k+2}\}_k$) is bounded. Let us study now what happens when the only bounded subsequence is $\{r_{n_k+1}\}_k$. If this is the situation, and besides one of the two subsequences $\{l_{n_k+1}\}_k$ or $\{l_{n_k+2}\}_k$ is bounded, then Ω is hyperbolic by Theorem 3.10(d). Otherwise, i.e., when both subsequences tend to infinity with k, we distinguish two cases:

(1) If $l_{n_k+2} - l_{n_k+1} + r_{n_k} \le C$ for every k, then Ω is hyperbolic by Theorem 3.10(e).
(2) If $\lim_{k\to\infty}(l_{n_k+2} - l_{n_k+1} + r_{n_k}) = \infty$, there are, again, two possibilities:
   (a) If $l_{n_k+2} - l_{n_k+1} \le C$ for every k, Ω is not hyperbolic by Theorem 3.9(d).
   (b) If the expression mentioned above tends to infinity with k, it means that $\{l_{n_k+1} - l_{n_k+2}\}_k$ is upper bounded, and then there are two options:
      (i) If $\lim_{k\to\infty}(l_{n_k+1} - l_{n_k+2} + r_{n_k+2}) = \infty$, then Ω is not hyperbolic by Theorem 3.9(d').
      (ii) Otherwise, $l_{n_k+1} - l_{n_k+2} + r_{n_k+2} \le C$ for every k, and Ω is hyperbolic by Theorem 3.10(e').

This finishes the proof. □

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